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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO THE BECKER-DÖRING EQUATIONS

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The asymptotics behaviour of solutions of the Becker-Döring cluster equations is determined for cases in which coagulation dominates fragmentation. We show that all non-zero solutions tend weak* to zero.

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1. Introduction

The Becker-Döring equations provide a model of the dynamics of a system consisting of a large number of identical particles. The particles can coagulate to form clusters, which in turn can fragment into smaller clusters. Let $c_r(t) \geq 0$, $r = 1, 2, \ldots$ denote the expected number of r-particle clusters per unit volume at time t. The Becker-Döring model [4] is based on the following assumptions: clusters coagulate and fragment by gaining or losing single particles; the coagulation rate of r-clusters and monomers into $(r+1)$-clusters is proportional to the numbers of r-clusters and monomers; and the fragmentation rate of r-clusters into $(r-1)$-clusters is proportional to the number of r-clusters. The coefficients of coagulation and fragmentation are the constants $a_r > 0$ and $b_{r+1} > 0$ $(r = 1, 2, \ldots)$ respectively, with $b_1 = 0$ since monomers cannot fragment. The Becker-Döring equations can then be written as

$$\left.\begin{aligned} \dot{c}_1(t) &= -J_1(c(t)) - \sum_{r=1}^{\infty} J_r(c(t)) \\ \dot{c}_r(t) &= J_{r-1}(c(t)) - J_r(c(t)) \end{aligned}\right\} \tag{1.1}$$

where the flux $J_r(c) = a_r c_1 c_r - b_{r+1} c_{r+1}$ represents the net rate of conversion of r-clusters into $(r+1)$-clusters.

From physical considerations, it is only relevant to consider solutions to (1.1) which are non-negative and have finite density ρ of the system given by

$$\rho = \sum_{r=1}^{\infty} r c_r(t).$$

Since each interaction preserves the number of particles, we expect that ρ is independent of time. Time independence can be formally verified by differentiating the above expression for ρ by t and substituting for each of the resulting $\dot{c}_r$ terms from (1.1). A fundamental theorem states that all solutions of the Becker-Döring equations conserve density [3].

If $c_r(t) = \tilde{c}_r$ is an equilibrium solution of (1.1) then $J_r(\tilde{c}_r) = 0$ so that

$$a_r\tilde{c}_r\tilde{c}_1 - b_{r+1}\tilde{c}_{r+1} = 0.$$

It follows that there is a one parameter family of equilibrium solutions of the form $\tilde{c}_r = Q_r z^r$ for each $r \geq 1$. The constants Q_r are given by $Q_1 = 1$ and

$$Q_r = \prod_{j=2}^{r}(a_{j-1}/b_j) \quad \text{for} \quad r \geq 2.$$

The density of the equilibrium solution is $\rho = \sum_{r=1}^{\infty} rQ_r z^r$. We will use z_s to denote the radius of convergence of the above series. Previous results on the asymptotic behaviour of solutions have been for the case $z_s > 0$. In order to outline the results for $z_s > 0$ we introduce the Banach space $X = \{c = (c_r) : \|c\| = \sum_{r=1}^{\infty} r|c_r| < \infty\}$.

If $z_s = \infty$ then fragmentation dominates coagulation. In this case the positive orbit $\mathcal{P}^+(c) = \{c(t) : t > 0\}$ is relatively compact in X, and $c(t)$ converges in X as $t \to \infty$ to the equilibrium with the same density as the initial data.

The case $0 < z_s < \infty$ corresponds to situations in which there is a balance between coagulation and fragmentation. The results here are particularly striking when $\rho_s = \sum_{r=1}^{\infty} rQ_r z_s^r < \infty$. Let ρ_0 be the density of the initial data. If $\rho_0 \leq \rho_s$ then the positive orbit $\mathcal{P}^+(c)$ is relatively compact in X and $c(t)$ converges in X to the equilibrium with density ρ_0. If $\rho_0 > \rho_s$ then the positive orbit $\mathcal{P}^+(c)$ is not relatively compact in X and $c(t)$ converges to the equilibrium with density ρ_s in the metric $d(x, y) = \sum_{r=1}^{\infty} |x_r - y_r|$ induced by weak* convergence on X.

In this paper we study situations in which coagulation dominates fragmentation. More precisely, we assume that $(b_{r+1} + b_r)/a_r \to 0$ as $r \to \infty$. Thus $\lim_{r\to\infty} Q_r^{1/r} = \infty$, $z_s = 0$ and the only equilibrium solution is $c_r = 0$ for all r. For this case, the positive orbit $\mathcal{P}^+(c)$ is not relatively compact in X if the initial data is non-zero. We prove that for each r, $\lim_{t\to\infty} c_r(t) = 0$. Note that for a solution with density $\rho > 0$,

$$\rho = \sum_{r=1}^{\infty} rc_r(t) > \sum_{r=1}^{\infty} \lim_{t\to\infty} rc_r(t) = 0.$$

This corresponds to the formation of larger and larger clusters as t increases.

Let

$$V(c) = \sum_{r=1}^{\infty} c_r\left[\ln\left(\frac{c_r}{Q_r}\right) - 1\right] \tag{1.2}$$

A formal calculation shows

$$\dot{V} = -\sum_{r=1}^{\infty}[a_r c_r c_1 - b_{r+1}c_{r+1}][\ln(a_r c_r c_1) - \ln(b_{r+1}c_{r+1})],$$

so that V is a Lyapunov function. Results on the asymptotic behaviour of solutions for the case $z_s > 0$ have been obtained by exploiting the Lyapunov function V. Crucial steps in this include:

(i) precompactness of orbits in a suitable metric space,

(ii) continuity of the Lyapunov function V with respect to the same metric as (i).

In general, if $z_s < \infty$, orbits are not compact in the metric induced by strong convergence on X. Thus we use the metric induced by weak* convergence on X. Unfortunately, in general V is not continuous in this metric. However, since density is conserved, $V_z(c) = V(c) - \ln z \sum_{r=1}^{\infty} rc_r$ is a Lyapunov function for each z, and when $z_s > 0$, for exactly one value of z, namely $z = z_s$, V_z is sequentially weak* convergent. This trick cannot be used for the case $z_s = 0$ and we use another technique to prove the weak* convergence of the solution to zero. If $c(t)$ does not converge weak* to 0 as $t \to \infty$ then we show that each $c_r(t)$ is bounded away from zero for all large t. Using this and the domination of coagulation over fragmentation, we show that for any $\tau > 0$, $q_n : [0, \infty) \to \mathbb{R}$, defined by

$$q_n(t) = \sum_{r=n+1}^{\infty}(r-n)c_r(t)$$

satisfies $\dot{q}_n(t) \geq 0$ for all $t \geq \tau > 0$ and all sufficiently large n. From this we can deduce that the assumption $c(t) \not\rightharpoonup 0$ as $t \to \infty$ leads to a contradiction.

In recent years a number of papers dealing with the mathematical theory of the dynamics of cluster growth have appeared. For the Becker-Döring equations, basic results and asymptotic behaviour for $z_s > 0$ are given in [3]; see also [**1**, **10**] for technical improvements and [9, 7] for a discussion of metastable solutions. Some of the results for the Becker-Döring equations have been extended to the more general coagulation-fragmentation equations [**2**, **5**, **6**, **8**, **11**].

This paper makes frequent use of results from [3]. Section 2 recalls the basic existence and density conservation results and the material needed on generalised flows. The main result on the asymptotic behaviour of solutions is given in Section 3. The final section shows that the energy equation for V holds when the initial data decays fast enough and that

$$\lim_{t\to\infty} V(c(t)) = \lim_{t\to\infty}\sum_{r=1}^{\infty} c_r(t)\left[\ln\left(\frac{c_r(t)}{Q_r}\right) - 1\right] = -\infty$$

while

$$\sum_{r=1}^{\infty} \lim_{t\to\infty} c_r(t)\left[\ln\left(\frac{c_r(t)}{Q_r}\right) - 1\right] = 0.$$

2. Preliminaries

We first introduce some notation. Let

$$X = \{x = (x_r) :< \infty\}, \quad \|x\| = \sum_{r=1}^{\infty} r|x_r|,$$

and let $X^+ = \{x \in X : \text{each } x_r > 0\}$. We say that c^n converges in the weak* sense to c in X, symbolically $c^n \xrightarrow{*} c$, if (i) $\sup_n \|c^n\| < \infty$ and (ii) $c_j^n \to c_j$ as $n \to \infty$ for each j. We can express weak* convergence as convergence in a metric space. Let

$$B_\rho = \left\{c \in X : \sum_{r=1}^{\infty} r|c_r| \le \rho\right\}.$$

Then (B_ρ, d) is a metric space where $d(\alpha, \beta) = \sum_{j=1}^{\infty} |\alpha_j - \beta_j|$. For $\rho > 0$ set $B_\rho^+ = B_\rho \cap X^+$ so that B_ρ^+ is a closed metric subspace of B_ρ (with metric d). Weak* convergence is useful because B_ρ is compact, equivalently, any bounded sequence in X has a weak* convergent subsequence.

The following definition of solution introduced in [3] is used.

Definition 2.1. Let $0 < T \le \infty$. A solution $c = (c_r)$ of (1.1) on $[0, T)$ is a function $c : [0, T) \to X^+$ such that

(i) each $c_r : [0, T) \to \mathbf{R}$ is continuous and $\sup_{t\in[0,T)} \|c(t)\| < \infty$;

(ii) $\int_0^t \sum_{r=1}^{\infty} a_r c_r(s)ds < \infty$, $\int_0^t \sum_{r=2}^{\infty} b_r c_r(s)ds < \infty$ for all $t \in [0, T)$;

(iii)

$$c_r(t) = c_r(0) + \int_0^t [J_{r-1}(c(s)) - J_r(c(s))]ds \quad \textit{for } r \ge 2$$

$$c_1(t) = c_1(0) - \int_0^t \left[J_1(c(s)) + \sum_{r=1}^{\infty} J_r(c(s))\right]ds \tag{2.1}$$

The next theorem contains three main results from [3].

Theorem 2.2. (i) *Let (g_r) be a positive sequence satisfying $g_{r+1} - g_r \ge \delta > 0$, for $r \ge 1$ where δ is a constant and let c_0 be a positive sequence satisfying $\sum_{r=1}^{\infty} g_r c_{0r} < \infty$. Assume that $a_r(g_{r+1} - g_r) = \mathcal{O}(g_r)$. Then there exists a solution c of* (1.1) *on* $[0, \infty)$ *with $c(0) = c_0$, for which $\sup_{t\in[0,T]} \sum_{r=1}^{\infty} g_r c_r(t) < \infty$, where $T > 0$.*

(ii) *Let c be a solution of* (1.1) *on some interval* $[0, T)$, $0 < T \leq \infty$. *Then for all* $t \in [0, T)$

$$\sum_{r=1}^{\infty} rc_r(t) = \sum_{r=1}^{\infty} rc_r(0) \tag{2.2}$$

and for $m \geq 2$

$$\sum_{r=m}^{\infty} c_r(t) - \sum_{r=m}^{\infty} c_r(\tau) = \int_{\tau}^{t} J_{m-1}(c(s))ds \qquad 0 \leq \tau \leq t < T \tag{2.3}$$

(iii) *Assume that* $a_r > 0$ *and* $b_r > 0$ *for each* $r \geq 1$. *Let c be a solution of* (1.1) *on some interval* $[0, T)$, $0 < T \leq \infty$, *with* $c(0) \neq 0$. *Then* $c_r(t) > 0$ *for all* $t \in (0, T)$ *and all* $r \geq 1$.

Let $c(t)$ be a solution of (1.1) with $c(0) \in B_\rho^+$. By (2.2), the positive orbit $\mathcal{P}^+(c)$ is relatively compact in B_ρ^+. The next theorem lists two results from [3] which concern the generalised flow formed by the set of all solutions of (1.1).

Theorem 2.3. (i) *Assume* $a_r = o(r)$ *and* $b_r = o(r)$. *For* $\rho > 0$, *let* $\mathcal{G}_\rho$ *denote the set of all solutions of* (1.1) *on* $[0, \infty)$ *with* $c(0) \in B_\rho^+$. *Then* $\mathcal{G}_\rho$ *is a generalised flow on* B_ρ.

(ii) *Let G be a generalised flow on Y, let* $\varphi \in G$ *and suppose* $\mathcal{P}^+(\varphi)$ *is relatively compact. Then* $\omega(\varphi)$ *is nonempty and quasi-invariant, and* $\operatorname{dist}(\varphi(t), \omega(\varphi)) \to 0$ *as* $t \to \infty$.

3. Main results

For the rest of this paper we assume that $a_r > 0$ and $b_{r+1} > 0$ for each $r \geq 1$. We shall make use of the following hypotheses:

H1 $a_r = o(r)$ and $b_r = o(r)$.

H2 $\lim_{r\to\infty} \frac{b_r}{a_r} = \lim_{r\to\infty} \frac{b_{r+1}}{a_r} = 0$.

From the previous section, the omega limit set $\omega(c)$ of a solution c is nonempty. We show that $\omega(c)$ contains only the zero solution so that $c(t) \stackrel{*}{\rightharpoonup} 0$ as $t \to \infty$. The proof is by contradiction. The first consequence of assuming that $c(t) \not\stackrel{*}{\rightharpoonup} 0$ is that all solutions are bounded away from zero as $t \to \infty$.

Theorem 3.1 *Assume H1 holds and that c is a solution of* (1.1) *such that* $c(t) \not\stackrel{*}{\rightharpoonup} 0$ *as* $t \to \infty$. *Then for any* $\tau > 0$ *and for each* $r = 1, 2, \ldots$ *there exists an* $\alpha_r > 0$ *such that* $c_r(t) \geq \alpha_r$ *for all* $t \geq \tau$.

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Proof. Suppose that there is a sequence t_k such that $t_k \to \infty$ as $k \to \infty$ and $c_m(t_k) \to 0$ as $k \to \infty$, for some $m \geq 1$. By Theorem 2.3(i), $\mathcal{G}_\rho$ is a generalised flow on B_ρ^+. The positive orbit of c is relatively compact in B_ρ^+, so Theorem 2.3(ii) implies that for any $\delta > 0$ there exists a subsequence of $\{t_k - \delta\}$, denoted $\{t_{k_j} - \delta\}$, for which $c(t_{k_j} - \delta) \stackrel{*}{\rightharpoonup} \bar{x} \in \omega(c)$. Let d be a solution of (1.1) with $d(0) = \bar{x}$, then Theorem 2.3(ii) also implies that $d(t) \in \omega(c)$ for all $t \in [0, \infty)$. Then the definition of a generalised flow implies that there is a subsequence of $\{t_{k_j} - \delta\}$, also denoted $\{t_{k_j} - \delta\}$, such that $c(t_{k_j} - \delta + t) \stackrel{*}{\rightharpoonup} d(t)$ as $j \to \infty$, for all $t \in [0, \infty)$. In particular $c_m(t_{k_j}) \stackrel{*}{\rightharpoonup} d_m(\delta)$. Since by 2.2(iii) $d_m(\delta) > 0$, this is a contradiction. □

Theorem 3.2. *Assume H1 holds. Then for each n,* $q_n(t) = \sum_{r=n+1}^{\infty} (r-n)c_r(t)$ *is bounded and differentiable with*

$$\dot{q}_n(t) = a_n c_1(t) c_n(t) + \sum_{r=n+1}^{\infty} (a_r c_1 - b_r) c_r(t).$$

If, in addition, H2 holds and $c \not\stackrel{*}{\rightharpoonup} 0$ *then for any* $\tau > 0$, *there exists* n_0 *such that* $\dot{q}_n(t) \geq 0$ *for all* $n \geq n_0$ *and* $t \geq \tau$.

Proof. The boundedness of q_n follows from:

$$0 \leq \sum_{r=n+1}^{\infty} c_r(t) \leq q_n(t) \leq \sum_{r=n+1}^{\infty} r c_r(t) \leq \rho.$$

By (2.2), we can write q_n as $q_n(t) = \rho - \sum_{r=1}^{n} r c_r(t) - n f_n(t)$, where $f_n(t) = \sum_{r=n+1}^{\infty} c_r(t)$. From formula (2.3), f_n is differentiable, and it follows from the definition of a solution that q_n is differentiable. Differentiation of q_n gives

$$\begin{aligned}
\dot{q}_n(t) &= -\sum_{r=1}^{n} r \dot{c}_r(t) - n J_n(c(t)) \\
&= \sum_{r=n+1}^{\infty} J_r(c(t)) \\
&= a_n c_1(t) c_n(t) + \sum_{r=n+1}^{\infty} (a_r c_1(t) - b_r) c_r(t).
\end{aligned}$$

If $c(t) \not\stackrel{*}{\rightharpoonup} 0$ as $t \to \infty$ then $c_1(t) \geq \alpha_1$ for some $\alpha_1 > 0$ and for all $t \in [\tau, \infty)$. If H2 holds then for large enough r, $a_r c_1(t) - b_r > 0$ for $t \geq \tau$ and $\dot{q}_n(t) \geq 0$. □

Theorem 3.3. *Assume that H1 and H2 hold. Let c be a solution of* (1.1) *on* $[0, \infty)$. *Then* $c(t) \stackrel{*}{\rightharpoonup} 0$ *as* $t \to \infty$.

Proof. Suppose $c(t) \not\rightharpoonup 0$. By Theorem 3.2 for large enough n, q_n is bounded and eventually non-decreasing. Hence $\dot{q}_n(t) \to 0$ as $t \to \infty$, so that $\lim_{t\to\infty} c_1(t)c_n(t) = 0$, which contradicts Theorem 3.1. □

4. The Lyapunov function

When the coefficients of the Becker-Döring equations are such that $\limsup_{r\to\infty} Q_r^{1/r} < \infty$ then it can be shown, once additional assumptions have been made, that the function V defined in equation (1.2) satisfies the energy equation

$$V(c(t)) + \int_0^t D(c(s))ds = V(c(0)), \tag{4.1}$$

where

$$D(c) := \sum_{r=1}^{\infty} (a_r c_1 c_r - b_{r+1} c_{r+1})(\ln(a_r c_1 c_r) - \ln(b_{r+1} c_{r+1})). \tag{4.2}$$

Writing $V(c) = G(c) - \sum_{r=1}^{\infty} r c_r \ln(Q_r^{1/r})$, where $G(c) = \sum_{r=1}^{\infty} c_r[\ln(c_r) - 1]$, we see that if H2 holds (so that $\lim_{r\to\infty} Q_r^{1/r} = \infty$), then $V(c)$ may not be defined for all solutions.

When H2 holds we show that if the initial data has a sufficiently fast decay, then $V(c(t))$ is defined and the energy equation (4.1) is satisfied. To establish the results about V it is necessary to make additional hypotheses on the coefficients:

H3 $a_r(\ln Q_{r+1} - \ln Q_r) = \mathcal{O}(\ln Q_r)$.

H4 $a_r = \mathcal{O}\left(\frac{r}{\ln r}\right)$.

By Lemma 4.2 in [3], G is weak* continuous, and is bounded above and below on B_ρ^+ for any $\rho \geq 0$. Thus $V(c(t))$ is defined if and only if

$$\sum_{r=1}^{\infty} \ln(Q_r) c_r(t) < \infty. \tag{4.3}$$

If H2 and H3 hold, $\ln(Q_r)$ satisfies the requirements for the sequence (g_r) in Theorem 2.2(i) and if the initial data satisfies

$$\sum_{r=1}^{\infty} \ln(Q_r) c_r(0) < \infty, \tag{4.4}$$

there is a solution of (1.1) on $[0, \infty)$ such that $V(c(t))$ is defined for $t \geq 0$. Since solutions of (1.1) are not known to be unique, we need to show that (4.3) holds for any solution with initial data satisfying (4.4).

Lemma 4.1. *Let H2 and H3 hold. If c is a solution of* (1.1) *on* $[0, \infty)$ *with initial data satisfying* (4.4) *then* (4.3) *holds.*

Proof. By H2, there exists m such that $\ln Q_{r+1} - \ln Q_r \geq 0$ for $r \geq m$. Let $h_n(t) = \sum_{r=m}^{n} \ln Q_r c_r(t)$. Then

$$\dot{h}_n = \sum_{r=m}^{n-1} (\ln Q_{r+1} - \ln Q_r)(a_r c_1 c_r - b_{r+1} c_{r+1}) + \ln Q_m J_{m-1} - \ln Q_n J_n. \tag{4.5}$$

This can be rearranged, and the inequalities $a_r(\ln Q_{r+1} - \ln Q_r) \leq K \ln Q_r$ for some constant K and $c_1 \leq \rho$ used to give:

$$\dot{h}_n + \sum_{r=m}^{n-1} (\ln Q_{r+1} - \ln Q_r) b_{r+1} c_{r+1} \leq K\rho h_n + \ln Q_m J_{m-1} - \ln Q_n J_n.$$

From equation (2.3) and H2

$$-\ln(Q_n) \int_0^t J_n(c(s))ds = -\ln(Q_n)\left[\sum_{r=n+1}^{\infty} (c_r(t) - c_r(0))\right] \leq \ln(Q_n) \sum_{r=n+1}^{\infty} c_r(0) \leq \sum_{r=n+1}^{\infty} \ln(Q_r) c_r(0).$$

It follows from Gronwall's inequality that for some constant M independent of n,

$$h_n(t) + \int_0^t \sum_{r=m}^{n-1} (\ln Q_{r+1} - \ln Q_r) b_{r+1} c_{r+1}(s)ds \leq M e^{K\rho t},$$

and the result follows. □

The proof of the energy equation (4.1) for the case $z_s > 0$ in [3] only uses $\limsup_{r\to\infty} Q_r^{1/r} < \infty$ to show

$$\lim_{n\to\infty} \ln Q_n \int_\tau^t J_n(c(s))ds = \lim_{n\to\infty} \ln Q_{n+1} \int_\tau^t J_n(c(s))ds = 0, \qquad 0 < \tau \leq t, \tag{4.6}$$

and that

$$\lim_{t\to 0} V(c(t)) = V(c(0)). \tag{4.7}$$

In the next lemma we prove that these limits hold without assuming that $Q_r^{1/r}$ is bounded.

Lemma 4.2. *Let H2 and H3 hold. If c is a solution of* (1.1) *on* $[0, \infty)$ *with initial data satisfying* 4.4 *then* (4.6) *and* (4.7) *hold.*

Proof. The proof is similar to the one given for Theorem 2.5 of [3]. From equation (2.3) we have that

$$\lim_{n\to\infty}\int_\tau^t J_n(c(s))\ln Q_{n+1}ds = \lim_{n\to\infty}\ln Q_{n+1}\left[\sum_{r=n+1}^{\infty}c_r(t) - \sum_{r=n+1}^{\infty}c_r(\tau)\right].$$

Using $\ln Q_{r+1} \geq \ln Q_r$ for large r and Lemma 4.1,

$$\lim_{n\to\infty}\ln Q_{n+1}\sum_{r=n+1}^{\infty}c_r(s) \leq \lim_{n\to\infty}\sum_{r=n+1}^{\infty}\ln Q_r c_r(s) = 0,$$

from which (4.6) follows. Let $h(t) = \sum_{r=m}^{\infty}\ln Q_r c_r(t)$. Using (4.6) and the dominated convergence theorem, it follows from (4.5) that

$$h(t) - h(0) = \int_0^t\left[\ln Q_m J_{m-1}(s) + \sum_{r=m}^{\infty}(\ln Q_{r+1} - \ln Q_r)J_r(s)\right]ds. \tag{4.8}$$

Since the integrand in the above equation is bounded, $h(t) \to h(0)$ as $t \to \infty$ from which (4.7) follows. □

It is now possible to state the equivalent of Theorem 4.7 in [3]; it can be proved by combining Lemma 4.2 and the proof of the original theorem.

Theorem 4.3. *Assume H2-H4 hold and let c be a solution of* (1.1) *on* $[0, \infty)$ *with initial data satisfying* $c(0) \neq 0$, *and* $\sum_{r=1}^{\infty}\ln Q_r c_r(0) < \infty$. *Then the energy equation* (4.1) *is satisfied for all* $t > 0$.

The next result shows that even though $V(c(t))$ is finite, it must be unbounded as $t \to \infty$.

Theorem 4.4. *Assume H2-H4 and let c be a solution of* (1.1) *on* $[0, \infty)$ *with initial data satisfying* $c(0) \neq 0$, *and* $\sum_{r=1}^{\infty}\ln Q_r c_r(0) < \infty$. *Then* $V(c(t)) \to -\infty$ *as* $t \to \infty$.

Proof. Recall that $V(c) = G(c) - \sum_{r=1}^{\infty} rc_r\ln(Q_r^{1/r})$ and that $G(c)$ is bounded. Since $\ln Q_r^{1/r} \to \infty$ as $r \to \infty$, if $V(c(t))$ is bounded then the positive orbit $\mathcal{P}^+(c)$ is relatively compact in X. This would contradict $c(t) \stackrel{*}{\rightharpoonup} 0$ and the result follows. □

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