ORIENTED TAIT GRAPHS

Dedicated to the memory of Hanna Neumann

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The four colour conjecture is well known to be equivalent to the proposition that every trivalent planar graph without an isthmus (i.e. an edge whose removal disconnects the graph) has an edge colouring in three colours ([1], p. 121). By an edge colouring we mean an assignment of colours to the edges of the graph so that no two edges of the same colour meet at a common vertex, and the graph is n-valent if n edges meet at each vertex. An edge colouring by three colours is called a Tait colouring; a trivalent graph which has a Tait colouring can be split in three edge-disjoint 1-factors, i.e. spanning monovalent subgraphs.

Let $G = \{G_1, G_2, G_3\}$ be a set of three monovalent graphs on a common vertex set $\{x_1, \cdots, x_{2n}\}$, each G_α containing n edges. We call such a set a *Tait graph* of order n; edges of G_α will be called α-edges and G_α, $\alpha = 1,2,3$ the *factors* of G. We allow edges of distinct factors to join the same pair of vertices, that is G (regarded as a trivalent graph with Tait colouring) may have multiple edges; no loops are admitted.

A Tait graph also defines three 2-factors (i.e. bivalent spanning subgraphs) $G_{\alpha\beta} = \{G_\alpha, G_\beta\}$, $1 \leqq \alpha < \beta \leqq 3$. They consist of disjoint elementary circuits formed by an even number of alternating α,β-edges. We refer to these as the circuits (in particular the (α,β) circuits) of G; the total number of circuits in G will be denoted by $\sigma(G)$. Each edge is associated with two circuits containing the edge.

Because of the relevance of trivalent Tait colourable graphs to the four colour problem, any non-trivial parity property of these graphs is of interest. The purpose of this note is to reveal such a property.

We define an *oriented Tait graph* G as one in which each circuit is provided with an orientation. In an oriented Tait graph each edge e has an orientation index $\omega(e)$, defined to be 0 or 1 according to whether the two associated circuits have the same or opposite orientation on e. The orientation index of G is defined $\omega(G) = \sum_e \omega(e)$ summed for all edges of G. Hence $\omega(G)$ is the number of edges which have opposite orientation in the two associated circuits. Similar definition applies to $\omega(C)$ when C is a circuit of G. Clearly $\omega(C)$ depends on the orientation of the circuit, but the parity of $\omega(C)$ is independent of orientation. For let $2k$ be the length of C. If we change the orientation of C then the number of oppositely oriented edges in the new orientation will be $2k - \omega(C)$, which is $\equiv \omega(C) \bmod 2$. In particular the parity of $\omega(G)$ depends only on G as a Tait graph, but not on the specific orientation of its circuits.

Our purpose is to prove:

THEOREM. $\omega(G) \equiv \sigma(G) \pmod 2$.

Or, the orientation index of a Tait graph (with any orientation of its circuits) has the same parity as the number of its distinct circuits.

By our previous remark it is sufficient to prove the theorem for any fixed orientation of the circuits. We may assume that G is connected; for if it is the union of disjoint oriented Tait graphs G_1 and G_2 then trivially $\omega(G) = \omega(G_1) + \omega(G_2)$, $\sigma(G) = \sigma(G_1) + \sigma(G_2)$, and the theorem is true for G if it is true for G_1 and G_2.

The following notation will be found convenient. We denote by $(x^\alpha y)$ an α-edge joining x and y, and by $\langle x^\alpha y\rangle$ an α-edge directed from x to y in a given orientation of the circuits. Similar notation will be used for arcs and circuits, e.g. $(x^\beta y^\alpha z)$ is the arc consisting of a β-edge $(x^\beta y)$ and an α-edge $(y^\alpha z)$, and $\langle x^\alpha y^\beta z^\alpha p^\beta x\rangle$ is an oriented (α, β) circuit.

We first verify the theorem in two simple cases: when $n = 1$ and when G is the complete 4-graph with Tait colouring.

If $n = 1$ then G has two vertices $\{x, y\}$ and three edges (x^1y), (x^2y), (x^3y). Take an orientation in which the circuits are $\langle x^1y^2x\rangle$, $\langle x^2y^3x\rangle$, $\langle x^3y^1x\rangle$; then $\omega(x^1y) = \omega(x^2y) = \omega(x^3y) = 1$, $\omega(G) = 3 = \sigma(G)$.

If G is the complete 4-graph with vertices $\{x, z, p, q\}$ and edges (x^1y), (x^2p), (x^3q), (y^3p), (y^2q), (p^1q), the circuits may be taken as $\langle x^1y^2q^1p^2x\rangle$, $\langle x^2p^3y^2q^3x\rangle$, $\langle x^3q^1p^3y^1x\rangle$, giving $\omega(x^1y) = \omega(x^2p) = \omega(x^3q) = 1$, $\omega(y^3p) = \omega(y^2q) = \omega(p^1q) = 0$. Hence $\omega(G) = 3 = \sigma(G)$.

In the general case we proceed by induction on the order of G. We have already verified the theorem for $n = 1$. Since a triple edge between two vertices x, y forms a Tait graph of order 1, disconnected from the rest, we may assume that G (as a trivalent graph) has no triple edges.

Suppose that G contains a double edge (x^1y), (x^2y), the other edges adjacent to x and y being (x^3p), (y^3q) where $p \neq q$. Consider the Tait graph G^* obtained from G by removing x, y and the four edges adjacent to them, and inserting a new edge (p^3q). Now $\langle p^3x^1y^3q\rangle$ forms part of an oriented $(1, 3)$ circuit $\langle p^3x^1y^3q^1U^1p\rangle$ in G, where U is a $(1, 3)$ arc of odd length in $G^* \cap G$. Similarly $\langle p^3x^2y^3q\rangle$ forms part of an oriented $(2, 3)$ circuit $\langle p^3x^2y^3V^2p\rangle$ in G, where V is a $(2, 3)$ arc of odd length in $G^* \cap G$. Hence $\langle p^3q^1U^1p\rangle$ is a $(1,3)$ circuit in G^*, $\langle p^3q^2V^2p\rangle$ is a $(2,3)$ circuit in G^*, and all other circuits in G^* are common to those in G. The only remaining circuit in G not contained in G^* is $\langle x^1y^2x\rangle$; thus $\sigma(G) = \sigma(G^*)+1$.

With the given orientations of the circuits, the orientation indices of the edges removed from G are $\omega(p^3x) = \omega(q^3y) = \omega(x^1y) = 0$, $\omega(x^2y) = 1$, and the orientation index of the new edge in G^* is $\omega(p^3q) = 0$; hence $\omega(G) = \omega(G^*) + 1$. So $\omega(G^*) \equiv \sigma(G^*) \pmod 2$ implies $\omega(G) \equiv \sigma(G) \pmod 2$, and the theorem is true for G provided it is true for G^*.

Assume now that all edges of G are simple, but G contains a triangle with edges (x^1y), (x^2p), (y^3p). We may assume that the edge (x^1y) is not adjacent to another triangle; for suppose that G has the property that every edge which is adjacent to one triangle is adjacent to another one. Then there is a vertex $q \neq p$ such that (x^3q), (y^2q) are edges, and then (since (y^3p), (x^1y) and (x^2p) form a triangle and (y^2q) is adjacent to (y^3p)), (p^1q) is also an edge. Hence the four vertices x, y, p, q span a Tait subgraph disconnected from the rest and we have already checked the theorem for this graph.

Assume then that (x^1y) is adjacent to exactly one triangle with edges (x^2p), (y^3p). Let the other two edges adjacent to x and y be (x^3q), (y^2r) with $p \neq q \neq r \neq p$. Again we consider G^* obtained from G by removing x, y and the five edges adjacent to them, and inserting the edges (p^2r), (p^3q). The circuits in G involving the removed edges are

$$\langle q^3x^2p^3y^2r^3U^2q\rangle,\ \langle q^3x^1y^3p^1V^1q\rangle,\ \langle p^3x^1y^2r^1W^1p\rangle$$

where U is a $(2,3)$ arc, V a $(1,3)$ arc, W a $(1,2)$ arc in $G^* \cap G$. The new circuits in G^* are

$$\langle q^3p^2r^3U^2q\rangle,\ \langle q^3p^1V^1q\rangle,\ \langle p^2r^1W^1p\rangle$$

hence $\sigma(G) = \sigma(G^*)$. The orientation indices in G are

$$\omega(x^1y) = \omega(x^3q) = \omega(y^2r) = 0,\ \omega(x^2p) = \omega(y^3p) = 1,$$

and in G^* $\omega(p^2r) = \omega(p^3q) = 0$, all other edges common to G and G^* have the same orientation indices in both graphs. Hence $\omega(G) = \omega(G^*) + 2 \equiv \omega(G^*) \pmod 2$, and the theorem follows by induction, provided that G contains at least one triangle.

Let us finally assume that G has no multiple edges and triangles. Then if (x^1y) is an edge, the other edges adjacent to x and y are (x^2p), (x^3q), (y^2r), (y^3s) with all six vertices x, y, p, q, r, s distinct. G^* is now obtained from G by removing these five edges and inserting (p^2r), (q^3s). Since $\langle p^2x^1y^2r\rangle$ is part of an oriented $(1,2)$ circuit in G, there is a $(1,2)$ arc $\langle r^1U^1p\rangle$ in $G^* \cap G$ (which may pass through q or s), and similarly there is a $(1,3)$ arc $\langle q^1V^1s\rangle$ in $G^* \cap G$.

For the $(2,3)$ arcs in $G^* \cap G$ there are three possibilities

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(i) $\langle p^3S^2q\rangle$, $\langle r^3T^2s\rangle$, where S, T are oriented $(2,3)$ arcs of even length. Circuits in G:

$$\langle p^3S^2q^3x^2p\rangle, \langle r^3T^2s^3y^2r\rangle, \langle r^1U^1p^2x^1y^2r\rangle, \langle q^1V^1s^3y^1x^3q\rangle.$$

Circuits in G^*:

$$\langle r^1U^1p^2r\rangle, \langle q^1V^1s^3q\rangle, \langle p^3S^2q^3s^2T'^3r^2p\rangle$$

where T' denotes T in opposite orientation. Hence

$$\sigma(G) = \sigma(G^*) + 1.$$

The relevant orientation indices are

$$\omega(x^1y) = \omega(x^2p) = \omega(x^3q) = 1, \ \omega(y^2r) = \omega(y^3s) = 0 \text{ in } G,$$

$$\omega(p^2r) = \omega(q^3s) = 1 \text{ in } G^*.$$

Also $\omega(T) \equiv \omega(T') \pmod 2$ since T is of even length, and we have $\omega(G) \equiv \omega(G^*) + 1 \pmod 2$, as required.

(ii) $\langle p^3S^3r\rangle$, $\langle s^2T^2q\rangle$, where S, T are oriented $(2,3)$ arcs of odd length in $G \cap G^*$. The relevant $(2,3)$ circuit in G is $\langle p^3S^3r^2y^3s^2T^2q^3x^2p\rangle$, and in G^* $\langle p^3S^3r^2p\rangle$, $\langle s^2T^2q^3s\rangle$, all other circuits as before. Hence $\sigma(G) = \sigma(G^*) - 1$. Furthermore

$$\omega(x^1y) = \omega(x^2p) = \omega(x^3q) = \omega(y^2r) = \omega(y^3s) = 1 \text{ in } G,$$

$$\omega(p^2r) = \omega(q^3s) = 1 \text{ in } G^*,$$

$$\omega(G) = \omega(G^*) + 3, \ \omega(G) - \omega(G) \equiv \omega(G^*) - \sigma(G^*) \pmod 2$$

(iii) $\langle p^3S^2s\rangle$, $\langle r^3T^2q\rangle$, where S, T are oriented $(2,3)$ arcs of even length in $G \cap G^*$. The relevant $(2,3)$ circuit in G is $\langle p^3S^2s^3y^2r^3T^2q^3x^2p\rangle$, and in G^* $\langle p^3S^2s^3q^2T'^3r^2p\rangle$, so that $\sigma(G) = \sigma(G^*)$. Furthermore

$$\omega(x^1y) = \omega(x^2p) = \omega(x^3q) = 1, \ \omega(y^2r) = \omega(y^3s) = 0 \text{ in } G,$$

$$\omega(p^2r) = 1, \ \omega(q^3s) = 0 \text{ in } G^*,$$

$$\omega(T') \equiv \omega(T) \pmod 2 \text{ hence } \omega(G) \equiv \omega(G^*) \pmod 2$$

as required. We have exhausted all possibilities, and the theorem is proved.

Reference

[1] O. Ore, *The Four-color Problem*, (Academic Press, New York, 1967).

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