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STABLY FREE MODULES OVER RINGS OF GENERALISED INTEGER QUATERNIONS

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ABSTRACT. In this note, we obtain, in a rather easy way, examples of stably free non-free right ideals. We also give an example of a stably free non-free two-sided ideal in a maximal $\mathbb{Z}$-order. These are obtained as applications of a theorem giving necessary and sufficient conditions for H/nH to be a complete 2×2 matrix ring, when H is a generalised quaternion ring.

1. **Introduction.** For some readers the most interesting part of this paper may be the construction in Theorem 6 of a maximal $\mathbb{Z}$-order with a stably free non-free two-sided ideal, but this was not the original intention of our work.

When $H_0 = \mathbf{Z}[i,j]$, the ring of Lipschitz quaternions, Chatters [1] asked whether the tiled matrix ring $\begin{pmatrix} H_0 & 3H_0 \\ H_0 & H_0 \end{pmatrix}$ is isomorphic to a complete 2×2 matrix ring. This question has led to quite a bit of activity ([2], [5], [6]). All three papers answer the question in the affirmative and show more generally that $\begin{pmatrix} H_0 & nH_0 \\ H_0 & H_0 \end{pmatrix}$ is a complete 2×2 matrix ring if and only if n is odd (far more general results are proved in [5]).

The original motivation for the present note was to determine how the above and related results carry over to generalised quaternion rings $H = \mathbf{Z}[i,j]$, where $i^2 = a$, $j^2 = b$ $(a, b \in \mathbf{Z})$ and $ij + ji = 0$. We do this in Section 2. Then, in Section 3, we use the material of Section 2 concerning the ring H to give many examples of stably free non-free right ideals of H (see also [7] and [8] for other examples with similar properties). More significantly, we also give an example of a maximal $\mathbb{Z}$-order with a stably free non-free two-sided ideal. The question of the existence of such ideals (in orders which are not necessarily maximal) was raised in [4] and the first examples were given in [3]. Our example has the advantages, we believe, of being easy and occurring in a maximal order.

2. **Matrices.** Crucial to our work is the following result of J. C. Robson.

PROPOSITION 1 ([6], THEOREM 2.2). *Let R be a ring with identity element 1, and suppose there are elements a and x of R such that $ax + xa = 1$ and $x^2 = 0$. Then R is a complete 2×2 matrix ring. More precisely, the following elements e_{ij} of R form a set of 2 by 2 matrix units: $e_{11} = ax$, $e_{12} = axa$, $e_{21} = x$ and $e_{22} = xa$.*

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To prove Theorem 3, we need the following number theoretic result. The proof follows the same lines as that of Lemma 2.2 in [2].

LEMMA 2. *Given integers a, b, m, n such that* $\gcd(n, 2ab) = 1$ *and* $\gcd(n, m) = 1$, *there exist integers v, r, s such that* $vn = m + ar^2 + bs^2$.

PROOF. We shall prove the result when n is a prime number. It is then routine to extend this to the case when n is a prime power (here the fact that $\gcd(n, m) = 1$ is required), and then the Chinese remainder theorem can be used for arbitrary n by combining the results for the various prime power factors of n.

So assume $n = p$ is an odd prime which does not divide ab. We shall work in $F = \mathbb{Z}/p\mathbb{Z}$. Note that there are $\frac{p+1}{2}$ distinct squares in F, so there are $\frac{p+1}{2}$ distinct elements of the form bs^2 and $\frac{p+1}{2}$ distinct elements of the form $-m-ar^2$. It follows that some element of F lies in both sets, *i.e.*, we have $bs^2 = -m - ar^2$ for some r, s in F. ∎

We are now ready to prove the main result of this section. Recall from the introduction that $H = \mathbb{Z}[i,j]$ is a generalised quaternion ring, where $i^2 = a, j^2 = b$ $(a, b \in \mathbb{Z})$ and $ij + ji = 0$.

THEOREM 3. *Let n be a positive integer. Then the following are equivalent:*

(i) H/nH *is isomorphic to* $M_2(S)$ *for some ring S.*
(ii) H/nH *is isomorphic to* $M_2(\mathbb{Z}/n\mathbb{Z})$.
(iii) H/pH *is isomorphic to* $M_2(\mathbb{Z}/p\mathbb{Z})$ *for every prime factor p of n.*
(iv) $\gcd(n, 2ab) = 1$.

PROOF. An easy counting argument shows that (i) implies (ii), and (ii) implies (iii) is clear.

Assume that H/pH is isomorphic to $M_2(\mathbb{Z}/p\mathbb{Z})$ for some prime p. Because $H/2H$ is commutative, we have $p \neq 2$. Set $R = H/pH$ and let x be the image of i in R. Then xR is a nonzero two-sided ideal of R and $(xR)^2 = aR$. Since R is semiprime, we conclude that p does not divide a. Similarly p does not divide b. It follows that (iii) implies (iv).

Finally, we show that (iv) implies (i). Using $m = -ab$ in Lemma 2, choose integers r, s such that $ar^2 + bs^2$ is congruent to $ab \pmod{n}$. Set $x = ri + sj + ij$ and $y = ri + sj$. Then x^2 is in nH and $yx + xy$ is congruent to $2ab \bmod (nH)$. Choose c such that $c(2ab)$ is congruent to $1 \pmod{n}$, and we have that $(cy)x + x(cy)$ is congruent to $1 \bmod (nH)$. Proposition 1 now gives the result. ∎

It is a special case of results in [5] that H/nH is a complete 2×2 matrix ring if and only if the same is true of the tiled matrix ring $\begin{pmatrix} H & nH \\ nH & H \end{pmatrix}$, and hence also $\begin{pmatrix} H & nH \\ H & H \end{pmatrix}$. We would like to note that the direct number-theoretic argument used in the last part of the proof of Theorem 3 can be adapted to give this as well. Starting with $\gcd(n, 2ab) = 1$, Lemma 2 allows us to choose integers u, r, s such that $ar^2 + bs^2 - ab = un^2$. With $\alpha = ri + sj$ and $\beta = ri + sj + ij$, set

$$X = \begin{pmatrix} \beta & un \\ -n & -\beta \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad C = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}.$$

Note that $X^2 = 0$, $XB + BX = \text{diag}\big(2(ab + un^2)\big)$ and $XC + CX = \text{diag}(-n^2)$. Since $\gcd\big(2(ab+un^2), -n^2\big) = 1$, we can choose integers f, g such that $X(fB+gC)+(fB+gC)X$ is the identity matrix. Proposition 1 then gives the result.

3. **Stably free modules.** As mentioned in the introduction, we will begin by applying the results of Section 2 to obtain examples of stably free non-free right ideals of H. Most important for our purposes is the equivalence of (ii) and (iv) in Theorem 3, namely the fact that $H/nH \simeq M_2(\mathbb{Z}/n\mathbb{Z})$ if and only if $\gcd(n, 2ab) = 1$.

Recall that an R-module M is stably free if $M \oplus F$ is isomorphic to G for some free R-modules F and G. Our approach is as follows (with a, b, H as in Section 2). If p is a prime number which does not divide $2ab$, then the principal maximal right ideals of H containing p correspond to elements of norm p in H. We shall show that a, b, p can be chosen so that there is at least one principal maximal right ideal of H containing p, but there are not enough elements of norm p to generate all the maximal right ideals containing p.

PROPOSITION 4. *Let p be a prime number which does not divide $2ab$. Then*

(i) There are $p + 1$ maximal right ideals of H which contain p.

(ii) Suppose that at least one of the maximal right ideals of H which contain p is principal. Then all the maximal right ideals of H which contain p are stably free.

PROOF. (i) By Theorem 3, H/pH is isomorphic to $M_2(\mathbb{Z}/p\mathbb{Z})$. The result now follows from the fact that $M_2(\mathbb{Z}/p\mathbb{Z})$ has $p + 1$ maximal right ideals.

(ii) Let K and L be maximal right ideals of H which contain p, $K \neq L$, and suppose that L is principal. Since $K + L = H$, if $\theta: K \oplus L \longrightarrow K + L$ is the obvious H-module epimorphism, the exact sequence $0 \longrightarrow \operatorname{Ker}\theta \longrightarrow K \oplus L \longrightarrow K + L \longrightarrow 0$ splits. But $\operatorname{Ker}\theta \simeq K \cap L = pH \simeq H$. Also, $L \simeq H$ as an H-module. We conclude that $K \oplus H \simeq H \oplus H$, so K is stably free. ∎

We can now easily construct stably free non-free right ideals. Here is one example.

EXAMPLE 5. Take $a = -1$, $b = -21$, $p = 31$. If $h = s + ti + uj + vij$ is in H ($s, t, u, v \in Z$), then the norm $N(h)$ of h is given by $N(h) = s^2 + t^2 + 21u^2 + 21v^2$. Thus H has some elements of norm 31, for instance $3 + i + j$, but not enough to generate all the 32 maximal right ideals of H which contain 31. Proposition 4 tells us that there is a stably free non-free maximal right ideal of H containing 31 (a specific example will be given in Theorem 6). ∎

Finally, we note the following concerning stably free non-free two-sided ideals. Recall again that the first examples of such ideals were given in [3].

THEOREM 6. *There is a maximal $\mathbb{Z}$-order with a non-principal stably free two-sided ideal.*

PROOF. Start with the generalised quaternion ring H in Example 5 and consider the two-sided ideal $I = 3H + jH$. With the object of obtaining a contradiction, we suppose that $I = xH$ for some x. Because xH contains the elements 3 and j, it follows that $N(x)$

divides both $N(3) = 9$ and $N(j) = 21$. Therefore $N(x)$ divides 3. But H has no elements of norm 3, and because $I \neq H$ we have $N(x) \neq 1$. This is the desired contradiction. Therefore I is not principal as a right ideal of H, and by symmetry it is also not principal as a left ideal. But note that $3H+jH = 3H+(3+2j)H$ is isomorphic as a right H-module to $(3-2j)\big(3H+(3+2j)H\big) = 3(3-2j)H+93H$, which is isomorphic to $31H+(3-2j)H = K$.

Now, K is a maximal right ideal of H which contains 31. It follows from Example 5 that K, and hence also I, is stably free.

Finally, note that, just as with the Lipschitz quaternions, the ring H which we have been discussing is contained in a maximal $\mathbb{Z}$-order.

$$S = \left\{ \frac{1}{2}(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 ij) \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z} \text{ are all even or all odd} \right\}$$

and all of the earlier calculations work equally well in S. ∎

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