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A NOTE ON GROUPS WHOSE PROPER LARGE SUBGROUPS HAVE A TRANSITIVE NORMALITY RELATION

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Abstract

A group G is said to have the T-property (or to be a T-group) if all its subnormal subgroups are normal, that is, if normality in G is a transitive relation. The aim of this paper is to investigate the behaviour of uncountable groups of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have a transitive normality relation. It is proved that such a group G is a T-group (and all its subgroups have the same property) provided that G has an ascending subnormal series with abelian factors. Moreover, it is shown that if G is an uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property, then every subnormal subgroup of G has only finitely many conjugates.

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1. Introduction

In a long series of recent papers, it has been shown that the structure of a group of infinite rank is strongly influenced by that of its proper subgroups of infinite rank (see, for instance, [4], where a full reference list on this subject can be found). The results in these papers suggest that the behaviour of *small* subgroups in a *large* group is neglectable, at least for an appropriate choice of the definition of largeness and within a suitable universe. This point of view was also adopted in [6–8], where the authors considered uncountable groups whose proper uncountable subgroups belong to certain relevant group classes, such as the class of nilpotent groups, groups with finite conjugacy classes and groups with modular subgroup lattice.

The aim of this paper is to give a further contribution to this topic, by investigating uncountable groups of cardinality $\aleph$ in which all proper subgroups of cardinality $\aleph$ have a transitive normality relation. The corresponding problem in the case of groups of infinite rank has been solved in [3].

Recall that a group G is said to have the *T-property* (or to be a *T-group*) if normality in G is a transitive relation, that is, if all subnormal subgroups of G are normal.

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The structure of soluble T-groups has been described by Gaschütz [5] in the finite case and by Robinson [10] for arbitrary groups. It turns out in particular that soluble groups with the T-property are metabelian and hypercyclic, and that finitely generated soluble T-groups are either finite or abelian. Although the class of T-groups is not subgroup closed (because any simple group is obviously a T-group), it is known that subgroups of finite soluble T-groups likewise have the T-property. A group G is called a $\overline{T}$*-group* if all its subgroups have the T-property. It follows easily from the properties of T-groups that any finite $\overline{T}$-group is soluble and so even supersoluble, while soluble nonperiodic groups with the $\overline{T}$-property are abelian.

The main obstacle in the study of groups of large cardinality is the existence of uncountable groups, of cardinality $\aleph$, say, in which all proper subgroups have cardinality strictly smaller than $\aleph$ (the so-called *Jónsson groups*). Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by Shelah [15] and Obraztsov [9]. It is known that if G is any Jónsson group of cardinality $\aleph$, then G is perfect and $G/Z(G)$ is a simple group of cardinality $\aleph$ (see, for instance, [6, Corollary 2.6]). Therefore, some suitable weak solubility assumption is enough in order to avoid Jónsson groups and other similar obstructions.

It turns out that an uncountable soluble group in which all proper normal subgroups have the T-property need not be a T-group. However, our first main result shows in particular that a group of this type has finite conjugacy classes of subnormal subgroups. The structure of soluble groups with this latter property was investigated by Casolo [1].

THEOREM 1.1. *Let G be an uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. Then every subnormal subgroup of G has only finitely many conjugates.*

Recall that a group is *subsoluble* if it has an ascending series with abelian factors consisting of subnormal subgroups; in particular, all hyperabelian groups are subsoluble. Our second main result deals with uncountable subsoluble groups in which every proper large subgroup is a T-group, and proves that these groups have the $\overline{T}$-property.

THEOREM 1.2. *Let G be an uncountable subsoluble group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the T-property. Then G is a $\overline{T}$-group.*

We mention finally that in Section 3 it will also be proved that an infinite locally graded group whose proper subgroups have the T-property is likewise a T-group.

Most of our notation is standard and can be found in [12].

2. The T-property for large normal subgroups

It seems to be unclear whether a subsoluble (or even hyperabelian) uncountable group of cardinality $\aleph$ must contain at least one proper normal subgroup of cardinality $\aleph$. However, this property obviously holds in the case of abelian groups and hence also for uncountable groups which properly contain their commutator subgroup.

Lemma 2.1. *Let G be an uncountable group of cardinality $\aleph$ such that $G' \neq G$. Then G contains a proper normal subgroup of cardinality $\aleph$.*

The imposition of the T-property to large proper normal subgroups of uncountable groups has a strong effect, at least when the commutator subgroup is not large. In fact, it turns out in particular that if G is an uncountable group whose proper normal uncountable subgroups have the T-property, then G itself is a T-group, provided that its commutator subgroup G' is countable.

Lemma 2.2. *Let G be an uncountable group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. If G/G' has cardinality $\aleph$, then G is a T-group.*

Proof. Let X be a subnormal subgroup of G such that XG'/G' has cardinality strictly smaller than $\aleph$ and let g be any element of G. Then the abelian group $G/\langle g, X, G'\rangle$ has cardinality $\aleph$ and so it contains a proper subgroup $H/\langle g, X, G'\rangle$ of cardinality $\aleph$. Then H is a normal subgroup of cardinality $\aleph$, so that it is a T-group and X is normal in H. It follows that $X^g = X$ and hence X is a normal subgroup of G.

Suppose now that X is any subnormal subgroup of G. If x is any element of X, the subgroup $\langle x, X'\rangle$ is subnormal in G and $\langle x, X'\rangle G'/G' = \langle x\rangle G'/G'$ is countable, so that $\langle x, X'\rangle$ is normal in G by the first part of the proof. Therefore also

$$X = \langle \langle x, X'\rangle \mid x \in X\rangle$$

is a normal subgroup of G and hence G is a T-group. □

Corollary 2.3. *Let G be an uncountable group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. If G' has cardinality strictly smaller than $\aleph$, then G is a T-group.*

Notice that in the latter two statements the assumption that the cardinal number $\aleph$ is uncountable cannot be omitted. In fact, in the direct product $G = \mathrm{Alt}(4) \times P$, where $\mathrm{Alt}(4)$ is the alternating group of degree 4 and P is a group of type p^∞ for some prime number $p > 3$, all infinite proper normal subgroups have the T-property but G is not a T-group, although its commutator subgroup is finite.

As we mentioned in the introduction, a soluble group in which all proper normal subgroups have the T-property need not be a T-group. To see this, let M be an abelian group of exponent 7 and let α be the automorphism of M defined by putting $a^\alpha = a^2$ for all $a \in M$. Then $\alpha^3 = 1$ and we may consider a homomorphism

$$\theta : \mathrm{Alt}(4) \longrightarrow \mathrm{Aut}(M)$$

such that $\mathrm{Im}\,\theta = \langle\alpha\rangle$. The semidirect product

$$G = \mathrm{Alt}(4) \ltimes_\theta M$$

is a periodic metabelian group whose proper normal subgroups are abelian, but clearly G is not a T-group. If the group M is chosen to be uncountable, this example also

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shows that under the assumptions of Theorem 1.1, the T-property may not hold for the group G. The above construction can be slightly modified in order to produce a soluble group of derived length 3 whose proper normal subgroups have the T-property; it is enough to replace M by the direct product $K = Q_8 \times M$, where Q_8 is a quaternion group of order 8, and extend α to K in such a way that it acts on Q_8 as an automorphism of order 3.

It is clear that if all proper normal subgroups of a group G have the T-property, then this property also holds for all proper subnormal subgroups of G. Our next lemma shows that this situation also occurs whenever G is an uncountable soluble group whose proper normal subgroups of high cardinality have the T-property.

Lemma 2.4. *Let G be an uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. Then all proper subnormal subgroups of G have the T-property.*

Proof. Let X be any proper subnormal subgroup of G. If the normal closure X^G of X has cardinality strictly smaller than $\aleph$, the factor group G/X^G has cardinality $\aleph$ and so by Lemma 2.1 it contains a proper normal subgroup of cardinality $\aleph$. Thus, in any case, X is contained in a proper normal subgroup H of G of cardinality $\aleph$. As H is a T-group and X is subnormal, it follows that also X has the T-property. □

Corollary 2.5. *Let G be an uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. Then G has derived length at most* 3.

Lemma 2.6. *Let G be a group whose proper normal subgroups have the T-property and let X be a subgroup of G such that $[X, G'] \leq X$. Then X has only finitely many conjugates in G.*

Proof. As the normaliser $N_G(X)$ contains G', it is normal in G and so the subgroup X is subnormal in G. Of course, it can be assumed that X is not normal in G. As all proper subgroups of G containing $N_G(X)$ have the T-property and so normalise X, it follows that $N_G(X)$ is a maximal subgroup of G. Therefore, the index $|G : N_G(X)|$ is finite and hence X has finitely many conjugates in G. □

Note that the proof of Lemma 2.6 actually shows that under the same assumptions the subgroup X either is normal or has a prime number of conjugates.

If G is any uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ are T-groups, it follows from Lemma 2.4 that each proper subnormal subgroup of G has the T-property and so Theorem 1.1 is a special case of our next result.

Theorem 2.7. *Let G be a soluble group whose proper normal subgroups have the T-property. Then every subnormal subgroup of G has only finitely many conjugates.*

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Proof. Assume for a contradiction that the statement is false and let X be a subnormal subgroup of G admitting infinitely many conjugates. As the commutator subgroup G' of G is a T-group, it follows from Lemma 2.6 that each subnormal subgroup of G' has finitely many conjugates in G. In particular, the normaliser $N_G(X \cap G')$ has finite index in G, so that X has infinitely many conjugates in $N_G(X \cap G')$. Moreover, $N_G(X \cap G')$ is normal in G, because it contains G', and so all its proper normal subgroups have the T-property. Therefore, the factor group $N_G(X \cap G')/X \cap G'$ is also a counterexample to the statement and hence it can be assumed without loss of generality that $X \cap G' = \{1\}$.

Another application of Lemma 2.6 yields that X is not normalised by G', so that XG' is not a T-group and hence $XG' = G$. It follows from Dedekind's modular law that the subnormal subgroup XG'' is properly contained in G, so that it has the T-property and X is normalised by G''. Then

$$G'' \leq N_G(X) \cap G' < G'$$

and so $N_G(X) \cap G'$ is normal in $G = XG'$. Let K be any normal subgroup of G such that

$$N_G(X) \cap G' \leq K < G'.$$

Thus, XK is a proper subnormal subgroup of G, so that it has the T-property and X is normalised by K and hence $K = N_G(X) \cap G'$. Therefore, $G'/N_G(X) \cap G'$ is a chief factor of G. Let g be an element of G' such that $X^g \neq X$, so that

$$G' = \langle g, N_G(X) \cap G' \rangle^G.$$

On the other hand, $\langle g, N_G(X) \cap G' \rangle$ is a normal subgroup of G', so that it has finitely many conjugates in G. It follows that the chief factor $G'/N_G(X) \cap G'$ is finitely generated and hence even finite. Therefore,

$$|G : N_G(X)| = |G' : G' \cap N_G(X)|$$

is finite and this contradiction completes the proof of the theorem. □

As soluble groups with finite conjugacy classes of subnormal subgroups are metabelian-by-finite (see [1]), the following statement is a consequence of Theorem 2.7; it applies in particular to the case of an uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property.

Corollary 2.8. *Let G be a soluble group whose proper normal subgroups have the T-property. Then G has derived length at most 3 and contains a metabelian subgroup of finite index.*

3. The T-property for large subgroups

Let $\mathfrak{X}$ be a class of groups. A group G is said to be *minimal non-$\mathfrak{X}$* if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. The structure of minimal non-$\mathfrak{X}$ groups has been studied for several different choices of the group class $\mathfrak{X}$. As the T-property is local, it is clear that any minimal non-T group is countable. Moreover,

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finite minimal non-T groups are soluble, since it is well known that any finite group with only supersoluble proper subgroups is soluble. It seems to be an open question whether there exist infinite minimal non-T groups. On the other hand, it was proved that a locally finite group whose proper subgroups have the T-property is either finite or a T-group (see [11]) and a similar result holds also in the case of groups with no infinite simple sections [2].

Recall that a group G is said to be *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index. Thus, locally graded groups form a large class of generalised soluble groups, containing in particular all locally (soluble-by-finite) and all residually finite groups. It is also clear that every group with no infinite simple sections is locally graded. Therefore, the following theorem extends both the above-quoted results about minimal non-T groups.

Theorem 3.1. *Let G be an infinite locally graded group whose proper subgroups have the T-property. Then G is a T-group.*

Proof. Assume for a contradiction that G is not a T-group, so that G is minimal non-T and hence it is finitely generated, because the T-property is local. If J is the finite residual of G, it follows that the group G/J is infinite.

Let N be any normal subgroup of finite index of G. Then the factor group G/N either is minimal non-T or has the $\overline{T}$-property, so that in any case it is soluble and has derived length at most 3. Therefore, also the infinite group G/J is soluble, so that it cannot be minimal non-T (see [2]) and hence it has the $\overline{T}$-property. Moreover, G/J cannot be periodic and so it is a finitely generated abelian group. In particular, there exist two maximal subgroups M_1 and M_2 of G both containing J. As M_1 and M_2 have the T-property, it follows that every subnormal subgroup of J is normal in G.

Let X be a subnormal nonnormal subgroup of G. Clearly, G can be generated by elements having infinite order with respect to J and so there exists an element g of G such that $X^g \neq X$ and the coset gJ has infinite order. The intersection $X \cap J$ is a normal subgroup of G and the factor group $G/X \cap J$ is likewise a counterexample, so that without loss of generality we may suppose that $X \cap J = \{1\}$. Then X is abelian and hence it is contained in the Baer radical B of G. Moreover, the subgroup $\langle B, g\rangle$ is subsoluble, so that it is properly contained in G and hence it is a nonperiodic group with the $\overline{T}$-property. Therefore, $\langle B, g\rangle$ is abelian, which is impossible as $X^g \neq X$. This contradiction proves the statement. □

We turn now to the proof of Theorem 1.2. The first lemma shows in particular that any uncountable abelian group of cardinality $\aleph$ has a residual system consisting of normal subgroups with indices strictly smaller than $\aleph$.

Lemma 3.2. *Let A be an uncountable abelian group of cardinality $\aleph$ and let B be a subgroup of A of cardinality strictly smaller than $\aleph$. Then A contains a subgroup C such that $B \cap C = \{1\}$ and $|A : C| < \aleph$.*

Proof. Let A^* be the divisible hull of A. Then A^* has cardinality $\aleph$ and so

$$A^* = \underset{i \in I}{\mathrm{Dr}}\, D_i^*,$$

where each D_i^* is isomorphic either to the additive group of rational numbers or to a Prüfer group and the index set I has cardinality $\aleph$. Clearly, I has a subset I' of cardinality strictly smaller than $\aleph$ such that

$$B^* = \underset{i \in I'}{\mathrm{Dr}}\, D_i^*$$

contains B. If

$$C^* = \underset{i \in I \setminus I'}{\mathrm{Dr}}\, D_i^*,$$

then $C = A \cap C^*$ is a subgroup of A such that $B \cap C = \{1\}$ and

$$|A : C| \leq |A^* : C^*| < \aleph.$$

The statement is proved. □

Lemma 3.3. *Let G be a group and let A be an uncountable abelian normal subgroup of G of cardinality $\aleph$. If each subgroup of A has only finitely many conjugates in G, then there exists a collection $(a_\alpha)_{\alpha \in \aleph}$ of nontrivial elements of A such that*

$$\langle a_\alpha \mid \alpha \in \aleph \rangle^G = \underset{\alpha \in \aleph}{\mathrm{Dr}} \langle a_\alpha \rangle^G.$$

Proof. Choose any nontrivial element a_0 of A and suppose that λ is an element of $\aleph$ for which a_α has been defined for all $\alpha < \lambda$ in such a way that

$$B = \langle a_\alpha \mid \alpha < \lambda \rangle^G = \underset{\alpha < \lambda}{\mathrm{Dr}} \langle a_\alpha \rangle^G.$$

Since every $\langle a_\alpha \rangle^G$ is countable, the subgroup B has cardinality strictly smaller than $\aleph$ and so, by Lemma 3.2, A contains a subgroup C such that $B \cap C = \{1\}$ and $|A : C| < \aleph$. But C has only finitely many conjugates in G and hence we have also $|A : W| < \aleph$, where W is the core of C in G. In particular, W cannot be trivial and so we can choose an element a_λ in W. The normal closure $\langle a_\lambda \rangle^G$ is contained in W, so that $B \cap \langle a_\lambda \rangle^G = \{1\}$ and hence

$$\langle a_\alpha \mid \alpha \leq \lambda \rangle = B \times \langle a_\lambda \rangle^G = \underset{\alpha \leq \lambda}{\mathrm{Dr}} \langle a_\alpha \rangle^G.$$

The proof of the statement can now be completed by transfinite induction on λ. □

We will also need the following property of abelian groups, for a proof of which we refer to [7, Lemma 2.2].

Lemma 3.4. *Let G be an abelian group which is not finitely generated. Then G has a countable homomorphic image which is not finitely generated.*

Uncountable groups in which all proper large subgroups are soluble with bounded derived length have been studied in [7]. In particular, the case $k = 2$ applies in the situation of Theorem 1.2.

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Lemma 3.5. *Let k be a positive integer and let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are soluble with derived length at most k. If G has no simple nonabelian homomorphic images, then it is soluble with derived length at most k.*

Corollary 3.6. *Let G be a subsoluble uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the T-property. Then G is metabelian.*

Proof. It is well known that any subsoluble T-group is metabelian, so that all proper subgroups of G of cardinality $\aleph$ are metabelian. Therefore, the group G is metabelian by Lemma 3.5. □

The next lemma is the main step in the proof of Theorem 1.2.

Lemma 3.7. *Let G be an uncountable subsoluble group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the T-property. Then G is a T-group.*

Proof. The group G is metabelian by Corollary 3.6, so that in particular each subgroup of G' has only finitely many conjugates in G by Lemma 2.6. Moreover, by Corollary 2.3, it can be assumed without loss of generality that the commutator subgroup G' of G has cardinality $\aleph$, so that it follows from Lemma 3.3 that G' contains a G-invariant subgroup of the form

$$U = \operatorname*{Dr}_{i \in I} U_i,$$

where each U_i is a countable nontrivial normal subgroup of G and the set I has cardinality $\aleph$.

Let X be any subnormal subgroup of G and suppose first that X has cardinality strictly smaller than $\aleph$. The above decomposition of U shows the existence of two G-invariant subgroups V and W of U of cardinality $\aleph$ such that

$$V \cap W = \langle V, W \rangle \cap X = \{1\}.$$

All proper subgroups of the factor groups G/V and G/W have the T-property, so that G/V and G/W are T-groups by Theorem 3.1. It follows that the subnormal subgroups XV and XW of G are normal, so that also $X = XV \cap XW$ is normal in G. In particular, all cyclic subnormal subgroups of G are normal and hence every subgroup of G' is normal in G.

Suppose now that the subnormal subgroup X has cardinality $\aleph$. If the commutator subgroup X' has cardinality strictly smaller than $\aleph$, the above argument shows that the subgroup $\langle x, X' \rangle$ is normal in G for each element x of X, so that also

$$X = \langle \langle x, X' \rangle \mid x \in X \rangle$$

is normal in G. Suppose finally that the abelian subgroup X' has cardinality $\aleph$, so that by Lemma 3.4 it contains a proper subgroup Y such that the index $|X' : Y|$ is countably infinite. Then Y is a normal subgroup of G of cardinality $\aleph$, so that all proper subgroups of the infinite group G/Y have the T-property and hence G/Y itself is a T-group. Therefore, X is normal in G and G is a T-group. □

Proof of Theorem 1.2. The group G has the T-property by Lemma 3.7 and in particular it is metabelian. Assume for a contradiction that G is not a $\overline{T}$-group, so that it contains a subgroup H of cardinality strictly smaller than $\aleph$ which has a subnormal nonnormal subgroup K. If G contains an abelian normal subgroup A of cardinality $\aleph$, an application of Lemma 3.3 shows that A contains two G-invariant subgroups V and W of cardinality $\aleph$ such that

$$V \cap W = \langle V, W\rangle \cap K = \{1\}.$$

Then the subgroups HV and HW have the T-property, so that the subgroups KV and KW are normal in HV and HW, respectively, and hence

$$K = KV \cap KW$$

is normal in H. This contradiction shows that G has no abelian normal subgroups of cardinality $\aleph$. In particular, the Fitting subgroup F of G has cardinality strictly smaller than $\aleph$, so that the index $|G : F|$ is infinite. It follows that F is periodic (see [10]) and hence also G' is a periodic subgroup of cardinality strictly smaller than $\aleph$. Of course, the intersection $K \cap G'$ is normal in G and the replacement of G by $G/K \cap G'$ allows the assumption that $K \cap G' = \{1\}$.

Suppose first that G is not periodic, so that G' is divisible and $[G', G] = G'$. Write

$$G' = \mathop{\mathrm{Dr}}_{i\in I} P_i,$$

where each P_i is a group of type p_i^∞ for some prime number p_i, and put

$$P_i^* = \mathop{\mathrm{Dr}}_{j\neq i} P_j$$

for each index i. Then

$$K = \bigcap_{i\in I} KP_i^*$$

and so there is an index j such that KP_j^* is not normalised by H. It follows that the factor group G/P_j^* is also a counterexample and hence it can be assumed without loss of generality that G' is a group of type p^∞ for some prime p. As $[G', G] = G'$, the cohomology class of the extension

$$G' \rightarrowtail G \twoheadrightarrow G/G'$$

has finite order and hence G nearly splits over G' (see [13] and [14]). This means that there exists a subgroup L of G such that $G = LG'$ and $L \cap G'$ is finite. Clearly, L contains an element a of infinite order and $[G', a] \neq \{1\}$, because $C_G(G') = F$ is periodic. Let x be an element of G' such that $L \cap G' \leq \langle x\rangle$ and $[x, a] \neq 1$. As $\langle x, L\rangle$ is a subgroup of G of cardinality $\aleph$, it has the T-property and hence also its normal subgroup $\langle x, a\rangle$ is a T-group. But finitely generated soluble T-groups are abelian, so that $xa = ax$ and this contradiction proves that the counterexample G must be periodic.

For each prime number p, let $G'(p')$ be the subgroup consisting of all elements of G' whose order is prime to p. As

$$K = \bigcap_p KG'(p'),$$

there exists a prime q such that H does not normalise the subgroup $KG'(q')$ and hence a further replacement of G by $G/G'(q')$ allows the assumption that G' is a q-group. It follows that the factor group $G/C_G(G')$ is finite, because it is isomorphic to a group of power automorphisms of G' and all periodic groups of power automorphisms of an abelian q-group are finite. This last contradiction completes the proof of Theorem 1.2. □

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