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Dynamics of the geodesic flow of a foliation

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Abstract. The geodesic flow of a foliated Riemannian manifold (M, F) is studied. The invariance of some smooth measure is established under some geometrical conditions on F. The Lyapunov exponents and the entropy of this flow are estimated. As an application, the non-existence of foliations with 'short' second fundamental tensors is obtained on compact negatively curved manifolds.

1. *Introduction*

The geodesic flows of leaves of a foliated Riemannian manifold (M, F) glue together with a flow on the unit tangent bundle of F. This is the geodesic flow of F, the main object of our interest here.

We have at least two reasons for studying this flow.

First, the geodesic flow of a Riemannian manifold M has been intensively studied during the last few years (see [**A, BBE, BBS, E, FM, Ma1, Ma2, K, OS**] and many others) and it has been established that the dynamical properties of this flow reflect the geometry of M. So, we may expect that the geodesic flow of a foliation could become a good tool in the study of the geometry of F.

Second, the last few years have seen some attempts concerning the dynamical theory of foliations. The idea is to adapt some notions and methods of the theory of dynamical systems to the theory of foliations. In this direction, we have, for example, studies of invariant measures and some ergodic properties of foliations [**Ga, Pl**] or an attempt [**GLW**] made at defining and studying the entropy of a foliation.

Also, let us note that the dynamics of the geodesic flow was used successfully [**Z**] to study totally geodesic foliations of locally symmetric spaces.

In this paper, after giving some preliminary information, we begin with varying geodesics on leaves among such geodesics. This leads to a suitable notion of Jacobi fields (§ 2). While studying natural Riemannian metrics on the unit tangent bundle of a foliation we ask when the Riemannian volume is preserved by our flow (§ 3). Jacobi fields are used to estimate Lyapunov exponents (§ 4) and the entropy (§ 5) of this flow. Using these estimations in § 6, we prove that there are no totally geodesic foliations on compact manifolds of negative sectional curvature. Moreover, for a manifold M like that, one can find a positive constant η such that either the norm of the second fundamental tensor of any foliation on M or the norm of its covariant derivative has to exceed η somewhere. We end with some questions and remarks (§ 7).

Finally, we would like to say that we began this work during a stay at the University of Sao Paulo where Question B of § 7 was posed and discussed with Fabiano Brito, Remi Langevin and Waldyr Oliva. Also, Problem D of § 7 had already been discussed with Etienne Ghys and Remi Langevin.

1. *Preliminaries*

Throughout the paper, M is an oriented C^∞-manifold with a C^3-Riemannian structure $g=\langle\cdot,\cdot\rangle$. F is an oriented C^3-foliation of M, $p=\dim F$, $n=\dim M$, $0<p<n$. For any x in M, L_x denotes the leaf of F which passes through x. TF and SF denote the tangent bundle and the unit tangent bundle of F, respectively.

We always assume that the leaves of F are complete with respect to the Riemannian structure induced from M. For any v of T_xF, $c_v : R \to L_x$ is the maximal geodesic satisfying $c_v(0)=x$ and $\dot c_v(0)=v$. The maps φ_t, $t\in R$, of TF into itself are defined by $\varphi_t(v)=c_v(t)$. They form a flow $\varphi=(\varphi_t)$ which is called the *geodesic flow* of F. Since $|\varphi_t v|=|v|$, φ preserves the bundle SF and induces the geodesic flow on SF which is also denoted by φ.

The Leví-Civita connection on M, its curvature tensor and the sectional curvature of M are denoted here by ∇, R and K, respectively. ∇ induces a connection ∇^F in the bundle TF. We have

$$\nabla^F_X Y=(\nabla_X Y)^\mathrm{T},$$

where X is a vector field on M, Y is a section of TF and

$$v=v^\mathrm{T}+v^\perp$$

is the decomposition of a vector $v\in TM$ into the components tangential to and orthogonal to F.

The second fundamental form of F can be considered either as a section A of the bundle $\mathrm{Hom}\,(T^\perp F\otimes TF, TF)$ or as a section B of $\mathrm{Hom}\,(TF\otimes TF, TM)$, where $T^\perp F$ is the orthogonal complement of TF in TM. We have

$$A(N,X)=-(\nabla_X N)^\mathrm{T}$$

and

$$B(X,Y)=(\nabla_X Y)^\perp.$$

for a section N of $T^\perp F$ and sections X and Y of TF. Similarly, the second fundamental tensor $B^\perp$ of $T^\perp F$ is given by

$$B^\perp(X,Y)=\tfrac{1}{2}(\nabla_X Y+\nabla_Y X)^\mathrm{T}$$

for sections X and Y of $T^\perp F$.

The connections ∇ and ∇^F induce a connection $\tilde\nabla$ in the bundle $\mathrm{Hom}\,(TF\otimes TF, TM)$. We have

$$(\tilde\nabla_Z B)(X,Y)=\nabla_Z B(X,Y) - B(\nabla^F_Z X, Y)-B(X,\nabla^F_Z Y). \tag{1}$$

Note that $\tilde\nabla$ differs from the connection $\bar\nabla$ which appears in the Codazzi equations [KN]. $\bar\nabla$ is the connection in the bundle $\mathrm{Hom}\,(TF\otimes TF, T^\perp F)$ induced by ∇^F and

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$\nabla^\perp$, the natural connection in $T^\perp F$. We have

$$(\tilde{\nabla}_Z B)(X, Y) = (\bar{\nabla}_Z B)(X, Y) + (\nabla_Z B(X, Y))^{\mathrm{T}}. \tag{2}$$

Now, given v in T_xF we can consider the linear transformation

$$T_xF \ni w \mapsto B(v, w).$$

Its norm is denoted here by $|B(v)|$. We have

$$|B(v)| = \sup \{|B(v, w)|; |w| = 1\}.$$

Similarly, $|\tilde{\nabla} B(v, w)|$ $(v, w \in T_xF)$ denotes the norm of the linear map

$$T_xF \ni u \mapsto (\tilde{\nabla}_u B)(v, w).$$

Also, if $v \in S_xF$, then we denote by $K(v)$ (respectively, by $|K(v)|$) the maximum of sectional curvatures of M (respectively, maximum of their absolute values) over the set of all planes $\sigma \subset T_xM$ containing v.

Finally, we equip TF (and SF as well) with two Riemannian metrics which arise in a natural way. The first one, $\tilde{g} = \langle \cdot, \cdot \rangle^{\sim}$, is defined as the metric induced from TM. Recall that TM carries the metric (denoted here also by $\tilde{g}$) given by

$$\tilde{g}(\xi, \eta) = \langle \pi_* \xi, \pi_* \eta \rangle + \langle C\xi, C\eta \rangle,$$

where $\xi, \eta \in T_vTM$, $\pi : TM \to M$ is the canonical projection and $C : TTM \to TM$ is the connection map of ∇. Recall also (see **[GKM]**) that C maps linearly T_vTM onto T_xM $(x = \pi(v))$, coincides with the canonical identification $T_v(T_xM) \approx T_xM$ on the space of vertical vectors and satisfies

$$C(X_* u) = \nabla_u X$$

for any u of TM and any vector field X on M. On the other hand, we can define the connection map C^F of ∇^F. C^F maps T_vTF linearly onto T_xF where $v \in T_xF$ coincides with the canonical identification $T_v(T_xF) \approx T_xF$ on the space of vertical vectors and satisfies

$$C^F(X_* u) = \nabla^F_u X$$

for any u of TM and any section X of TF. With this map we may put

$$\tilde{g}_F(\xi, \eta) = \langle \pi_* \xi, \pi_* \eta \rangle + \langle C^F \xi, C^F \eta \rangle$$

for all ξ and η of T_vTF $(v \in TF)$. $g_F = \langle \cdot, \cdot \rangle_{\tilde{F}}$ is also a Riemannian metric on TF. The metrics $\tilde{g}$ and $\tilde{g}_F$ coincide on TF when the foliation F is totally geodesic (i.e., $B = 0$). If M is compact, the metrics induced by $\tilde{g}$ and $\tilde{g}_F$ on SF are quasi-isometric, so from the dynamical point of view they are equivalent. In this paper, we shall sometimes use one of them, sometimes the other.

Let us note that the vector field V generated on TF by the flow φ has the following properties:

(i) $\pi_* \circ V = \mathrm{id}_{TF}$,

(ii) $C^F \circ V = 0$,

(iii) $V \circ \mu_r = r \cdot \mu_{r*} \circ V$, $r \in \mathbb{R}$, where $\mu_r : TF \to TF$ is multiplication by r.

These properties of V follow immediately from the analogous properties (see **[GKM]**) of geodesic flows of Riemannian manifolds because our flow φ restricted to the tangent bundle TL of a leaf L coincides with the geodesic flow of L.

2. *Jacobi fields*

Let us consider a curve $v:(-\varepsilon,\varepsilon)\to TF$ and the mapping $f:\mathbb{R}\times(-\varepsilon,\varepsilon)\to M$ given by $f(t,s)=\exp^F tv(s)$, where $\exp^F$ is the exponential map of $\nabla^F:\exp^F(u)=c_u(1)$ for any u of TF. Put

$$X=f_*(d/dt),\quad Y=f_*(d/ds)$$

and $Z(t)=Y(t,0)$ for $t\in\mathbb{R}$. Z is a vector field along the geodesic $c=c_{v(0)}$.

Since the vector fields d/dt and d/ds commute, and the torsion of ∇ vanishes, we have

$$R(Y,X)X=\nabla_{d/ds}\nabla_{d/dt}X-\nabla_{d/dt}\nabla_{d/ds}X$$

and

$$\nabla_{d/ds}X=\nabla_{d/dt}Y.$$

Also,

$$\nabla_{d/dt}X=\nabla^F_{d/dt}X+B(X,X)=B(X,X),$$

because the trajectories of X are geodesics on leaves of F. These equalities together with (1) yield the equality

$$R(Y,X)X=(\tilde{\nabla}_{d/ds}B)(X,X)+2B(\nabla^F_{d/dt}Y,X)-\nabla_{d/dt}\nabla_{d/dt}Y,$$

which assumes the shape of

$$Z''-2B(Z'^{\mathrm{T}},\dot c)-(\tilde{\nabla}_Z B)(\dot c,\dot c)-R(\dot c,Z)\dot c=0 \tag{3}$$

along c. Here $Z'=\nabla_{d/dt}Z$ and $Z''=(Z')'$.

Now, put $\xi=\dot v(0)$. Then

$$Z(0)=(\pi\circ v)(0)=\pi_*\xi$$

and

$$\begin{aligned}Z'(0)&=\nabla_{d/dt}Y(0,0)=\nabla_{d/ds}X(0,0)\\&=\nabla_{d/ds}v(0)=C(v_*(d/ds))=C\xi.\end{aligned}$$

From the theory of ordinary differential equations, it follows that for any ξ of TTM there exists a unique solution Z_ξ of (3) satisfying $Z_\xi(0)=\pi_*\xi$ and $Z'_\xi(0)=C\xi$. However, only those with $\xi\in TTF$ correspond to variations of geodesic considered above. We call them *Jacobi fields* (for F) here. It is clear that Jacobi fields form a vector space of dimension $n+p$.

Remark 1. If Z is a Jacobi field for F tangential everywhere to F, then Z is a Jacobi field on a leaf of F. This can be seen easily from the construction of the Jacobi fields described above. Also, using equality (2) and the Codazzi and Gauss equations one can see that if W and Z are vector fields along a geodesic c and W is tangential to F, then

$$\begin{aligned}&\langle Z''-2B(Z'^{\mathrm{T}},\dot c)-(\tilde{\nabla}_Z B)(\dot c,\dot c)-R(\dot c,Z)\dot c,\ W\rangle\\&\quad=\langle\nabla^F_{\dot c}\nabla^F_{\dot c}Z-R^F(\dot c,z^{\mathrm{T}})\dot c,\ W\rangle\\&\qquad-2\langle(\bar{\nabla}_{\dot c}B)(W,\dot c),Z^\perp\rangle+\langle(\bar{\nabla}_W B)(\dot c,\dot c),Z^\perp\rangle\\&\qquad+\langle B(\dot c,\dot c),\nabla_{Z^\perp}W\rangle-2\langle B(\dot c,W),\nabla^\perp_{\dot c}Z^\perp\rangle.\end{aligned}$$

This equality shows that: (1) if $Z^{\perp}=0$ and Z is a Jacobi field for F, then it is a Jacobi field on a leaf, (2) if F is totally geodesic, then the tangent component Z^{T} of any Jacobi field for F is a Jacobi field on a leaf.

LEMMA 1. *If Z is a Jacobi field (for F) along a geodesic c, then* $\langle Z', \dot c\rangle=$ constant.

Proof. With the notation used in the beginning of this section we have

$$\begin{aligned}\langle(\tilde{\nabla}_Y B)(X,X),X\rangle&+\langle\nabla_{d/dt}Y, B(X,X)\rangle\\ &=\langle\nabla_{d/ds}B(X,X),X\rangle-2\langle B(\nabla^F_{d/ds}X,X),X\rangle+\langle\nabla_{d/dt}Y,B(X,X)\rangle\\ &=\langle B(X,X),\nabla_{d/dt}Y-\nabla_{d/ds}X\rangle=0.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\langle Z',\dot c\rangle&=\langle Z'',\dot c\rangle+\langle Z', B(\dot c,\dot c)\rangle\\ &=\langle(\tilde{\nabla}_Z B)(\dot c,\dot c),\dot c\rangle+\langle Z',B(\dot c,\dot c)\rangle=0.\end{aligned}$$ □

Remark 2. Since $(d/dt)\langle Z,\dot c\rangle=\langle Z',\dot c\rangle+\langle B(\dot c,\dot c),Z\rangle$ we cannot claim (as for Jacobi fields on Riemannian manifolds) that Z is constantly orthogonal to c if $Z(0)$ and $Z'(0)$ are. This makes our calculation slightly different from that for the geodesic flow of a Riemannian manifold.

LEMMA 2. *For any ξ of TTF and any t of $\mathbb{R}$ we have*

$$\pi_*\varphi_{t*}\xi=Z_\xi(t)\quad\text{and}\quad C\varphi_{t*}\xi=Z'_\xi(t). \tag{4}$$

Proof. If $\xi=\dot v(0)$ for a curve $v:(-\varepsilon,\varepsilon)\to TF$, then

$$\begin{aligned}\pi_*\varphi_{t*}\xi&=(d/ds)(\pi\circ\varphi_t\circ v)|_{s=0}\\ &=(d/ds)(\exp^F tv(s))|_{s=0}=Z_\xi(t)\end{aligned}$$

and

$$\begin{aligned}C\varphi_{t*}\xi&=C((s\mapsto\varphi_t(v(s)))^{\cdot}(0))\\ &=C((s\mapsto X(t,s))^{\cdot}(0))=(\nabla_{d/ds}X)(t,0)\\ &=(\nabla_{d/dt}Y)(t,0)=Z'_\xi(t).\end{aligned}$$ □

COROLLARY 1. $\langle\varphi_{t*}\xi,\varphi_{t*}\eta\rangle^{\sim}=\langle Z_\xi(t),Z_\eta(t)\rangle+\langle Z'_\xi(t),Z'_\eta(t)\rangle$ *for any* $\xi,\ \eta\in T_vTF$ *and* $v\in TF$. □

Lemma 2 and Corollary 1 show that to study the dynamical properties of the flow φ we should investigate the behaviour of Jacobi fields.

3. *An invariant measure*

The existence of invariant measures for a dynamical system is an old and still important question. For example, in **[Pl]** and **[GLW]** one can find conditions sufficient for a pseudo-group of local diffeomorphisms to admit a non-trivial invariant measure. In the case of a flow on a manifold, a reasonable question is to look for a smooth (i.e. absolutely continuous with respect to the Riemannian volume) invariant measure. The geodesic flow on a Riemannian manifold (M,g) always preserves the Riemannian volume of $(TM, \tilde{g})$. Zeghib [Z] proved that the geodesic flow of a totally geodesic foliation of a compact locally symmetric space always admits an invariant smooth measure. In this context we prove here the following.

THEOREM 1. *The geodesic flow φ of a foliation F preserves the Riemannian volume of $(SF, \tilde{g}_F)$ if and only if F is transversely minimal, i.e.* trace $B^{\perp}=0$.

Proof. Denote by Ω and ω, respectively, the volume forms on TF and SF with the Riemannian metric $\tilde{g}_F$. On SF we have

$$\Omega = \omega \wedge \theta,$$

where θ is the one-form on SF given by $\theta(\xi) = \langle W, \xi \rangle \tilde{g}_F$ with W being the vertical field on TF given by

$$W(u) = (t \mapsto tu)^{\cdot}(1) \quad (u \in TF).$$

Since the vector field W corresponds to the flow (μ_{e^t}) and

$$L_V\Omega = L_V\omega \wedge \theta + \omega \wedge L_V\theta,$$

so

$$[V, W] = \lim_{t \to 0} \frac{1}{t}(V - \mu_{e^t *} V \mu_{e^{-t}}) = V,$$

$L_V\theta = 0$ and consequently

$$L_V\Omega = 0 \quad \text{iff} \quad L_V\omega = 0.$$

Now, take a local orthonormal frame $X_1, \ldots, X_n$ of vector fields on M such that X_i is tangential to F for $i = 1, \ldots, p$. Denote by E_i $(i = 1, \ldots, p)$ the vertical lift of X_i and by E_{p+j} $(j = 1, \ldots, n)$ the horizontal lift of X_j. Then

$$\pi_* E_i = 0 \quad \text{and} \quad C_F E_i = X_i \quad (i = 1, \ldots, p)$$

and

$$\pi_* E_{p+j} = X_j \quad \text{and} \quad C_F E_{p+j} = 0 \quad (j = 1, \ldots, n).$$

The fields $E_1, \ldots, E_{n+p}$ form a local orthonormal frame on TF, so

$$\Omega(E_1, \ldots, E_{n+p}) = 1$$

and

$$L_V\Omega(E_1, \ldots, E_{n+p}) = -\sum_{i=1}^{2p} \Omega(E_1, \ldots, [V, E_i], \ldots, E_{n+p}) \\ - \sum_{j=2p+1}^{n+p} \Omega(E_1, \ldots, [V, E_j], \ldots, E_{n+p}). \qquad (5)$$

The first sum in (5) vanishes because it is equal to

$$L_{V|L}\Omega_L(E_1, \ldots, E_{2p}),$$

where Ω_L is the Riemannian volume form on the tangent bundle TL of a leaf L, and the geodesic flow on TL preserves Ω_L. The second sum in (5) is equal to

$$\sum_{j=p+1}^{n} \langle \pi_*[V, E_{n+j}], X_j \rangle. \qquad (6)$$

To calculate it we need the following.

LEMMA 3. *Let X be a vector field on M and X^h its horizontal lift to TF. Then for any v of T_xF we have*

$$\pi_*[V, X^{\mathrm{h}}](v) = [Z, X](x),$$

where Z is any vector field on M satisfying $Z(\psi_t x) = \tilde{\psi}_t v$ $(-\varepsilon < t < \varepsilon)$ for the (local) flows (ψ_t) of X and $(\tilde{\psi}_t)$ of X^{h}. Consequently,

$$\langle \pi_*[V, X^{\mathrm{h}}](v), X(x)\rangle = \langle B^\perp(X, X)(x), v\rangle \tag{7}$$

when X is orthogonal to F and $|X| = \text{constant}$.

Proof. Since $\pi_* \circ X^{\mathrm{h}} = X \circ \pi$, $\pi \circ \tilde{\psi}_t = \psi_t \circ \pi$ for any t. Therefore,

$$\pi_*([V, X^{\mathrm{h}}](v)) = \lim_{t\to 0} \frac{1}{t}(\pi_* V(v) - \pi_* \tilde{\psi}_{t*} V(\tilde{\psi}_{-t} v))$$

$$= \lim_{t\to 0} \frac{1}{t}(v - \psi_{t*}\pi_* V(\tilde{\psi}_{-t} v)) = \lim_{t\to 0} \frac{1}{t}(v - \psi_{t*}\tilde{\psi}_{-t} v)$$

$$= \lim_{t\to 0} \frac{1}{t}(Z(x) - \psi_{t*}Z(\psi_{-t} x)) = [Z, X](x).$$

Consequently,

$$\langle \pi_*[V, X^{\mathrm{h}}](v), X(x)\rangle = \langle [Z, X], X\rangle(x)$$

$$= \langle \nabla_Z X, X\rangle(x) - \langle \nabla_X Z, X\rangle(x) = \langle Z, \nabla_X X\rangle(x)$$

because X is orthogonal to F while Z is tangential to F along the trajectory of X passing through x. Moreover,

$$\langle Z, \nabla_X X\rangle(x) = \langle B^\perp(X, X)(x), v\rangle.$$

In order to complete the proof of Theorem 1 we should just apply formula (7) to the sum (6). □

Remark 3. The condition trace $B^\perp = 0$ is equivalent to the following:

(*) $L_X \omega^\perp = 0$ for any vector field X tangential to F, where $\omega^\perp$ is the volume form of $T^\perp F$ induced by g, i.e. $\omega^\perp$ is a q-form on M ($q = \operatorname{codim} M$) given by

$$\omega^\perp(v_1, \ldots, v_q) = \det[(v_i, e_j); i, j \le q],$$

$e_1, \ldots, e_q$ being an orthonormal frame of $T^\perp F$. In other words, $\omega^\perp$ determines a holonomy invariant measure in the sense of [Pl]. More precisely, if T_k $(k = 1, 2)$ is a submanifold transverse to F and $h: T_1 \to T_2$ is a holonomy map of F, then

$$h^*(\omega^\perp | T_2) = \omega^\perp | T_1.$$

Examples. Any Riemannian foliation is transversely geodesic ($B^\perp = 0$), so it is transversely minimal. In codimension one, the converse is true: every transversely minimal foliation of (M, g) is Riemannian with g being bundle-like. If the bundle $T^\perp F$ is integrable, then F is transversely minimal iff the leaves of the orthogonal foliation are minimal.

4. *Lyapunov exponents*

Let (f_t) be an arbitrary flow on a Riemannian manifold (M, g). If $x \in M$ and $0 \neq v \in T_xM$, then we put

$$\lambda(v) = \limsup_{t \to +\infty} \frac{1}{t} \log |f_{t*}v|. \tag{8}$$

Recall that $\lambda(v)$ is called the *Lyapunov exponent* of (f_t) in the direction of v. From the definition it is clear that the function $\lambda : T_xM \backslash \{0\} \to \mathbb{R}$ admits at most $n = \dim M$ values $\lambda_1 < \cdots < \lambda_k$ and that there is a filtration $L_1(x) \subset L_2(x) \subset \cdots \subset L_k(x) = T_xM$ such that each of $L_j(x)$ is a linear subspace of T_xM and $\lambda(v) = \lambda_j$ iff $v \in L_j(x) \backslash L_{j-1}(x)$ $(L_0(x) = 0)$. Clearly, $\lambda(v) = 0$ when v is tangential to the flow.

If M is compact, we have Oseledets' Multiplicative Ergodic Theorem ([**O**], see also [**M**] and [**Wa**]) which says that M contains a Borel set Λ such that (i) $\mu(\Lambda) = 1$ for any Borel f_t-invariant probability measure on M and (ii) for any x in Λ there are unique constants $\lambda_1(x) < \cdots < \lambda_k(x)$ and unique decomposition $T_xM = E_1(x) \oplus \cdots \oplus E_k(x)$ for which

$$\lim_{t \to \pm\infty} \frac{1}{t} \log |f_{t*}v| = \lambda_j(x)$$

if and only if $v \in E_j(x)$.

In [**M**] elements of the set Λ are called *regular* for the flow (f_t).

In this section, we will estimate Lyapunov exponents of the geodesic flow of F. For this purpose we shall apply Corollary 1 of § 2. Our calculation is analogous to that of [**S**] with one difference: we have to remember that Jacobi fields orthogonal to a geodesic at one point do not remain orthogonal to it all the time.

First, let us put

$$\Phi(u) = 4|B(u)| + |\tilde{\nabla} B(u, u)| + |K(u)| \tag{9}$$

for $u \in TF$, take $\xi \in T_vTF$ $(\xi \neq 0)$ and put

$$z_a(t) = a|Z(t)|^2 + \frac{1}{a}|Z'(t)|^2 \quad (t \in \mathbb{R}),$$

where a is a positive constant and $Z = Z_\xi$ is a Jacobi field along the geodesic $c = c_v$. We have

$$\begin{aligned} z'_a &= 2a\langle Z, Z'\rangle + \frac{2}{a}\langle Z', Z''\rangle = 2a\langle Z, Z'\rangle \\ &+ \frac{4}{a}\langle B(Z'^{\mathrm{T}}, \dot{c}), Z'\rangle + \frac{2}{a}\langle(\tilde{\nabla}_Z B)(\dot{c}, \dot{c}), Z'\rangle \\ &+ \frac{2}{a}\langle R(\dot{c}, Z)\dot{c}, Z'\rangle \leq 2a|Z| \cdot |Z'| \\ &+ \frac{4}{a}|B(\dot{c})| \cdot |Z'|^2 + \frac{2}{a}|\tilde{\nabla} B(\dot{c}, \dot{c})| \cdot |Z| \cdot |Z'| \\ &+ \frac{2}{a}|K(\dot{c})| \cdot |Z| \cdot |Z'| \leq \left(a + \frac{1}{a}\Phi(\dot{c})\right) \cdot z_a. \end{aligned}$$

Therefore,

$$(\log z_a)' \leq a + \frac{1}{a}\Phi(\dot{c})$$

and

$$\limsup_{t\to+\infty} \frac{1}{t}\log z_a(t) \leq a + \frac{1}{a}\limsup_{t\to+\infty}\frac{1}{t}\int_0^t \Phi(\dot{c}(s))\,ds.$$

Now, if $f(t) = |Z(t)|^2 + |Z'(t)|^2$, then $f \leq \alpha \cdot z_a$ for some positive constant α. Therefore,

$$\limsup_{t\to+\infty} \frac{1}{t}\log f(t) \leq a + \frac{1}{t}\limsup_{t\to+\infty}\frac{1}{t}\int_0^t \Phi(\dot{c}(s))\,ds \tag{10}$$

for any $a > 0$. Estimation (10) is the best when

$$a = \left[\limsup_{t\to+\infty}\frac{1}{t}\int_0^t \Phi(\dot{c}(s))\,ds\right]^{1/2}.$$

With this constant we get

$$\limsup_{t\to+\infty}\frac{1}{t}\log f(t) \leq 2\left[\limsup_{t\to+\infty}\frac{1}{t}\int_0^t \Phi(\dot{c}(s))\,ds\right]^{1/2}. \tag{11}$$

Next, let us assume that $\pi_*\xi \perp \dot{c}(0)$ and consider the function $y_a : \mathbb{R} \to \mathbb{R}$ given by

$$y_a(t) = a|Z(t)|^2 - \langle Z(t), \dot{c}(t)\rangle^2,$$

where $a \in (0; 1)$ is fixed. Assume also that $\langle Z'(0), \dot{c}(0)\rangle = 0$, i.e. that (Lemma 1) $\langle Z', \dot{c}\rangle \equiv 0$. Then

$$y_a' = 2a\langle Z, Z'\rangle - 2\langle Z, \dot{c}\rangle\langle Z, B(\dot{c}, \dot{c})\rangle$$

and

$$\tfrac{1}{2}y_a'' = a\langle Z, Z''\rangle + a|Z'|^2 - \langle Z, B(\dot{c}, \dot{c})\rangle^2$$
$$-\langle Z, \dot{c}\rangle\langle Z', B(\dot{c}, \dot{c})\rangle - \langle Z, \dot{c}\rangle\langle Z, (\tilde{\nabla}_{\dot{c}}B)(\dot{c}, \dot{c})\rangle.$$

From (3) we get

$$\tfrac{1}{2}y_a'' \geq -aK(\dot{c})y_a + [|Z|^2((a^2 - a)K(\dot{c})$$
$$-(a+1)|\tilde{\nabla}B(\dot{c}, \dot{c})| - |B(\dot{c})|^2)$$
$$+|Z|\cdot|Z'|(2a-1)\cdot|B(\dot{c})| + |Z'|^2 a]. \tag{12}$$

Assume now that the sectional curvature of M is negative, $K_M \leq K_0 < 0$. Then for any $\delta > 0$ there exists $\eta > 0$ such that the sum in the square brackets in (12) is always non-negative for any $a \in (\delta, 1-\delta)$ if only $|B(\dot{c})| < \eta$ and $|\tilde{\nabla}B(\dot{c}, \dot{c})| < \eta$. For these a we have

$$y_a'' \geq -2aK(\dot{c})y_a \geq -2aK_0 y_a.$$

If, moreover, ξ is horizontal ($C\xi = 0$), then $y_a(0) = a|\pi_*\xi|^2 > 0$ and $y_a'(0) = 0$. So, it is easy to see that in this case

$$y_a(t) \to +\infty \quad \text{when} \quad t \to +\infty. \tag{13}$$

Now, put

$$\Psi_a(u)=(1-|B(u)|)(K(u)(a-1)-|B(u)|-|\tilde{\nabla}B(u,u)|) \quad (u\in TF) \tag{14}$$

and

$$g(t)=\langle Z(t),Z'(t)\rangle \quad (t\in\mathbb{R}).$$

Then

$$\begin{aligned} g'=\langle Z,Z''\rangle+|Z'|^2\geq -K(\dot c)y_a \\ +[|Z|^2(K(\dot c)(a-1)-|\tilde{\nabla}B(\dot c,\dot c)|) \\ -2|Z|\cdot|Z'|\cdot|B(\dot c)|+|Z'|^2] \end{aligned}$$

and if ξ and a are such that condition (13) holds, then for t big enough we have

$$\begin{aligned} g'(t)&\geq|Z(t)|^2(K(\dot c(t))(a-1)-|\tilde{\nabla}B(\dot c(t),\dot c(t))| \\ &\quad -|B(\dot c(t))|)+|Z'(t)|^2(1-|B(\dot c(t))|) \\ &\geq 2g(t)\sqrt{\Psi_a(\dot c(t))} \end{aligned}$$

if $|B(\dot c)|\leq 1$ and $|B(\dot c)|+|\tilde{\nabla}B(\dot c,\dot c)|\leq K(\dot c)(a-1)$. In this case, $g(t)>0$ for t big enough and

$$\limsup_{t\to+\infty}\frac{1}{t}\log g(t)\geq 2\limsup_{t\to+\infty}\frac{1}{t}\int_0^t(\Psi_a(\dot c(s)))^{1/2}\,ds.$$

Finally, since $f(t)\geq 2g(t)$,

$$\limsup_{t\to+\infty}\frac{1}{t}\log f(t)\geq 2\limsup_{t\to+\infty}\frac{1}{t}\int_0^t(\Psi_a(\dot c(s)))^{1/2}\,ds. \tag{15}$$

Let us recapitulate the above results as follows.

THEOREM 2. *The Lyapunov exponents $\lambda(\xi)$ of the flow φ satisfy*

$$\lambda(\xi)\leq\left[\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\Phi(\varphi_s v)\,ds\right]^{1/2}, \tag{16}$$

where $\xi\in T_vSF$ and Φ is given by (9). *Moreover, if the sectional curvature of M is negative, if $|B(\varphi_s v)|$ and $|\tilde{\nabla}B(\varphi_s v,\varphi_s v)|$ are small enough for any s, and if ξ is horizontal, then there exists $a\in(0;1)$ such that*

$$\lambda(\xi)\geq\limsup_{t\to+\infty}\frac{1}{t}\int_0^t(\Psi_a(\varphi_s v))^{1/2}\,ds \tag{17}$$

with Ψ_a given by (14).

Proof. From Corollary 1 and (8), it follows immediately that (16) is equivalent to (11) while (17) is to (15).

5. *Entropy estimate*

In this section, M is assumed to be compact. Denote by Λ the set of all points regular for φ. For any v of Λ denote by $\chi(v)$ the sum of all positive Lyapunov exponents of φ counted with their multiplicities:

$$\chi(v)=\sum_{\{i;\lambda_i(v)>0\}}\lambda_i(v)\cdot\dim E_i(v)$$

with notation of § 4. The measure entropy $h_\mu(\varphi)$ of φ with respect to any φ-invariant Borel probability measure μ satisfies (see [**R**])

$$h_\mu(\varphi)\geq\int_{SF}\chi\,d\mu.$$

THEOREM 3. *The measure entropy of φ with respect to any invariant measure μ satisfies*

$$h_\mu(\varphi)\leq\tfrac{1}{2}(n+p-2)\left[\int_{SF}\Phi\,d\mu\right]^{1/2}. \tag{18}$$

Therefore, the topological entropy of φ satisfies

$$h_{\text{top}}(\varphi)\leq\tfrac{1}{2}(n+p-2)\max_{SF}\Phi^{1/2}. \tag{19}$$

Proof. First, for any point v regular for φ put

$$E^u(v)=\bigoplus_{\{i;\lambda_i(v)>0\}}E_i(v),$$

$$E^s(v)=\bigoplus_{\{i;\lambda_i(v)<0\}}E_i(v)$$

and

$$E^0(v)=\left\{\xi;\ \lim_{t\to\pm\infty}\frac{1}{t}\log|\varphi_{t*}\xi|=0\right\}.$$

Denote by σ the isometry of SF given by $\sigma(v)=-v$. Then

$$\varphi_{-t}\circ\sigma=\sigma\circ\varphi_t\quad(t\in\mathbb{R})$$

and therefore

$$\sigma_*E^u(v)=E^s(\sigma(v)),\ \sigma_*E^s(v)=E^u(\sigma(v))\quad\text{and}\quad\sigma_*E^0(v)=E^0(\sigma(v)). \tag{20}$$

Next, observe that the Birkhoff Ergodic Theorem asserts that the limit

$$\alpha(v)=\lim_{t\to+\infty}\frac{1}{t}\int_0^t\Phi(\varphi_s v)\,ds \tag{21}$$

exists almost everywhere and if μ is an ergodic invariant measure, then

$$\alpha(v)=\int_{SF}\Phi\,d\mu. \tag{22}$$

for μ-almost all v.

From (16), we get

$$\chi(v)\leq\dim E^u(v)\cdot\alpha(v)^{1/2}\qquad\text{(a.e.)}$$

and therefore

$$\chi(v)+\chi(\sigma(v))\leq[\dim E^u(v)+\dim E^u(\sigma(v))]\cdot\alpha(v)^{1/2}$$

almost everywhere. Since $\dim E^0(v)\geq1$ and $\dim E^u(\sigma(v))=\dim E^s(v)$ according to (20), we have

$$\chi(v)+\chi(\sigma(v))\leq(n+p-2)\alpha(v)^{1/2}\qquad\text{(a.e.)}$$

Also,

$$\lim_{t\to\infty}\frac{1}{t}\log|\varphi_{-t*}\sigma_*\xi|=\lim_{t\to\infty}\frac{1}{t}|\varphi_{t*}\xi|,$$

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so Lyapunov exponents of the flow $\varphi^{-1}=(\varphi_{-t})$ at $\sigma(v)$ coincide with those of φ at v. Therefore

$$h_\mu(\varphi^{-1})\le\int_{SF}\chi\circ\sigma\,d\mu.$$

Finally,

$$\begin{aligned}2h_\mu(\varphi)&=h_\mu(\varphi)+h_\mu(\varphi^{-1})\\&\le\int_{SF}(\chi+\chi\circ\sigma)\,d\mu\le(n+p-2)\int_{SF}\alpha^{1/2}\,d\mu\\&\le(n+p-2)\left[\int_{SF}\alpha\,d\mu\right]^{1/2}=(n+p-2)\left[\int_{SF}\Phi\,d\mu\right]^{1/2}\end{aligned}$$

proving (18) for ergodic measures. Inequality (18) for an arbitrary measure follows now directly from the ergodic decomposition and Jacobs theorem (see, for example, [**Wa**]). Inequality (19) follows immediately from (18) because

$$h_{\mathrm{top}}(\varphi)=\sup_\mu h_\mu(\varphi). \qquad \square$$

Remark 3. If F is a trivial foliation with the single leaf M, then $p=n$ and (18) reduces to

$$h_\mu(\varphi)\le(n-1)\left[\int_{SF}|K(u)|\,d\mu(u)\right]^{1/2},$$

the inequality proved in [S] in the case $K<0$.

6. *Foliations of hyperbolic manifolds*

In this section, we get the following application of estimates obtained in § 4.

THEOREM 4. *Let M be a compact manifold of negative sectional curvature. Then there exists a positive number η such that there are no non-trivial foliations of M with the second fundamental form B satisfying $|B|<\eta$ and $|\tilde{\nabla}B|<\eta$ everywhere on M. In particular, there are no totally geodesic foliations of M.*

Proof. Assume that η is such that the Lyapunov exponents $\lambda(\xi)$ of the geodesic flow φ in the direction of horizontal vectors satisfy (17) for some $a>0$ if $|B|<\eta$ and $|\tilde{\nabla}B|<\eta$. In this case, $\lambda(\xi)>0$ for any horizontal vector ξ of T_vSF such that $\pi_*\xi\perp v$ and therefore

$$\dim E^u(v)\ge n-1$$

for any regular point v. From (20), it also follows that

$$\dim E^s(v)\ge n-1$$

and since $\dim E^0(v)\ge 1$ we get

$$n+p-1=\dim T_vSF\ge 2n-1$$

and $n=p$.

Let us note that this argument is possible because – according to the Multiplicative Ergodic Theorem – the set of regular points is non-empty. If M is non-compact, this is no longer true, so there exist many totally geodesic foliations of compete non-compact hyperbolic spaces.

7. *Final remarks*

(A) Our entropy estimates are obtained in the way analogous to that of Sarnak [S] who also gives estimates from below. In the same way, one would use inequality (17) and the Pesin formula ([**P**], see also [**M**]) to estimate $h_\mu(\varphi)$ under some conditions on M and F. However, one should be careful since, as we could see in the proof of Theorem 4, we arrived at the conclusion $\dim F = \dim M$ assuming inequality (17) for all horizontal vectors. Also, the entropy estimates from [S] have been improved by Freire–Mane [**FM**] and Osserman–Sarnak [**OS**]. In both papers, the problem is reduced to the study of some Riccati type matrix equation of the form

$$U'(s)+U^2(s)+R(s)=0, \tag{23}$$

where $R(s)$ is a symmetric matrix. The symmetry of $R(s)$ allows Green's results to be applied [**Gr**] to get estimates of the growth rate of Jacobi fields. In our case, the Riccati type equation deduced from (3) has the form

$$U'(s)+U^2(s)+A(s)U(s)+B(s)=0, \tag{24}$$

where $B(s)$ is, in general, nonsymmetric and indefinite. In fact, a straightforward but lengthy calculation shows that $B(s)$ is symmetric and definite if and only if F is totally geodesic, the sectional curvature of M vanishes for all planes spanned by a vector tangent to F and by a vector orthogonal to F, and the sectional curvature of the leaves has a constant sign. So, one could follow this procedure in some very special cases only and therefore estimating $h_\mu(\varphi)$ from below (for a smooth invariant measure μ) seems to be promising.

(B) One may also ask: when is φ Anosov? Some necessary conditions are obvious: codim F should be even and all geodesic flows of the leaves of F should be uniformly hyperbolic, therefore the leaves could not admit conjugate points. Trying to study this problem by the methods analogous to those of [*E*] we find the difficulty mentioned in (A): the matrix $B(s)$ in (24) need not be neither symmetric nor definite. It seems that the problem becomes even more interesting when φ is the geodesic flow of a non-integrable subbundle of *TM* (equipped with the connection induced from *TM* by the orthogonal projection). In this case, φ cannot be decomposed into the family of the geodesic flows of leaves.

(C) Let us note that there are several results concerning the existence of totally geodesic foliations of Riemannian manifolds. For example, it is known that codimension-one totally geodesic foliations of closed Riemannian manifolds of strictly positive (or, negative) Ricci (or, k-sectional) curvature do not exist [**Br**]. Also, there are no complete totally geodesic foliations of locally symmetric negatively curved Riemannian manifolds of finite volume [**Z**]. (A foliation of a Riemannian manifold is said to be complete when all its leaves are complete with respect to the induced metric.)

In this context several questions arise. For example, do totally geodesic foliations (of codimension greater than one) exist on compact manifolds of negative k-sectional curvature? Or, when it is the set of regular points (in the sense of Oseledets' Ergodic Theorem) of φ non-empty (if the foliated manifold is not compact)? Is this true when M has negative curvature and finite volume, and F is close to geodesic?

An answer to these questions could allow us to reprove (or, to improve) Zeghib's result.

(D) In [GLW], the geometric entropy of a foliation of a compact Riemannian manifold is defined. It is shown that this entropy depends on the transverse structure of a foliation, more precisely, on its holonomy pseudogroup. Therefore, the authors expect that the topological entropy of the geodesic flow of a foliation F could be related to the geometric entropy of F, the rate of growth of leaves and the second fundamental tensor of F.

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