A NOTE ON A SEQUENCE OF CONTRACTION MAPPINGS

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Let E be a metric space. A mapping T of the space E into itself is said to be a contraction if there exists a number k, with $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for any two points $x, y \in E$. Every contraction mapping is continuous.

The well-known Banach contraction principle is the following: if T is a contraction mapping of a complete metric space E into itself, then T has a unique fixed point. i.e. $Tx = x$ has a unique solution.

Contraction mappings on metric space have been of interest for many years. In the present note we study a theorem on a sequence of contraction mappings and fixed points.

The following result is proved in [1, page 6].

THEOREM 1. Let E be a complete metric space, and let T and $T_n (n = 1, 2, \ldots)$ be contraction mappings of E into itself with the same Lipschitz constant $k < 1$, and with fixed points u and u_n respectively. Suppose that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in E$. Then $\lim_{n\to\infty} u_n = u$.

Definition. Let (E, d) be a metric space and $\varepsilon > 0$. A finite sequence $x_0, x_1, \ldots, x_n$ of points of E is called an ε-chain joining x and x_n if

$$d(x_{i-1}, x_i) < \varepsilon, \qquad (i = 1, 2, \ldots, n).$$

The metric space (E, d) is said to be ε-chainable if, for each pair (x, y) of its points, there exists an ε-chain joining x and y.

The well-known result due to Edelstein [2, page 76] is the following.

THEOREM 2. Let T be a mapping of a complete ε-chainable metric space (E, d) into itself, and suppose that there is a real number k with $0 \leq k < 1$ such that

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$$d(x, y) < \varepsilon \implies d(Tx, Ty) \le kd(x, y).$$

Then T *has a unique fixed point* u *in* E, *and* $u = \lim_{n\to\infty} T^n x_0$ *where* x_0 *is an arbitrary element of* E.

In the above theorem Edelstein has taken an ε-chainable metric space and has considered contraction mappings.

We now construct and prove a theorem by considering a sequence of such mappings.

THEOREM 3. *Let* (E, d) *be a complete* ε-*chainable metric space, and let* T_n $(n = 1, 2, \ldots)$ *be mappings of* E *into itself, and suppose that there is a real number* k *with* $0 \le k < 1$ *such that*

$$d(x, y) < \varepsilon \implies d(T_n x, T_n y) \le kd(x, y) \text{ for all } n.$$

If u_n $(n = 1, 2, \ldots)$ *are the fixed points for* T_n *and* $\lim_{n\to\infty} T_n x = Tx$ *respectively for every* $x \in E$, *then* T *has a unique fixed point* u *and* $\lim_{n\to\infty} u_n = u$.

Proof. (E, d) being ε-chainable we define for $x, y \in E$,

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{p} d(x_{i-1}, x_i)$$

where the infimum is taken over all ε-chains $x_0, x_1, \ldots, x_p$ joining $x_0 = x$ and $x_p = y$. Then d_ε is a distance function on E satisfying

(i) $d(x, y) \le d_\varepsilon(x, y)$

(ii) $d(x, y) = d_\varepsilon(x, y)$ for $d(x, y) < \varepsilon$.

From (ii) it follows that a sequence $\{x_n\}$, $x_n \in E$ is a Cauchy sequence with respect to d_ε if and only if it is a Cauchy sequence with respect to d and is convergent with respect to d_ε if and only if it converges with respect to d. Hence, (E, d) being complete, (E, d_ε) is also a complete metric space. Moreover T_n $(n = 1, 2, \ldots)$ are contraction mappings with respect to d_ε. Given $(x, y) \in E$, and any ε-chain $x_0, x_1, \ldots, x_p$ with $x_0 = x$, $x_p = y$, we have $d(x_{i-1}, x_i) < \varepsilon$ $(i = 1, 2, \ldots, p)$, so that $d(T_n x_{i-1}, T_n x_i) \le k d(x_{i-1}, x_i) < \varepsilon$ $(i = 1, 2, \ldots, p)$. Hence $T_n x_0, \ldots, T_n x_p$ is an ε-chain joining $T_n x$ and $T_n y$ and

$$d_\varepsilon(T_n x, T_n y) \le \sum_{i=1}^{p} d(T_n x_{i-1}, T_n x_i) \le k \sum_{i=1}^{p} d(x_{i-1}, x_i) .$$

$x_0, x_1, \ldots, x_p$ being an arbitrary ε-chain, we have

$$d_\varepsilon(T_n x, T_n y) \le k d_\varepsilon(x, y) .$$

Now since T_n $(n = 1, 2, \ldots)$ are contraction mappings with respect to d_ε and (E, d_ε) is a complete metric space, then $Tx = \lim_{n\to\infty} T_n x$ is a contraction mapping with respect to d_ε. Moreover T has a unique fixed point u and $\lim_{n\to\infty} u_n = u$ by Theorem 1.

This unique fixed point is given by

$$\lim_{n\to\infty} d_\varepsilon(T^m x_0, u) = 0 \text{ for } x_0 \in E \text{ arbitrary}.$$

But (i) at the beginning of this proof implies

$$\lim_{n\to\infty} d(T^m x_0, u) = 0 .$$

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