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A NOTE ON MODULAR FORMS MOD p

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§1. Introduction

In this paper we shall study relations among the spaces of modular forms mod p attached to $S_k(Np, \psi\chi)$ and $S_{k'}(N, \psi)$ by using certain identities between dimensions of these spaces.

Let N be a positive integer and χ be an arbitrary character of $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$. Let $\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \,|\, c \equiv 0 \pmod N\right\}$. Let $f(z)$ be a cusp form of weight k satisfying

$$f(\sigma(z)) = (cz + d)^k \chi(d) f(z) \qquad \text{for all } \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\,.$$

Then we call $f(z)$ a cusp form of type (k, χ) on $\Gamma_0(N)$, and we denote by $S_k(N, \chi)$ the space of all cusp forms of type (k, χ) on $\Gamma_0(N)$. If χ is trivial, we simply denote $S_k(N)$.

From now we fix a rational prime p, $p \geq 5$. Let N be a positive integer such that $(p, N) = 1$. Let ψ and χ be arbitrary characters of $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$ and $(\boldsymbol{Z}/p\boldsymbol{Z})^\times$ respectively such that

$$\psi\chi(-1) = 1\,. \tag{1.1}$$

Let t denote the order of χ and put

$$\kappa = \frac{(p-1)(t-a)}{t}\,. \tag{1.2}$$

with any integer a such that $1 \leq a \leq t$, $(a, t) = 1$. Let k be any even positive integer. Then we shall prove the following simple identities between dimensions of spaces of cusp forms by using Hijikata's explicit trace formula [2]:

THEOREM 1.1. *The notation being as above, we have*

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$$(1.3)\qquad \dim_C S_k(Np, \psi\chi) = \dim_C S_{(p+1)(k/2)-\kappa}(N, \psi) + \dim_C S_{(p+1)(k/2)-(p-1-\kappa)}(N, \psi)\,.$$

In § 3, we shall prove a statement analogous to Theorem 1.1 when k is odd, $k \geq 3$.

Let $\widetilde{S_k(1)}$ and $\widetilde{S_k(p)}$ denote the space of modular forms mod p attached to $S_k(1)$ and $S_k(p)$ respectively. Some of fundamental results on modular forms mod p, due to Serre and Swinnerton-Dyer are the followings:

$$(1.4)\qquad \widetilde{S_k(1)} \subset \widetilde{S_{k+p-1}(1)}\,,$$

$$(1.5)\qquad \widetilde{S_2(p)} = \widetilde{S_{p+1}(1)}\,.$$

It is essential to prove (1.5) that it holds

$$\dim_C S_2(p) = \dim_C S_{p+1}(1)$$

which is a special case of Theorem 1.1. Hence next to do is to generalize (1.5), namely to get relations between the spaces of modular forms mod p attached to $S_k(Np, \psi\chi)$ and $S_{k'}(N, \psi)$ as an application of Theorem 1.1.

To state further results, we fix N, ψ and k as above. Take an algebraic number field K of finite degree over the rational number field which contains all eigenvalues of all Hecke operators acting on $S_k(Np, \psi\chi)$ for all characters χ of $(Z/pZ)^\times$ and on $S_{k'}(N, \psi)$ for all $k' \leq (k/2)(p+1)$, and all $p(p-1)$-th roots of unity. We fix a prime divisor $\mathfrak{p}$ of K lying over p. Let ν be the valuation of K attached to $\mathfrak{p}$ normalized by $\nu(p) = p^{-1}$. Also we write $\mathfrak{o} = \{\alpha \in K \,|\, \nu(\alpha) \leq 1\}$ and $F = \mathfrak{o}/\mathfrak{p}$.

For any character χ of $(Z/pZ)^\times$, we define V_χ and $V_{k'}$ by

$$(1.6)\qquad V_\chi = \left\{f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(Np, \psi\chi) \,|\, a_n \in K \text{ for all } n \geq 1\right\},$$

$$(1.7)\qquad V_{k'} = \left\{g(z) = \sum_{n=1}^{\infty} b_n q^n \in S_{k'}(N, \psi) \,|\, b_n \in K \text{ for all } n \geq 1\right\}.$$

where $q = e^{2\pi i z}$. Then V_χ and $V_{k'}$ are vector spaces over K with same dimensions as those of $S_k(Np, \psi\chi)$ and $S_{k'}(N, \psi)$ over the complex number field respectively. For any subspace V of V_χ or $V_{k'}$, we define

$$(1.8)\qquad V(\mathfrak{o}) = \left\{f(z) = \sum_{n=1}^{\infty} a_n q^n \in V \,|\, a_n \in \mathfrak{o} \text{ for all } n \geq 1\right\}.$$

For any $f = \sum_{n=1}^{\infty} a_n q^n$ in $V(\mathfrak{o})$, we put

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$$\tilde{f} = \sum_{n=1}^{\infty} \bar{a}_n q^n \in F[[q]]$$

where $\bar{a}_n = a_n \bmod \mathfrak{p}$. $\tilde{f}$ is a cusp form mod p in slightly more generalized sense than that of Serre and Swinnerton-Dyer. Let $\tilde{V}$ denote the space over F spanned by $\tilde{f}$ for all $f \in V(\mathfrak{o})$. Then $\tilde{V}$ has the same dimension over F as that of V over K. $\tilde{V}$ is called the space of modular forms mod p attached to V.

Let $W_p = \begin{pmatrix} px, & 1 \\ pNy, & p \end{pmatrix}$ with some rational integers x and y, be a matrix with determinant p. We define a linear operator W_p on V_χ by

$$(f \mid W_p)(z) = p^{k/2}(pNyz + p)^{-k} f\left(\frac{pxz + 1}{pNyz + p}\right). \tag{1.9}$$

Then we shall prove that W_p gives an isomorphism between V_χ and $V_{\bar{\chi}}$ where $\bar{\chi}$ is the character of $(Z/pZ)^\times$ defined by $\bar{\chi}(n) =$ the complex conjugate of $\chi(n)$.

Since we fix a prime divisor $\mathfrak{p}$, there exists a unique character ω of $(Z/pZ)^\times$ such that

$$\omega(n) \equiv n \bmod \mathfrak{p}$$

for all integers $n \in Z$, $(n, p) = 1$. Since χ is a character of $(Z/pZ)^\times$ of order t, there exists a unique rational integer a, $(1 \le a \le t, (a, t) = 1)$ such that

$$\chi = \omega^{(p-1)(t-a)/t}. \tag{1.10}$$

Put

$$\kappa = (p-1)(t-a)/t. \tag{1.11}$$

THEOREM 1.2. *The notation being as above, there is a decomposition of V_χ into a direct sum of subspaces $V_{1,\chi}$ and $V_{2,\chi}$ satisfying following properties*:

$$\dim_K V_{1,\chi} = \dim_C S_{(p+1)(k/2)-(p-1-\kappa)}(N, \psi), \tag{1.12}$$
$$\dim_K V_{2,\chi} = \dim_C S_{(p+1)(k/2)-\kappa}(N, \psi).$$

$$\widetilde{V_{1,\chi}} = \widetilde{V_{(p+1)(k/2)-(p-1-\kappa)}}. \tag{1.13}$$

$$(\widetilde{V_{2,\chi}} \mid W_p) = \widetilde{V_{(p+1)(k/2)-\kappa}}. \tag{1.14}$$

When $N = 1$, $k = 2$ and χ is trivial, Theorem 1.2 implies $\widetilde{S_2(p)} = \widetilde{S_{p+1}(1)}$.

As a corollary of Theorem 1.2, we shall prove

COROLLARY 1.3. *For any Hecke operator $T(n)$ with $(n, Np) = 1$, the following congruence holds*:

$$\begin{aligned}(1.15)\qquad \operatorname{tr} T(n) \ on\ S_k(Np, \psi\chi) \equiv \operatorname{tr} T(n) \quad on\ S_{(p+1)(k/2)-(p-1-\kappa)}(N, \psi) \\ + n^{\kappa} \operatorname{tr} T(n) \quad on\ S_{(p+1)(k/2)-\kappa}(N, \psi) \quad (\bmod \mathfrak{p}) .\end{aligned}$$

Some results of this note were already announced in [5] without proofs. For general terminology on automorphic forms, we refer to Shimura's textbook [9].

When $k = 2$ and ψ and χ are trivial characters, Dr. Hatada obtained earlier similar results to Theorem 1.1 and Theorem 1.2 in Part 2 of his doctoral thesis at University of Tokyo, April 1979.

§ 2. Proof of Theorem 1.1

We shall prove Theorem 1.1.

Hijikata's trace formula for $\dim_C S_k(N, \chi)$ can be read as follows [2]: let N be a positive integer and $k \geq 2$ be an integer. Let χ be an arbitrary $C^{\times}$-valued character of $(Z/NZ)^{\times}$ such that $\chi(-1) = (-1)^k$.

Then we have

$$(2.1)\qquad \begin{aligned}\dim_C S_k(N, \chi) = t_v(k, N, \chi) + t_p(k, N, \chi) + t_{e1}(k, N, \chi) \\ + t_{e2}(k, N, \chi) + t_{\delta}(k, N, \chi) ,\end{aligned}$$

where

$$(2.2)\qquad t_v(k, N, \chi) = \frac{k-1}{12} \prod_{p|N} p^{\nu}\left(1 + \frac{1}{p}\right),$$

$$(2.3)\qquad t_{e1}(k, N, \chi) = \begin{cases} -\dfrac{1}{3} \dfrac{\omega^{k-1} - \omega'^{k-1}}{\omega - \omega'} \displaystyle\prod_{p|N} \frac{1}{2}\left(1 + \left(\frac{-3}{p}\right)\right)(\chi_p(\omega) + \chi_p(\omega')) & \text{if } 3^2 \nmid N , \\ 0 & \text{if } 3^2 \mid N , \end{cases}$$

$$(2.4)\qquad t_{e2}(k, N, \chi) = \begin{cases} -\dfrac{1}{4} i^{k-2} \displaystyle\prod_{p|N} \frac{1}{2}\left(1 + \left(\frac{-4}{p}\right)\right)(\chi_p(i) + \chi_p(-i)) & \text{if } 4 \nmid N , \text{ and } k \text{ even}, \\ 0 & \text{otherwise} , \end{cases}$$

$$(2.5)\qquad t_{\delta}(k, N, \chi) = \begin{cases} 1 & \text{if } k = 2 \text{ and } \chi \text{ is trivial} , \\ 0 & \text{otherwise} , \end{cases}$$

$$\chi = \prod_{P|N} \chi_p , \quad N = \prod_{P|N} p^{\nu} , \quad X^2 + X + 1 = (X - \omega)(X - \omega') ,$$

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and all products run over all prime divisors p of N. $t_p(k, N, \chi)$ denotes the term which comes from the contribution of the parabolic elements. Later we shall describe it clearly.

Remark. If χ does not satisfy the condition $\chi(-1) = (-1)^k$, $\dim_C S_k(N, \chi)$ is automatically equal to zero. But the right hand side does not vanish generally. Hence (2.1) does not hold in general if $\chi(-1) \neq (-1)^k$.

Now we fix a rational prime $p \geq 5$. Let N be a positive rational integer which is prime to p. Let $k \geq 2$ be an even rational integer. Let ψ and χ be arbitrary $C^\times$-valued characters of $(Z/NZ)^\times$ and $(Z/pZ)^\times$ respectively satisfying

$$\psi\chi(-1) = 1 .$$

Let t denote the order of χ. Then t is a divisor of $p-1$. For any integer a such that $1 \leq a \leq t$ and $(a, t) = 1$, we put $\kappa = ((p-1)(t-a))/t$. Put

$$k_1 = \frac{k}{2}(p+1) - (p-1-\kappa) ,$$

and

$$k_2 = \frac{k}{2}(p+1) - \kappa .$$

We shall prove

THEOREM 2.1. *The notation being as above, we have*

$$t_\alpha(k, Np, \psi\chi) = t_\alpha(k_1, N, \psi) + t_\alpha(k_2, N, \psi) \tag{2.6}$$

for $\alpha = v, p, e1, e2, \delta$.

Remark 2.2. It is clear that Theorem 1.1 follows from Theorem 2.1.

Proof. We should remark that $\psi(-1) = (-1)^{k_1} = (-1)^{k_2}$. Because if $\psi(-1) = 1$, then $\chi(-1) = 1$. Hence κ is even. So k_1 and k_2 are also even. If $\psi(-1) = -1$, then $\chi(-1) = -1$. Hence κ is odd. So k_1 and k_2 are also odd. We should remark that, for $\alpha = v, e1, e2$, the terms corresponding to prime factors ℓ of N are common in both hand sides of (2.6).

For $\alpha = v$, we have

$$t_v(k_1, 1, \psi) + t_v(k_2, 1, \psi) = \frac{1}{12}\{k_1 + k_2 - 2\}$$
$$= \frac{1}{12}\{k(p+1) - (p-1) - 2\}$$
$$= \frac{1}{12}(k-1)(p+1)$$
$$= t_v(k, p, \chi) .$$

For $\alpha = \delta$, the left hand side of (2.6) is equal to 1 if and only if $k = 2$ and ψ and χ are trivial characters. Then we have $t = 1$ and $\kappa = 0$. Hence $k_1 = 2$ and $k_2 = p + 1$. Therefore the right hand side of (2.6) is equal to 1. Otherwise $k_1 \neq 2$ and $k_2 \neq 2$. Hence the right hand side of (2.6) is equal to zero.

For $\alpha = e1$, it is sufficient to prove:

$$(\omega^{k-1} - \omega'^{k-1})\frac{1}{2}\left(1 + \left(\frac{-3}{p}\right)\right)(\chi(\omega) + \chi(\omega'))$$
$$= \omega^{k_1-1} - \omega'^{k_1-1} + \omega^{k_2-1} - \omega'^{k_2-1} .$$

When $p \equiv 2 \pmod 3$, the left hand side of (2.6) is equal to zero. On the other hand, we have $k_1 \equiv \kappa - 1 \pmod 3$ and $k_2 \equiv -\kappa \pmod 3$. Hence $\omega^{k_1-1} = \omega^{\kappa-2}$ and $\omega^{k_2-1} = \omega^{-\kappa-1}$. To prove (2.6), it is sufficient to prove the following lemma:

LEMMA 2.3. *For any integer κ, we have*

$$\omega^{\kappa-2} - \omega'^{\kappa-2} = \omega'^{-\kappa-1} - \omega^{-\kappa-1} \tag{2.7}$$

Proof. If $\kappa \equiv 0 \pmod 3$, we have $\omega^{\kappa-2} = \omega$ and $\omega'^{-\kappa-1} = \omega'^{-1} = \omega$. If $\kappa \equiv 1 \pmod 3$, we have $\omega^{\kappa-2} = \omega^{-1} = \omega'$ and $\omega'^{-\kappa-1} = \omega'^{-2} = \omega'$. If $\kappa \equiv 2 \pmod 3$, we have $\omega^{\kappa-2} = 1$ and $\omega'^{-\kappa-1} = 1$. Hence Lemma 2.3 is proved.

When $p \equiv 1 \pmod 3$, the left hand side of (2.6) is equal to $(\chi(\omega) + \chi(\omega'))(\omega^{k-1} - \omega'^{k-1})$. In this case, ω^κ is proved to be equal to $\chi(\omega)$ or $\chi(\omega')$. Since $\omega^{k_1-1} = \omega^\kappa \cdot \omega^{k-1}$ and $\omega^{k_2-1} = \omega^{-\kappa}\omega^{k-1}$, we have

$$\omega^{k_1-1} - \omega'^{k_1-1} + \omega^{k_2-1} - \omega'^{k_2-1} = (\chi(\omega) + \chi(\omega'))(\omega^{k-1} - \omega'^{k-1}) .$$

Hence Theorem 2.1 is proved for $\alpha = e1$. Now we shall prove (2.6) for $\alpha = e2$. If $\chi(-1) = -1$, κ is odd. Hence k_1 and k_2 are also odd. Hence the both hand sides of (2.6) are equal to zero. Therefore we may assume that $\chi(-1) = 1$. Hence k_1 and k_2 are even, so it is sufficient to prove:

$$(2.8)\qquad \left(1+\left(\frac{-4}{p}\right)\right)\chi(i)i^{k-2}=i^{k_1-2}+i^{k_2-2}\,.$$

When $p\equiv 3 \pmod 4$, the left hand side of (2.8) is equal to zero. Since $k_1-k_2\equiv p-1\equiv 2 \pmod 4$, we have $i^{k_1-2}+i^{k_2-2}=0$. Hence (2.8) is true. When $p\equiv 1 \pmod 4$, the left hand side of (2.8) is equal to $2\chi(i)i^{k-2}$. Since $\chi(-1)=1$, $\chi(i)$ is equal to ± 1. Hence $\chi(i)=i^{\kappa}=i^{-\kappa}$. Therefore we have $i^{k_1-2}=i^{\kappa}i^{k-2}$ and $i^{k_2-2}=i^{-\kappa}i^{k-2}=i^{\kappa}\cdot i^{k-2}$, so

$$i^{k_1-2}+i^{k_2-2}=2i^{\kappa}\cdot i^{k-2}=2\chi(i)i^{k-2}\,.$$

Now we calculate $t_p(k,N,\chi)$ following the formula in [2]. We use the same notation as in [2]. Let $s=2$ and $\Phi(X)=(X-1)^2$. For $1\le f\le N$, we calculate $c(2,f)$. Let p be a prime divisor of N and $\rho=\mathrm{ord}_p f$. Then $\nu=\mathrm{ord}_p N$. Let p^m denote the conductor of χ_p. Put

$$\tilde{A}=\{x\in \boldsymbol{Z}\,|\,(x-1)^2\equiv 0 \bmod p^{\nu+2\rho},\ 2x\equiv 2 \bmod p^{\rho}\}$$
$$\tilde{B}=\{x\in \tilde{A}\,|\,(x-1)^2\equiv 0 \bmod p^{\nu+1+2\rho}\}\,.$$

Let A (resp. B) be a complete system of representatives of $\tilde{A}$ (resp. $\tilde{B}$) modulo $p^{\nu+\rho}$. $c(2,f)$ and $c(2,f,p)$ are defined by the followings:

$$c(2,f)=\prod_{p|N}c(2,f,p)\,,$$
$$c(2,f,p)=\sum_{x\in A}\chi_p(x)+\sum_{y\in B}\chi_p(2-y)\,.$$

(*Case of* $\nu=2n, n\in \boldsymbol{Z}$). We have $\tilde{A}=\{x\in\boldsymbol{Z}\,|\,x\equiv 1 \bmod p^{n+\rho}\}$ and $\tilde{B}=\{x\in\boldsymbol{Z}\,|\,x\equiv 1 \bmod p^{n+1+\rho}\}$. Hence we have $A=\{x \bmod p^{2n+\rho},\ x\equiv 1 \bmod p^{n+\rho}\}$ and $B=\{x \bmod p^{2n+\rho},\ x\equiv 1 \bmod p^{n+\rho+1}\}$, so $|A|=p^n$ and $|B|=p^{n-1}$. Therefore we have

$$(2.9)\qquad c(2,f,p)=\begin{cases}p^n+p^{n-1} & \text{if } m\le n+\rho\,,\\ p^{n-1} & \text{if } m=n+\rho+1\,,\\ 0 & \text{if } m>n+\rho+1\,.\end{cases}$$

(*Case of* $\nu=2n+1, n\in\boldsymbol{Z}$). We have $\tilde{A}=\tilde{B}=\{x\in\boldsymbol{Z}\,|\,x\equiv 1 \bmod p^{n+1+\rho}\}$. Hence we have $A=B=\{x \bmod p^{2n+1+\rho},\ x\equiv 1 \bmod p^{n+1+\rho}\}$ and $|A|=|B|=p^n$. Therefore we have

$$(2.10)\qquad c(2,f,p)=\begin{cases}2p^n & \text{if } m\le n+\rho+1\,,\\ 0 & \text{if } m>n+\rho+1\,.\end{cases}$$

Since $\chi(-1)=(-1)^k$, $t_p(k,N,\chi)$ is given by the following:

$$t_p(k, N, \chi) = -\frac{2}{4N}\sum_f c(2, f)\,, \tag{2.11}$$
$$= -\frac{2}{4N}\sum_f \prod_{p|N} c(2, f, p)\,,$$

where f ranges from 1 to N.

We should write $c(2, f, N, \chi)$ or $c(2, f, p, N, \chi)$ instead of $c(2, f)$ or $c(2, f, p)$ to make explicit contributions of the level N and the character χ. It is clear that $t_p(k, N, \chi)$ is independent of k.

LEMMA 2.4. *Let f be an integer such that $1 \le f \le N$. Put $f' = f + iN$ for $0 \le i \le p-1$. Then, for any prime divisor ℓ of N, we have*

$$c(2, f', \ell, Np, \psi\chi) = c(2, f, \ell, N, \psi)\,. \tag{2.12}$$

Proof. Put $\rho = \mathrm{ord}_\ell f$, $\rho' = \mathrm{ord}_\ell f'$ and $\nu = \mathrm{ord}_\ell N$. Let ℓ^m denote the conductor of ψ_ℓ. Since $\ell \neq p$, we have $(\psi\chi)_\ell = \psi_\ell$. If $\rho < \nu$, ρ' is equal to ρ. Hence (2.12) is true. If $\rho \ge \nu$, it also holds $\rho' \ge \nu$. But it is clear that $m \le \nu$, so we have

$$n + \rho \ge m \quad \text{and} \quad n + \rho' \ge m\,.$$

Hence (2.12) is true.

LEMMA 2.5. *If $\nu = 1$, we have*

$$c(2, f, p) = 2$$

for any integer f.

Proof. When $\nu = 1$, we have $n = 0$. Hence $n + \rho + 1 \ge 1$ and $m \le 1$, so $n + \rho + 1 \ge m$. Therefore $c(2, f, p) = 2$.

Now we shall prove Theorem 2.1 for $\alpha = p$. We have

$$t_p(k, Np, \psi\chi) = -\frac{2}{4Np}\sum_{\substack{f' = f + iN\\ 1 \le f \le N\\ 0 \le i \le p-1}} c(2, f', p, Np, \psi\chi)\prod_{\ell|N} c(2, f', \ell, Np, \psi\chi)$$
$$= -\frac{4}{4Np}\sum_{\substack{f' = f + iN\\ 1 \le f \le N\\ 0 \le i \le p-1}} \prod_{\ell|N} c(2, f, \ell, N, \psi)$$
$$= -\frac{4}{4N}\sum_{1 \le f \le N} c(2, f, N, \psi)$$
$$= t_p(k_1, N, \psi) + t_p(k_2, N, \psi)\,.$$

This completes the proof.

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§ 3. Supplements to Theorem 1.1

Here we shall give several supplements to Theorem 1.1.

Let $S_k^0(N, \chi)$ denote the subspace of all new forms in $S_k(N, \chi)$. We consider the same situation as in Theorem 1.1. Let M denote the conductor of ψ. If the conductor of χ is p, we have

$$S_k(Np, \psi\chi) = \bigoplus_{M} \bigoplus_{d|(N/M)} S_k^0(Mp, \psi\chi)^d \qquad \text{(direct sum)}$$

where $S_k^0(Mp, \psi\chi)^d = \{f(dz) \,|\, f \in S_k^0(Mp, \psi\chi)\}$ and the summation ranges over all positive divisors M of N which are multiples of M and over all positive divisors d of N/M.

If χ is the trivial character, we have $S_k(Np, \psi\chi) = S_k(Np, \psi)$ and

$$S_k(Np, \psi) = \bigoplus_{M} \bigoplus_{d|(N/M)} \{S_k^0(M, \psi)^d \oplus S_k^0(M, \psi)^{dp}\} \oplus \bigoplus_{M} \bigoplus_{d|(N/M)} S_k^0(Mp, \psi)^d \qquad \text{(direct sum)}$$

where the summation ranges over the same sets as above. Therefore, using inductions on levels and Theorem 1.1, we can easily prove the following:

THEOREM 3.1. *We use the same notation as in Theorem 1.1. We assume that χ is not trivial, then we have*

$$\dim_C S_k^0(Np, \psi\chi) = \dim_C S^0_{(k/2)(p+1)-(p-1-\kappa)}(N, \psi) + \dim_C S^0_{(k/2)(p+1)-\kappa}(N, \psi)\,.$$

We also have

$$\dim_C S_k^0(Np, \psi) + 2\dim_C S_k^0(N, \psi) = \dim_C S^0_{(k/2)(p+1)-(p-1)}(N, \psi) + \dim_C S^0_{(k/2)(p+1)}(N, \psi)\,,$$

where ψ is an arbitrary character of $(Z/NZ)^\times$ such that $\psi(-1) = 1$.

Combining the above theorem and Hijikata's results in [3], we can prove identities between dimensions of spaces of cusp forms with respect to Fuchsian groups which are obtained from split orders of indefinite division quaternion algebras.

In Theorem 1.1, we assume that k is even. Now we consider the case that k is odd and $k \geq 3$. Let ψ and χ be arbitrary characters of $(Z/NZ)^\times$ and $(Z/pZ)^\times$ respectively satisfying $\psi\chi(-1) = -1$. Let t denote the order of χ. Put $\kappa = ((p-1)(t-a))/t$ with any integer a such that $1 \leq a \leq t$ and $(a, t) = 1$. Put

$$k_1 = \frac{k}{2}(p+1) - (p-1-\kappa)$$

and

$$k_2 = \frac{k}{2}(p+1) - \kappa\,.$$

We have to check whether $\psi(-1) = (-1)^{k_1}$ or not.

LEMMA 3.2. *We have*

$$\psi(-1) = \begin{cases} (-1)^{k_1} = (-1)^{k_2} & \text{if } p \equiv 1 \pmod 4\,, \\ -(-1)^{k_1} = -(-1)^{k_2} & \text{if } p \equiv 3 \pmod 4\,. \end{cases}$$

Proof. If $p \equiv 3 \pmod 4$, we have $k_1 \equiv \kappa \pmod 2$ and $k_2 \equiv \kappa \pmod 2$. Since $\chi(-1) = (-1)^\kappa$, we have $\psi(-1) = -\chi(-1) = -(-1)^\kappa = -(-1)^{k_1} = -(-1)^{k_2}$. If $p \equiv 1 \pmod 4$, we have $k_1 \equiv 1 + \kappa$ and $k_2 \equiv 1 + \kappa \pmod 2$. Since $\chi(-1) = (-1)^\kappa$, we have $\psi(-1) = -\chi(-1) = (-1)^{1+\kappa} = (-1)^{k_1} = (-1)^{k_2}$.

Considering a statement analogous to Theorem 1.1 in the case k is odd, we have almost no problem if $p \equiv 1 \pmod 4$. But, if $p \equiv 3 \pmod 4$, the right hand side of (1.3) is automatically equal to zero, because $\dim_C S_{k_1}(N, \psi) = 0$ if $\psi(-1) \neq (-1)^{k_1}$. So we have to define the followings: let N a positive integer and $k \geq 2$ be an integer. Let χ be an arbitrary $C^\times$-valued character of $(Z/NZ)^\times$. We don't assume that $\chi(-1) = (-1)^k$. For the triple (k, N, χ), we define $t_v(k, N, \chi)$, $t_{e1}(k, N, \chi)$, $t_{e2}(k, N, \chi)$ and $t_\delta(k, N, \chi)$ by (2.2) $\sim$ (2.5). For the definition of $t_p(k, N, \chi)$, we use (2.11). The definitions of these numbers don't need the condition $\chi(-1) = (-1)^k$.

Then we put

$$d(k, N, \chi) = \sum_{\alpha = v, e1, e2, p, \delta} t_\alpha(k, N, \chi)\,. \tag{3.1}$$

Hence if $\chi(-1) = (-1)^k$, $d(k, N, \chi)$ coincides with $\dim_C S_k(N, \chi)$.

THEOREM 3.3. *We suppose k is odd, $k \geq 3$ and $\psi\chi(-1) = -1$. The notation being as above, we have*

$$d(k, Np, \psi\chi) = d(k_1, N, \psi) + d(k_2, N, \psi)\,. \tag{3.2}$$

Proof. In proving Theorem 2.1, we don't use the condition k is even except for the proof in the case $\alpha = e2$. Since k is odd, the left hand side is always equal to zero. If $p \equiv 3 \pmod 4$, we have $(-4/p) = -1$, so the right hand side is equal to zero. We may assume $p \equiv 1 \pmod 4$. Then we have $k_1 \equiv k_2 \equiv 1 + \kappa \pmod 2$. If κ is even, namely $\chi(-1) = 1$, the right hand side is equal to zero. If $\chi(-1) = -1$, we have $\chi(i) + \chi(-i) = 0$.

This completes the proof.

EXAMPLE 3.4. We calculate $d(k) = d(k, 1, \chi^0)$ for any odd k and the trivial character χ^0. We have

$$
\begin{aligned}
d(k) &= \frac{k-1}{12} - \frac{1}{2} - \frac{1}{3}\,\frac{\omega^{k-1} - \omega'^{k-1}}{\omega - \omega'} \\
&= \begin{cases} \dfrac{m}{2} - \dfrac{1}{2} & \text{if } k = 6m + 1\,, \\[2ex] \dfrac{m}{2} & \text{if } k = 6m + 3\,, \\[2ex] \dfrac{m}{2} - \dfrac{1}{2} & \text{if } k = 6m + 5\,. \end{cases}
\end{aligned}
\tag{3.3}
$$

Let p be a rational prime such that $p \equiv 3 \pmod 4$ and $p \geq 5$. We apply Theorem 3.3 for $\dim_C S_k(p, \chi)$ for an odd k, $k \geq 3$ and an arbitrary character χ of $(Z/pZ)^\times$ such that $\chi(-1) = -1$. Let t denote the order of χ. Then we have

$$
\begin{aligned}
\dim_C S_k(p, \chi) &= d\Big(\Big\{\frac{k}{2}(p+1) - \Big(p - 1 - \frac{p-1}{t}\Big)\Big\}\Big) \\
&\quad + d\Big(\Big\{\frac{k}{2}(p+1) - \frac{p-1}{t}\Big\}\Big)\,.
\end{aligned}
\tag{3.4}
$$

When $\chi(n) = (n/p)$, we have

$$
\dim_C S_k\Big(p, \Big(\frac{\ }{p}\Big)\Big) = 2d\Big(\Big\{\frac{k-1}{2}(p+1) + 1\Big\}\Big)\,.
$$

§ 4. Proof of Theorem 1.2

We use the same notation as in the introduction. We prove the following lemma.

LEMMA 4.1. *W_p gives the isomorphism between V_χ and $V_{\bar\chi}$.*

Proof. By virtue of results of Miyake [7] and Asai [1], W_p gives the isomorphism between $S_k(Np, \psi\chi)$ and $S_k(Np, \psi\bar\chi)$. Therefore it is sufficient to prove that W_p maps V_χ into $V_{\bar\chi}$, or W_p maps a suitable basis of V_χ into $V_{\bar\chi}$. As a suitable basis of V_χ, we can choose following elements:

(i) $f_i(z) = \sum_{n=1}^\infty a_n q^n \in S_k^0(Np, \psi\chi)$ are common eigenfunctions of all

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Hecke operators such that $a_1 = 1$ which are called primitive eigenforms in $S_k(Np, \psi\chi)$,

(ii) $g_j(z)$ are primitive eigenforms in $S_k(M, \psi\chi)$ for any positive divisors M of Np such that $\psi\chi$ is defined modulo M,

(iii) $g_j(dz)$ for any positive divisors d of Np/M.

By Lemma 3 in [1], W_p maps $f_i(z)$ into $V_{\bar{\chi}}$. If $g_j(z)$ is a primitive eigenform in $S_k(Mp, \psi\chi)$ where M is a positive divisor of N, for any positive divisors d of N/M, we have

$$\begin{aligned} g_j(dz)\,|\,W_p &= d^{-k/2} g_j(z)\Big|\begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} px, & 1 \\ pNy, & p \end{bmatrix} \\ &= d^{-k/2} g_j(z)\Big|\begin{bmatrix} px, & d \\ p(N/d)y, & p \end{bmatrix}\begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Hence we can apply Lemma 3 in [1] to $g_j(z)$, so W_p maps $g_j(dz)$ into $V_{\bar{\chi}}$. If $g_j(z)$ is a primitive eigenform in $S_k(M, \psi)$ where M is a positive divisor of N, χ must be the trivial character. Hence we have, for any positive divisors d of N/M,

$$\begin{aligned} g_j(dz)\,|\,W_p &= d^{-k/2} g_j(z)\Big|\begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} px, & 1 \\ pNy, & p \end{bmatrix} \\ &= d^{-k/2} g_j(z)\Big|\begin{bmatrix} x, & d \\ (N/d)y, & p \end{bmatrix}\begin{bmatrix} pd & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Since $\begin{bmatrix} x, & d \\ (N/d)y, & p \end{bmatrix}$ belongs to $\Gamma_0(N/d)$, W_p maps $g_j(dz)$ into $V_{\bar{\chi}}$. Hence Lemma 4.1 is completely proved.

Let $K_\mathfrak{p}$ denote the completion of K at $\mathfrak{p}$ and $\mathfrak{o}_\mathfrak{p}$ the closure of $\mathfrak{o}$ in $K_\mathfrak{p}$. We mainly consider $V_\chi \otimes K_\mathfrak{p}$ and $V_{k'} \otimes K_\mathfrak{p}$ (over K) in this section. It is clear that W_p is uniquely extended to the $K_\mathfrak{p}$-linear isomorphism between $V_\chi \otimes K_\mathfrak{p}$ and $V_{\bar{\chi}} \otimes K_\mathfrak{p}$, so we also denote this by W_p. We define $K_\mathfrak{p}$-norms N_χ and N'_χ on $V_\chi \otimes K_\mathfrak{p}$ as follows: for any $f = \sum_{n=1}^{\infty} a_n q^n \in V_\chi \otimes K_\mathfrak{p}$, $a_n \in K_\mathfrak{p}$, we define N_χ and N'_χ by

(4.1) $$N_\chi(f) = \sup_n \nu(a_n)\,,$$

and

(4.2) $$N'_\chi(f) = N_{\bar{\chi}}(f\,|\,W_p)\,.$$

Hence by virtue of Proposition 4 of Chap II, § 2 in [10], there exists the following decomposition

$$V_\chi \otimes K_\mathfrak{p} = V_1 \oplus \cdots \oplus V_{d_\chi} \tag{4.3}$$

of $V_\chi \otimes K_\mathfrak{p}$ into a direct sum of subspaces V_i of dimension 1 such that

$$N_\chi(\sum v_i) = \sup_i N_\chi(v_i) ,$$

and

$$N'_\chi(\sum v_i) = \sup_i N'_\chi(v_i) ,$$

whenever $v_i \in V_i$ for $1 \leq i \leq d_\chi$ where d_χ denotes the dimension of $V_\chi \otimes K_\mathfrak{p}$ over $K_\mathfrak{p}$. Here we may assume that $V_i = K_\mathfrak{p} \cdot v_i$ with $v_i \in V_i$ and $N_\chi(v_i) = 1$ for $1 \leq i \leq d_\chi$. Changing the order of i suitably, we may assume

$$\begin{cases} N'_\chi(v_i) < p^{\kappa/(p-1)} & \text{for } 1 \leq i \leq r_\chi , \\ N'_\chi(v_i) \geq p^{\kappa/(p-1)} & \text{for } r_\chi < i \leq d_\chi . \end{cases} \tag{4.4}$$

THEOREM 4.2. *The notation being as above, we have*

$$r_\chi = \dim_C S_{(k/2)(p+1)-(p-1-\kappa)}(N, \psi) . \tag{4.5}$$

Proof. We use the same idea as in the proof of Theorem (3.4) in [4], so we quote several results from [4]. For any odd Dirichlet character η mod p, let

$$E_{1,\eta} = 1 + c_\eta \sum_{\substack{m>0 \\ m_1>0}} \eta(m) e^{2\pi i m m_1 z} ,$$

$$c_\eta = -\frac{2p}{\sum_{1\leq a\leq p-1} \eta(a)a}$$

denote the normalized Eisenstein series of type $(1, \eta)$ on $\Gamma_0(p)$. Then we have

$$E_{1,\bar\omega} \left| \begin{bmatrix} 0 & -1 \\ p & 0 \end{bmatrix} \right. = \frac{c_{\bar\omega}}{c_\omega} \frac{\sqrt{p}}{C(\omega)} E_{1,\omega}$$

where $C(\omega) = \sum_{a=1}^{p-1} \omega(a) e^{2\pi i (a/p)}$ is the Gauss sum.

For each v_i, $1 \leq i \leq d_\chi$, put

$$h_i = v_i(E_{1,\bar\omega})^{(k/2)(p-1)-(p-1-\kappa)} . \tag{4.6}$$

Then h_i is written by a $K_\mathfrak{p}$-linear sum of elements in $S_{k_1}(Np, \psi)$. Hence we can define

$$T_r(h_i) = h_i + \psi(p)^{-1} p^{1-(k_1/2)} (h_i \,|\, W_p) \,|\, U(p) ,$$

as in [4].

Then $T_r(h_i)$ is written by a $K_\mathfrak{p}$-linear sum of elements in $S_{k_1}(N, \psi)$. We shall prove

$$T_r(h_i) \equiv h_i \pmod{\mathfrak{p}} \tag{4.7}$$

for i, $1 \leq i \leq r_\chi$.

For any element $f = \sum_{n=1}^{\infty} a_n q^n \in K_\mathfrak{p}[[q]]$, we define

$$N(f) = \sup_n \nu(a_n) .$$

Then, in order to prove (4.7), it sufficient to prove

$$N(p^{1-(k_1/2)} h_i \mid W_p) < 1 \tag{4.8}$$

for $1 \leq i \leq r_\chi$. It is clear that

$$\begin{aligned} h_i \mid W_p &= (v_i \mid W_p)\left(E_{1,\bar{\omega}} \middle| \begin{bmatrix} 0 & -1 \\ p & 0 \end{bmatrix}\right)^{k_1 - k} \\ &= (v_i \mid W_p) \cdot \left(\frac{\sqrt{p}\, c_{\bar{\omega}}}{C(\omega) c_\omega}\right)^{k_1 - k} E_{1,\omega}^{k_1 - k} . \end{aligned}$$

Therefore, using Proposition (1.2) in [4], we have

$$\begin{aligned} N(p^{1-(k_1/2)} h_i \mid W_p) &= p^{-(1-(k_1/2))} N'_\chi(v_i) p^{(-(1/2)-(1/p-1))(k_1 - k)} \\ &= p^{-(\kappa/(p-1))} N'_\chi(v_i) . \end{aligned}$$

Hence (4.8) holds for $1 \leq i \leq r_\chi$.

Since $h_i \equiv v_i \pmod{\mathfrak{p}}$, we have

$$T_r(h_i) \equiv v_i \pmod{\mathfrak{p}}$$

for $1 \leq i \leq r_\chi$. Hence $\{\tilde{v}_i, 1 \leq i \leq r_\chi\}$ are contained in $\tilde{V}_{k_1}$. Since $\{\tilde{v}_i, 1 \leq i \leq r_\chi\}$ are linearly independent over F, we have

$$r_\chi \leq \dim_C S_{k_1}(N, \psi) . \tag{4.9}$$

We apply the same argument to $V_{\bar{\chi}}$. Let

$$u_i = N'_\chi(v_i)^{-1}(v_i \mid W_p) \tag{4.10}$$

for $1 \leq i \leq d_\chi$. Then $\{u_i, 1 \leq i \leq d_\chi\}$ forms a basis of $V_{\bar{\chi}} \otimes K_\mathfrak{p}$. For $\bar{\chi}$, κ changes to $p - 1 - \kappa$.

Hence we have

$$N'_\chi(u_i) < p^{1-(\kappa/(p-1))}$$

for $r_\chi + 1 \leq i \leq d_\chi$. Therefore we have

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$$d_\chi - r_\chi \le \dim_C S_{k_2}(N, \psi) . \tag{4.11}$$

By virtue of Theorem 1.1, we have

$$d_\chi = \dim_C S_{k_1}(N, \psi) + \dim_C S_{k_2}(N, \psi) . \tag{4.12}$$

Combining (4.9), (4.11) and (4.12), we obtain the proof of Theorem 4.2.

Proof of Theorem 1.2. For each v_i, $1 \le i \le d_\chi$, there exists an element f_i in V_χ satisfying $v_i \equiv f_i \pmod{\mathfrak{p}}$. As $V_{1,\chi}$ and $V_{2,\chi}$, we can take the spaces spanned by $\{f_i, 1 \le i \le r_\chi\}$ and $\{f_i, r_\chi + 1 \le i \le d_\chi\}$ respectively over K. Then the proof of Theorem 4.2 shows that (1.12), (1.13) and (1.14) are true.

§ 5. Proof of Corollary 1.3

We use the same notation as in the proof of Theorem 4.2, except for d_χ and r_χ. We simply write d and r instead of d_χ and r_χ. Let $\{v_i, 1 \le i \le d\}$ be the basis of $V_\chi \otimes K_\mathfrak{p}$. For any positive integer n such that $(n, Np) = 1$, we simply denote by $T(n)$ the Hecke operator of degree n acting on $S_k(Np, \psi\chi)$ or $S_{k'}(N, \chi)$. Since $T(n)$ are K-linear endomorphisms of V_χ or $V_{k'}$, $T(n)$ are uniquely extended to $K_\mathfrak{p}$-linear endomorphisms of $V_\chi \otimes K_\mathfrak{p}$ or $V_{k'} \otimes K_\mathfrak{p}$, which we also denote by $T(n)$. Let ℓ be a prime such that $\ell \nmid Np$. Then, for any $v = \sum_{n=1}^\infty a(n)q^n \in V_\chi \otimes K_\mathfrak{p}$, $a(n) \in K_\mathfrak{p}$, we have

$$v \mid T(\ell) = \sum_{n=1}^\infty \left\{a(n\ell) + \ell^{k-1}\psi\chi(\ell)a\left(\frac{n}{\ell}\right)\right\}q^n , \tag{5.1}$$

where $a(n/\ell) = 0$ if $\ell \nmid n$.

Theorem 1.2 implies that, for each v_i, $1 \le i \le r$, there exists an element f_i in $V_{k_1} \otimes K_\mathfrak{p}$ such that

$$v_i \equiv f_i \pmod{\mathfrak{p}} . \tag{5.2}$$

Then it is clear that $\{f_i, 1 \le i \le r\}$ forms a basis of $V_{k_1} \otimes K_\mathfrak{p}$. For any $f = \sum_{n=1}^\infty b(n)q^n \in V_{k_1} \otimes K_\mathfrak{p}$, $b(n) \in K_\mathfrak{p}$, we have

$$f \mid T(\ell) = \sum_{n=1}^\infty \left\{b(n\ell) + \ell^{k_1-1}\psi(\ell)b\left(\frac{n}{\ell}\right)\right\}q^n . \tag{5.3}$$

Since $\ell^{k_1-1} = \ell^{(k/2)(p+1)-(p-1-\kappa)-1} \equiv \ell^{k-1}\cdot\ell^\kappa \equiv \chi(\ell)\ell^{k-1} \pmod{\mathfrak{p}}$, by (5.1), (5.2) and (5.3), we have

$$v_i \mid T(n) \equiv f_i \mid T(n) \qquad \pmod{\mathfrak{p}} \tag{5.4}$$

for any i, $1 \le i \le r$ and for any positive integers n such that $(n, Np) = 1$.

We should remark that V_{k_1} is closed under the action of $T(n)$, but the space spanned by v_i, $1 \leq i \leq r$ may not be closed under the action of any $T(n)$.

Take suitable constants $c_i \in K_\mathfrak{p}$ satisfying

$$\begin{cases} v'_i = v_i \mid W_p , \\ u_i = c_i v'_i, \quad N_{\bar{\chi}}(u_i) = 1 \qquad \text{for } 1 \leq i \leq d . \end{cases} \tag{5.5}$$

For any $u = \sum_{n=1}^{\infty} c(n)q^n \in V_{\bar{\chi}} \otimes K_\mathfrak{p}$, $c(n) \in K_\mathfrak{p}$, we have

$$u \mid T(\ell) = \sum_{n=1}^{\infty} \left\{ c(n\ell) + \ell^{k-1}\psi(\ell)\bar{\chi}(\ell)c\left(\frac{n}{\ell}\right)\right\} q^n . \tag{5.6}$$

Theorem 1.2 implies that for each i, $r + 1 \leq i \leq d$, there exists an element g_i in $V_{k_2} \otimes K_\mathfrak{p}$ such that

$$u_i \equiv g_i \qquad (\text{mod } \mathfrak{p}) . \tag{5.7}$$

For any $g = \sum_{n=1}^{\infty} d(n)q^n \in V_{k_2} \otimes K_\mathfrak{p}$, $d(n) \in K_\mathfrak{p}$, we have

$$g \mid T(\ell) = \sum_{n=1}^{\infty} \left\{ d(n\ell) + l^{k_2-1}\psi(\ell)d\left(\frac{n}{\ell}\right)\right\} q^n . \tag{5.8}$$

Since $\ell^{k_2-1} = \ell^{(k/2)(p+1)-\kappa-1} \equiv \ell^{k-1}\cdot\ell^{-\kappa} \equiv \ell^{k-1}\bar{\chi}(\ell)$ (mod $\mathfrak{p}$), by (5.6), (5.7) and (5.8), we have

$$u_i \mid T(n) \equiv g_i \mid T(n) \qquad (\text{mod } \mathfrak{p}) , \tag{5.9}$$

for any i, $r + 1 \leq i \leq d$ and for any positive integer n such that $(n, Np) = 1$. Let

$$\begin{aligned} &[v_{r+1}, \cdots, v_d] \mid T(n) = [v_1, \cdots, v_d]A(n) , \\ &[u_{r+1}, \cdots, u_d] \mid T(n) = [u_1, \cdots, u_d]C(n) , \\ &[g_{r+1}, \cdots, g_d] \mid T(n) = [g_{r+1}, \cdots, g_d]D(n) , \end{aligned} \tag{5.10}$$

where $A(n)$, $C(n)$, $D(n)$ are $d \times d - r$, $d \times d - r$, $d - r \times d - r$ matrices whose elements are in $\mathfrak{o}_\mathfrak{p}$. Then (5.9) shows that

$$C(n) \ (\text{mod } \mathfrak{p}) = \left[\frac{O}{D(n)}\right]\}_r \qquad (\text{mod } \mathfrak{p}) . \tag{5.11}$$

In [1], Asai proved that, for any positive integer n such that $(n, Np) = 1$, it holds

$$\chi(n)\{v_i \mid W_p\} \mid T(n) = \{v_i \mid T(n)\} \mid W_p , \tag{5.12}$$

for any i, $1 \leq i \leq d$. We should remark that he considered only when the level is square free but it is easily shown that his method is applicable to our case.

Therefore we have

$$\begin{aligned}[v_{r+1}, \cdots, v_d] | T(n) | W_p &= [v_1, \cdots, v_d] A(n) | W_p \\ &= ([v_1, \cdots, v_d] | W_p) A(n) \\ &= [u_1, \cdots, u_d] \begin{bmatrix} c_1^{-1} & & 0 \\ & \ddots & \\ 0 & & c_d^{-1} \end{bmatrix} A(n)\end{aligned}$$

and

$$\begin{aligned}\chi(n)[v_{r+1}, \cdots, v_d] | W_p | T(n) &= \chi(n)[u_{r+1}, \cdots, u_d] \left[\begin{bmatrix} c_{r+1}^{-1} & & 0 \\ & \ddots & \\ 0 & & c_d^{-1} \end{bmatrix}\right| T(n) \\ &= \chi(n)[u_1, \cdots, u_d] C(n) \begin{bmatrix} c_{r+1}^{-1} & & 0 \\ & \ddots & \\ 0 & & c_d^{-1} \end{bmatrix}.\end{aligned}$$

Hence we have

$$\begin{bmatrix} c_1^{-1} & & 0 \\ & \ddots & \\ 0 & & c_d^{-1} \end{bmatrix} A(n) = \chi(n) C(n) \begin{bmatrix} c_{r+1}^{-1} & & 0 \\ & \ddots & \\ 0 & & c_d^{-1} \end{bmatrix},$$

so

$$\begin{aligned}C(n) &= \chi(n)^{-1} \begin{bmatrix} c_1^{-1} & & 0 \\ & \ddots & \\ 0 & & c_d^{-1} \end{bmatrix} A(n) \begin{bmatrix} c_{r+1} & & 0 \\ & \ddots & \\ 0 & & c_d \end{bmatrix} \\ &= \chi(n)^{-1} \left[\begin{array}{ccc} & * & \\ \hline a_{r+1,r+1} & & * \\ & \ddots & \\ * & & a_{d,d} \end{array}\right] \begin{matrix} \}r \\ \\ \\ \\ \end{matrix},\end{aligned} \tag{5.12}$$

where $A(n) = (a_{ij})$.

Then combining (5.4), (5.9), (5.11) and (5.12) we have

(5.13) $\quad \operatorname{tr} T(n) \quad \text{on } V_\chi \equiv \operatorname{tr} T(n) \quad \text{on } V_{k_1} + \chi(n) \operatorname{tr} T(n) \quad \text{on } V_{k_2} \pmod{\mathfrak{p}}.$

This completes the proof.

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Remark 5.1. In subsequent papers [6], we shall give another method to prove these congruences. It is to compute $\operatorname{tr} T(n)$ on $V_\chi \pmod{\mathfrak{p}}$, $\operatorname{tr} T(n)$ on $V_{k'} \pmod{\mathfrak{p}}$ directly by using Hijikata's explicit trace formulae for Hecke operators [2], and to compare both hands sides of (5.13).

Remark 5.2. If (5.13) is proved by an another method, we can get simpler proof of Theorem 1.1 applying (5.13) to the case $n = 1$.

§ 6. Supplements to Theorem 1.2

Here we assume that $p \equiv 1 \pmod 4$, k is odd, $k \geq 3$ and $\psi\chi(-1) = -1$. In Theorem 3.3, § 3, we proved certain identities of dimensions of spaces of cusp forms analogous to Theorem 1.1. Hence it is natural to study a statement analogous to Theorem 1.2 in this case. We slightly modify the proof of Theorem 1.2 at following points:

(i) K must be enlarged to contain $\sqrt{d}$ for all positive divisors d of Np.

(ii) We consider $S_k(Np, \psi\chi\omega^{(p-1)/2})$ instead of $S_k(Np, \psi\chi)$. Since $p \equiv 1 \pmod 4$, we have $\omega^{(p-1)/2}(-1) = 1$ and $\psi\chi\omega^{(p-1)/2}(-1) = -1$. Hence the order of $\chi\omega^{(p-1)/2}$ coincides with that of χ. Therefore we have

$$\dim_C S_k(Np, \psi\chi\omega^{(p-1)/2}) = \dim_C S_k(Np, \psi\chi) .$$

Since $\omega^{(p-1)/2}$ is a real valued character, we have

$$(\overline{\chi\omega^{(p-1)/2}}) = \bar{\chi}\omega^{(p-1)/2} .$$

(iii) V_χ should be changed as follows:

$$V_\chi = \left\{ f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(Np, \ \psi\chi\omega^{(p-1)/2}) \,|\, a_n \in K \ \text{ for all } \ n \geq 1 \right\} .$$

(iv) Since k is odd, we have

$$\bar{\omega}^{(p-1)(k/2)-(p-1-\kappa)} = \bar{\omega}^{(p-1)/2}\bar{\omega}^{\kappa} = \omega^{(p-1)/2}\bar{\omega}^{\kappa} .$$

Hence, following the arguments in § 4, § 5, we can prove

THEOREM 6.1. *The notation being as above, there is a decomposition of V_χ into a direct* sum *of subspaces $V_{1,\chi}$ and $V_{2,\chi}$ satisfying following properties*:

$$\dim_K V_{1,\chi} = \dim_C S_{k_1}(N, \psi) , \tag{6.1}$$
$$\dim_K V_{2,\chi} = \dim_C S_{k_2}(N, \psi) .$$

(6.2) $$\widetilde{V_{1,\chi}} = \tilde{V}_{k_1}\,.$$

(6.3) $$(\widetilde{V_{2,\chi} | W_p}) = \tilde{V}_{k_2}\,.$$

COROLLARY 6.2. *For any Hecke operator $T(n)$, with $(n, Np) = 1$, the following congruence holds*:

(6.4) $$\operatorname{tr} T(n) \quad on \quad S_k(Np, \psi\chi\omega^{(p-1)/2}) \equiv \operatorname{tr} T(n) \quad on \quad S_{k_1}(N, \psi) + n^{\varepsilon} \operatorname{tr} T(n) \quad on \quad S_{k_2}(N, \psi) \pmod{\mathfrak{p}}\,.$$

REFERENCES

[1] T. Asai, On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution, J. Math. Soc. Japan, **28** (1976), 48–61.

[2] H. Hijikata, Explicit formula of the traces of Hecke operators for $\Gamma_0(N)$, J. Math. Soc. Japan, (1974), 56–82.

[3] ——, On certain identities between the traces of Hecke operators, Proc. Japan Acad., **48** (1972), 585–588.

[4] M. Koike, Congruences between cusp forms and linear representations of the Galois group, Nagoya Math. J., **64** (1976), 63–85.

[5] ——, A note on modular forms mod p, Proc. Japan Acad., **55** (1979), 313–315.

[6] ——, Eigenvalues of Hecke operators mod p, preprint.

[7] T. Miyake, On automorphic forms on GL_2 and Hecke operators, Ann. of Math., **94** (1971), 174–189.

[8] J.-P. Serre, Formes modulaires et fonctions zêta p-adiques, Modular functions of one variable. III. Proc. Intern. Summer School, Univ. Antwerp. Lect. Notes in Math., **350**, Springer, (1972) 191–268.

[9] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Iwanami Shoten and Princeton University Press, 1971.

[10] A. Weil, Basic Number Theory, Grundl. Math. Wiss., 144, Berlin-Heidelberg-New York, 1967.

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