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CONTINUITY OF WEIGHTED COMPOSITION OPERATORS BETWEEN WEIGHTED BLOCH TYPE SPACES

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Abstract

Let $\phi : D \to D$ and $\psi : D \to \mathbb{C}$ be analytic maps. These induce a weighted composition operator ψC_ϕ acting between weighted Bloch type spaces. Under some assumptions on the weights we give a necessary as well as a sufficient condition when such an operator is continuous.

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1. Introduction

Let D denote the open unit disk D of the complex plane and let $\phi : D \to D$ and $\psi : D \to \mathbb{C}$ be analytic maps. These induce a linear weighted composition operator $\psi C_\phi(f) = \psi(f \circ \phi)$. Furthermore, let v and w be strictly positive continuous and bounded functions (*weights*) on D. We denote by B_v the weighted Bloch type space of functions $f \in H(D)$ satisfying $\|f\|_{B_v} := \sup_{z \in D} v(z)|f'(z)| < \infty$. Provided that we identify functions that differ by a constant, $\|.\|_{B_v}$ becomes a norm and B_v a Banach space.

We are interested in weighted composition operators acting between weighted Bloch type spaces. Composition operators and weighted composition operators acting between various spaces of analytic functions have been investigated by several authors; see, for example, [2–4, 7, 8, 10, 11]. The aim of this paper is to give a necessary and a sufficient condition for a weighted composition operator acting between weighted Bloch type spaces to be continuous. These conditions are given in terms of the weights as well as the analytic functions ϕ and ψ.

2. Notation and auxiliary results

For notation on composition operators we refer the reader to the monographs [5, 12]. To treat the described problem we need some information on weighted Bergman spaces of infinite order, defined as

$$H_v^\infty := \left\{ f \in H(D);\ \|f\|_v = \sup_{z \in D} v(z)|f(z)| < \infty \right\}.$$

Let $B^{v,\infty}$ denote the closed unit ball of the space H_v^∞. In the setting of weighted spaces of analytic functions such as the weighted Bergman spaces of infinite order or – as in the present case – weighted Bloch type spaces the so-called *associated weights* turn out to be very useful. For a weight v we can define the associated weight as follows:

$$\widetilde{v}(z) = \frac{1}{\sup\{|f(z)|;\ f \in B^{v,\infty}\}} = \frac{1}{\|\delta_z\|_{H_v^{\infty\prime}}},$$

where δ_z denotes the point evaluation of z. By [1] we know that the associated weight $\widetilde{v}$ has the following properties:

(i) $\widetilde{v}$ is continuous;
(ii) $\widetilde{v} \geq v > 0$;
(iii) for every $z \in D$ we can find $f_z \in B^{v,\infty}$ such that $|f_z(z)| = 1/\widetilde{v}(z)$.

It will be helpful to recall an auxiliary result.

THEOREM 1 (Harutyunyan–Lusky [6, Theorem 2.1]). *Let v be a radial weight such that v is continuously differentiable with respect to $|z|$, $\lim_{|z|\to 1} v(z) = 0$ and such that H_v^∞ is isomorphic to l_∞. If*

$$\limsup_{r\to 1} \left(-\frac{v'(r)}{v(r)} \right) < \infty,$$

then $D : H_v^\infty \to H_v^\infty$, $f \to f'$ is bounded.

For conditions when H_v^∞ is isomorphic to l^∞ we refer the reader to [6, 9]. By [6] we know that the weights $v(z) = (1 - |z|)^\alpha$, $\alpha > 0$, and $v(z) = e^{-1/(1-|z|)}$, $z \in D$, have the desired properties.

3. Main result

PROPOSITION 2. *Let w be an arbitrary weight and v be a radial weight such that v is continuously differentiable with respect to $|z|$, $\lim_{|z|\to 1} v(z) = 0$ and such that H_v^∞ is isomorphic to l_∞. Moreover, we assume that*

$$\limsup_{r\to 1} \left(-\frac{v'(r)}{v(r)} \right) < \infty.$$

If the weighted composition operator $\psi C_\phi : B_v \to B_w$ is continuous, then:

(a) $\sup_{z\in D} |\psi'(z)|(w(z)/\widetilde{v}(\phi(z))^{1/2}) < \infty$;
(b) $\sup_{z\in D} |\psi(z)||\phi'(z)|(w(z)/\widetilde{v}(\phi(z))) < \infty$.

PROOF. Let us assume that $\psi C_\phi : B_v \to B_w$ is continuous. First, we show (a). Fix $a \in D$. Now we choose $f_{a,1/2} \in B^{v^{1/2},\infty}$ with $f_{a,1/2}(\phi(a)) = 1/\widetilde{v}(\phi(a))^{1/2}$ and set $g_a(z) = 2f_{a,1/2}(z) - \widetilde{v}(\phi(a))^{1/2} f_{a,1/2}(z)^2$. Then

$$g_a'(z) = 2f_{a,1/2}'(z) - 2\widetilde{v}(\phi(a))^{1/2} f_{a,1/2}(z) f_{a,1/2}'(z).$$

Obviously $g_a(\phi(a)) = 1/\widetilde{v}(\phi(a))^{1/2}$ and $g_a'(\phi(a)) = 0$.

Then

$$|\psi'(a)| \frac{w(a)}{\widetilde{v}(\phi(a))^{1/2}} \le \|\psi C_\phi g_a\|_{B_w} \le \|\psi C_\phi\| \|g_a\|_{B_v}.$$

By Theorem 1 we know that under the given assumptions $D : H_v^\infty \to H_v^\infty$, $f \to f'$ is bounded. Taking into account that there exists $M > 0$ such that $\sup_{z\in D} v(z)^{1/2} \le M$, we obtain

$$\begin{aligned}
\|g_a\|_{B_v} &= \sup_{z\in D} v(z)|g_a'(z)| \\
&= \sup_{z\in D} v(z)|2f_{a,1/2}'(z) - 2\widetilde{v}(\phi(a))^{1/2} f_{a,1/2}(z) f_{a,1/2}'(z)| \\
&\le 2\sup_{z\in D} v(z)^{1/2}|f_{a,1/2}'(z)| \sup_{z\in D} v(z)^{1/2} \\
&\quad + 2\widetilde{v}(\phi(a))^{1/2} \sup_{z\in D} v(z)|f_{a,1/2}(z) f_{a,1/2}'(z)| \\
&\le 2\|f_{a,1/2}'\|_{v^{1/2}} M + 2M\|f_{a,1/2}\|_{v^{1/2}} \|f_{a,1/2}'\|_{v^{1/2}} \\
&\le 2M\|D\| \|f_{a,1/2}\|_{v^{1/2}} + 2M\|f_{a,1/2}\|_{v^{1/2}} \|D\| \le 4M\|D\|.
\end{aligned}$$

Thus

$$|\psi'(a)| \frac{w(a)}{\widetilde{v}(\phi(a))} \le 4M\|D\| \|\psi C_\phi\| < \infty,$$

which establishes (a).

Next, we show that (b) holds. Fix $a \in D$. Then choose $f_a' \in B^{v,\infty}$ such that $f_a'(\phi(a)) = 1/\widetilde{v}(\phi(a))$. Set $g_a(z) := f_a(z) - f_a(\phi(a))$. Then $g_a'(z) = f_a'(z)$. Obviously $g_a(\phi(a)) = 0$ and $g_a'(z) = 1/\widetilde{v}(\phi(a))$ as well as

$$\|g_a\|_{B_v} = \|f_a'\|_v \le 1.$$

We obtain

$$\begin{aligned}
w(a)|\psi(a)| \frac{|\phi'(a)|}{\widetilde{v}(\phi(a))} &= w(a)|(\psi C_\phi g_a)'(a)| \\
&\le \|\psi C_\phi\| \|g_a\|_{B_v} \le \|\psi C_\phi\| < \infty.
\end{aligned}$$

Thus the claim follows. $\square$

PROPOSITION 3. *Let w be an arbitrary weight and v be a radial weight such that $1/V$ is a primitive of $1/v$ with respect to $|z|$. If:*

(a) $\sup_{z\in D} w(z)|\psi'(z)|(1/V(\phi(z))) < \infty$;

(b) $\sup_{z\in D} w(z)|\psi'(z)| < \infty$;

(c) $\sup_{z\in D} |\psi(z)|(w(z)|\phi'(z)|/v(\phi(z))) < \infty$;

then $\psi C_\phi : B_v \to B_w$ is continuous.

PROOF. We fix $f \in B_v$ such that $\|f\|_{B_v} \le 1$. Obviously $|f'(z)| \le \|f\|_{B_v}/v(z) \le 1/v(z)$. Integration yields

$$|f(z) - f(0)| \le \int_0^1 |z||f'(zt)|\,dt \le \int_0^1 \frac{|z|}{v(t|z|)}\,dt$$
$$= \left[\frac{1}{V(t|z|)}\right]_0^1 = \frac{1}{V(|z|)} - \frac{1}{V(0)} = \frac{V(0) - V(|z|)}{V(0)V(|z|)} \le \frac{K}{V(z)}.$$

Finally, we get

$$\begin{aligned} w(z)|(\psi C_\phi)'(z)| &= w(z)|\psi'(z) f(\phi(z)) + \psi(z)\phi'(z) f'(\phi(z))| \\ &\le w(z)|\psi'(z)||f(\phi(z)) - f(\phi(0))| \\ &\quad + w(z)|\psi'(z)||f(\phi(0))| + w(z)|\psi(z)|\frac{|\phi'(z)|}{v(\phi(z))} \\ &\le w(z)|\psi'(z)|\frac{K}{V(\phi(z))} + Cw(z)|\psi'(z)| + w(z)|\psi(z)|\frac{|\phi'(z)|}{v(\phi(z))}, \end{aligned}$$

since $\|f\|_{B_v} \le 1$ and by the above conditions we conclude that there is a constant $L > 0$ such that $\|\psi C_\phi f\|_{B_v} \le L$. Thus, ψC_ϕ maps B_v continuously into B_w. □

Finally, we consider some examples in order to apply the conditions obtained.

EXAMPLE 4. (i) Select $w(z) = e^{-1/(1-|z|)}$, $v(z) = e^{-2/(1-|z|)} = \tilde{v}(z)$ (see [1]) as well as $\phi(z) = (z+1)/2$ and $\psi(z) = 1 - z$, $z \in D$. Then $\psi'(z) = -1$. For $z = r \in \mathbb{R}$ we get

$$|\psi'(r)|\frac{w(r)}{\tilde{v}(\phi(r))} = e^{1/(1-r)} \to \infty \quad \text{if } r \to 1.$$

Thus, by Proposition 2 the weighted composition operator $\psi C_\phi : B_v \to B_w$ is not continuous.

(ii) Now choose $w(z) = e^{-3/(1-|z|)}$, $v(z) = (1-|z|)^2 e^{-1/(1-|z|)} = \tilde{v}(z)$ as well as $\phi(z) = (z+1)/2$ and $\psi(z) = 1 - z$, $z \in D$. Then $1/V(z) = e^{1/(1-|z|)}$, $\phi'(z) = 1/2$ and $\psi'(z) = -1$. We have to check conditions (a)–(c) of the previous proposition:

(a)

$$\sup_{z\in D} w(z)|\psi'(z)|\frac{1}{V(\phi(z))} = \sup_{z\in D} e^{-3/(1-|z|)}e^{1/(1-|(z+1)/2|)} < \infty;$$

(b)

$$\sup_{z\in D} w(z)|\psi'(z)| = \sup_{z\in D} e^{-3/(1-|z|)} < \infty;$$

(c)

$$\sup_{z\in D} |w(z)|\frac{\psi(z)|\phi'(z)|}{v(\phi(z))} = \sup_{z\in D} \frac{1}{2}\frac{e^{-3/(1-|z|)}}{(1-|(z+1)/2|)^2}e^{1/(1-|(z+1)/2|)} < \infty.$$

Thus, the weighted composition $\psi C_\phi : B_v \to B_w$ operator is continuous.

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