APPLICATION OF THE RITZ METHOD TO NON-STANDARD EIGENVALUE PROBLEMS

A. L. ANDREW

(Received 3 October 1967)

1. Introduction

There is an extensive literature on application of the Ritz method to eigenvalue problems of the type

$$L_1 w = \lambda L_2 w \tag{1}$$

where L_1, L_2 are positive definite linear operators in a Hilbert space (see for example [1]). The classical theory concerns the case in which there exists a minimum (or maximum) eigenvalue, and subsequent eigenvalues can be located by a well-known mini-max principle [2; p. 405]. This paper considers the possibility of application of the Ritz method to eigenvalue problems of the type (1) where the linear operators L_1, L_2 are not necessarily positive definite and a minimum (or maximum) eigenvalue may not exist. The special cases considered may be written with the eigenvalue occurring in a non-linear manner.

This investigation was suggested by results [3] recently obtained when the Ritz method was applied to a variational formulation by Chandrasekhar [4] of the boundary value problem describing non-radial oscillations of stars. This problem, which is described in section 2, does not have a minimum (or maximum) eigenvalue and hence is not dealt with by the classical theory. It is the aim of section 2 to indicate the use that has been made by astrophysicists of the methods whose validity is examined in the succeeding sections, to discuss the results they have obtained, and to examine the mathematical properties of the eigenvalue problem they considered, so that later the similarities between that problem and the one considered in section 4 may become apparent.

Some elementary results for quite general operators are established in section 3. Section 4 considers a simpler problem for which an exact solution may be obtained but which is in many ways similar to that studied in [3]. Like the problem considered in [3], this simpler problem has no minimum or maximum eigenvalue, though its eigenvalues are bounded from below. Some theorems, which it is hoped are also interesting in their own right, are proved concerning the results of formal application of the Ritz method to this simpler problem. Section 5 reconsiders the astrophysical problem in the light of the results of section 4 which are also extended slightly.

2. Non-radial oscillations of stars

The fourth order system of differential equations governing small adiabatic non-radial oscillations of stars is derived and its properties discussed in [5; p. 509]. This system, and the simpler second order system obtained when, as in [3] and many other papers, the Eulerian perturbation $\delta\phi$ of gravitational potential is neglected, will be denoted here, as in [3], by IA and IB respectively.

When $\delta\phi$ is neglected, Chandrasekhar's variational principle for oscillations corresponding to the spherical harmonic ℓ may be written

$$\sigma^2 \int_0^R \rho \left(\frac{\psi^2}{r^2} + \frac{(\chi')^2}{\ell(\ell+1)}\right) dr = \int_0^R \left\{\Gamma_1 p \left(\frac{dp}{dr}\,\frac{\psi}{\Gamma_1 p} + \psi' - \chi'\right)^2 + A\,\frac{dp}{dr}\,\psi^2\right\} \frac{dr}{r^2} \tag{2}$$

where $2\pi/\sigma$ is the period, R is the radius,

$$A = \frac{1}{\rho}\,\frac{d\rho}{dr} - \frac{1}{\Gamma_1 p}\,\frac{dp}{dr}$$

and the components ψ, χ' of the vector-valued eigenfunction, which depend on the radial and transverse components of displacement, are defined in [4]. The density ρ, the total pressure p, and the adiabatic exponent Γ_1 are functions of the distance r from the centre. These functions depend on the stellar model chosen. It is required that ψ/r^2 and χ'/r be bounded at $r = 0$ and $r = R$. If the stellar model satisfies certain conditions that are satisfied by all the stellar models considered in this paper, Lebovitz has shown [6] that if $\sigma^2 \neq 0$, the eigenfunctions of IA must satisfy stronger conditions (which are satisfied by (6)) at the centre. An identical proof establishes the same result for IB.

The Euler-Lagrange equations obtained from (2) by considering variations in ψ and χ' respectively are

$$\sigma^2 \rho\psi/r^2 = -F'(r) + (p'/\Gamma_1 p)F(r) + Ap'\psi/r^2 \tag{3a}$$

$$\sigma^2 \rho\chi'/\ell(\ell+1) = -F(r) \tag{3b}$$

where $F(r) = (p'\psi + \Gamma_1 p\psi' - \Gamma_1 p\chi')/r^2$. Equations (12), (31), (33) and (34) of [4] show that equations (3a) and (3b) above are scalar multiples of the radial and transverse components of the basic equation (9) of [4] when $\delta\phi$ is neglected.

Equation (3) is of the form (1). Denote the linear operators L_1, L_2 in this case by A_1, A_2 respectively. In this case w corresponds to the vector valued function $\{\psi, \chi'\}$, the inner product is defined by

$$(\{\psi_1, \chi_1'\}, \{\psi_2, \chi_2'\}) = \int_0^R (\psi_1\psi_2+\chi_1'\chi_2')dr,$$

$\lambda = \sigma^2$ and A_1 and A_2 are symmetric, but not positive definite. (That the operators are symmetric, as is necessary for the variational formulation, was proved in [4].) As A_2 is given by $A_2\{\psi, \chi'\} = \{\rho\psi/r^2, \rho\chi'/\ell(\ell+1)\}$, the orthogonality relation satisfied by eigenfunctions $\{\psi_1, \chi_1'\}$, $\{\psi_2, \chi_2'\}$ corresponding to distinct eigenvalues is

$$\int_0^R (\rho\psi_1\psi_2/r^2+\rho\chi_1'\chi_2'/\ell(\ell+1))dr = 0.$$

Since $\ell > 0$, and $\rho > 0$ for $r < R$ in all models considered here (and all physical models), the operator A_2 is positive (that is for all non-zero w in the domain of A_2, $(A_2w, w) > 0$). Hence (3) satisfies the conditions of both Theorems 1 and 2.

Substituting for $F(r)$ from (3b) in (3a) shows that when $\sigma^2 \neq 0$ equation (3) may be written in standard matrix form as

$$(4)\qquad \begin{pmatrix} \psi \\ \chi' \end{pmatrix}' = \begin{pmatrix} -\dfrac{1}{\Gamma_1 p}\dfrac{dp}{dr} & 1-\dfrac{\sigma^2\rho r^2}{\Gamma_1 p\ell(\ell+1)} \\ \dfrac{\ell(\ell+1)}{r^2}\left(1-\dfrac{A}{\rho\sigma^2}\dfrac{dp}{dr}\right) & -A \end{pmatrix} \begin{pmatrix} \psi \\ \chi' \end{pmatrix}.$$

This differs only in notation from the form given for IB in [5]. When Schwarzschild's stability criterion $A < 0$ is satisfied throughout the star (except perhaps on a set of measure zero) it is obvious from (2) that, as $dp/dr < 0$ and $\Gamma_1 p > 0$, all eigenvalues are strictly positive so that the above formulation is valid. Although (3) is written with σ^2 occurring in a linear manner, when that equation is written as the second order differential equation (4) (instead of as a third order one), σ^2 occurs in a non-linear manner. This property, which is also exhibited by the equation considered in section 4, appears to be intimately related to the properties of the spectrum.

Numerical solutions of IA and IB have been obtained for a number of stellar models. Generally these follow the pattern predicted by Cowling [7]. The spectrum is discrete, there being two families of modes — the p-modes with eigenvalues tending to $+\infty$ and the g-modes with eigenvalues tending to zero. In most cases the nth p-mode and the nth g-mode both have

https://doi.org/10.1017/S144678870000762X Published online by Cambridge University Press

exactly n zeros in ψ and n zeros in χ'. In between these two families is a single mode, the f-mode, with no zeros in ψ or χ'. Variations in this pattern have been noted for some models. (See for example [8], [9] and [5; p. 515].)

An exception relevant to what follows is the convective model, in which $A \equiv 0$. In this case it is obvious that $\sigma^2 = 0$ is an eigenvalue of (3) of infinite multiplicity, with the corresponding space of eigenfunctions given by $\chi' = \psi' + p'\psi/\Gamma_1 p$, where ψ/r^2 is any bounded function sufficiently differentiable on $(0, R)$. In this case (3) may be written

$$\sigma^2 \left[\left(\frac{r^2 p_1 \chi''}{\ell(\ell+1)} \right)' + \left(\frac{p_1 \rho r^2 \sigma^2}{\Gamma_1 p \ell(\ell+1)} - p_1 \right) \chi' \right] = 0 \tag{5}$$

where $p_1(r) = \exp\left(\int_0^r (p'/\Gamma_1 p) dr\right)$. The non-zero eigenvalues are given by the equation obtained by dividing (5) by σ^2. With the boundary conditions this is a singular Sturm-Liouville problem in χ' [2; p. 324], since ρ, $\Gamma_1 p$, p_1 and ℓ are all strictly positive in $[0, R)$. The singularities are regular (compare [5; p. 461]) when, as is usual, $\rho/\Gamma_1 p \sim (R-r)^{-1}$ as $r \to R$. It seems likely, as is generally accepted [5; p. 518], that with the convective model IB has an f-mode and an infinite family of p-modes but that instead of the infinite family of g-modes there is an infinite family of eigenfunctions with eigenvalue zero.

In [3] the Ritz method was applied to (2) by assuming $\psi/r^{\ell+1}$ and χ'/r^ℓ to be continuous and piecewise linear. Good results were obtained for the f-mode and the p-modes but no trace of the g-modes was obtained. Of the eigenvalues and eigenvectors of the matrix equation yielded by the Ritz method, those which did not correspond to p-modes or the f-mode seemed completely spurious.

The only model considered in [3] was one constructed by Van der Borght [10]. It has a convective core and a radiative envelope with $A = 0$ for $r < r_f < R$ and $A < 0$ for $r_f \leqq r \leqq R$. (The possible effect on the solution of IB, of the vanishing of A throughout a region, is considered in section 5.) For this model, Wan and Van der Borght [11] have solved both IA and IB by numerical integration and obtained both p-modes and g-modes in both cases.

The Ritz method, using similar continuous piecewise linear coordinate functions, has been applied successfully to the problem of purely radial oscillations of this model [12]. In that case however the associated differential operator is positive definite and the variational principle is of the classical minimum type. The existence of the g-modes with eigenvalues tending to zero ensures that the operator A_1 associated with non-radial oscillations is not positive definite. (A symmetric linear operator L is said to be positive definite if there is a strictly positive constant k such that $(Lw, w) \geqq k(w, w)$ for all w in the domain of L.) The variational principle

(2) is not a classical minimum principle as there is no minimum eigenvalue.

A second application of the Ritz method to (2) in [3] assumed ψ and χ' to be given by

$$\psi = \sum_{i=0}^{n} C_i r^{\ell+2i+1}, \quad \chi' = (\ell+1)C_0 r^{\ell} + \sum_{i=1}^{n} C_i^* r^{\ell+2i} \tag{6}$$

where the C_i, C_i^* are parameters. Coordinate functions of type (6) were initially used in [13] where only the case $n = 1$ was considered and only the f-mode determined. However those numerical results in [3] which used (6) have subsequently been found to be incorrect. Amended results using values of n up to $n = 6$ are given in a later paper [20]. These amended results show that when (6) is used with this model crude approximations of g-modes may be obtained for both IB and IA but that they are far less accurate than the approximations obtained for f- and p-modes. These calculations, like those in [3] and [11], consider only the spherical harmonic $\ell = 2$.

Robe and Brandt [14] have applied the Ritz method to both IA and IB for several polytropic models. For $\ell = 2$ they obtained both p- and g-modes in both cases although, except for models of unrealistically low central condensation, the values they obtained, especially for g-modes, differed considerably from those obtained by numerical integration of the differential equation. They assumed ψ and χ' to be given by (6), using values of n up to $n = 2$. Their relatively good results for models of low central condensation may be explained by the fact that (6) is the form of the exact eigenfunctions for the homogeneous (constant density) model. For a model with ρ proportional to $1-(r/R)^2$, which also has very low central condensation, Tassoul [15] obtained encouraging results for both p- and g-modes of IA using the Ritz method, again with ψ and χ' given by (6). However in [15] comparison of the results of the Ritz method with accurate results is limited to the case of purely radial oscillations when, as already noted, the Ritz method presents no difficulty. It cannot be assumed that the Ritz method will be as successful for the more realistic model used in [3] as it was for Tassoul's model or the low index polytropes. As well as having higher central condensation, Van der Borght's model has $A \equiv 0$ throughout an extensive core and Cowling's argument [7] is weakest when A vanishes throughout some region.

No rigorous mathematical analysis has yet been undertaken for either IA or IB except with some highly unrealistic models [5, p. 514] and the numerical solutions obtained have not been rigorously justified. Accordingly before considering in more detail the reason for the partial failure of the Ritz method in this incompletely understood case, it would seem useful to consider a related problem which can be solved explicitly and shown to have cluster points of eigenvalues at $+\infty$ and zero. This is done in section 4.

3. The general equation

First it would seem of interest to consider the general eigenvalue problem (1) where L_1, L_2 are symmetric linear operators in a Hilbert space. This equation may be regarded as the Euler-Lagrange equation of the variational problem $(L_1 w, w) = \lambda(L_2 w, w)$. Application of the Ritz method to this yields the matrix equation

$$Px = \Lambda Qx \tag{7}$$

where the matrices P and Q are symmetric, and if the operators L_1 and L_2 are positive then P and Q are positive definite. The Ritz method has been rigorously justified [1] when the variational principle is a true mini-max principle, but this is not the case with the problems considered in sections 2 and 4 and is not assumed in this section.

Throughout the rest of this paper, eigenvalues and eigenfunctions yielded by formal application of the Ritz method to some problem will be termed *Ritz approximate* eigenvalues and eigenfunctions of that problem.

The following Theorem, which emphasises the importance of the choice of coordinate functions, will be required for later work. The proof is straightforward and is omitted.

THEOREM 1. *Let L_1 be any symmetric linear operator and L_2 any symmetric positive linear operator in a real Hilbert space of functions. Let $\lambda_1, \cdots, \lambda_n$ be any distinct eigenvalues of* (1) *and $w_1, \cdots, w_n$ the corresponding eigenfunctions* (*which are linearly independent since the λ_i are distinct.*) *Let $\xi_1, \cdots, \xi_n$ be any basis of the space spanned by $w_1, \cdots, w_n$. Then when the Ritz method is applied to* (1) *using the ξ_i as coordinate functions, the Ritz approximate eigenvalues obtained are the exact λ_i and the corresponding Ritz approximate eigenfunctions are the corresponding exact w_i.*

A particular case of Theorem 1 is the result obtained by Robe [16] when he applied the Ritz method to *IA* for the homogeneous stellar model using as coordinate functions a basis of the space spanned by certain exact eigenfunctions, whose form in that very special (and non-physical) case is known [5; p. 514]. However this gives little indication of the reliability of the Ritz method when the solution is not known in advance.

In the theory of convergence of the Ritz method in the classical case, an important fact is that the Ritz approximate eigenfunctions corresponding to distinct eigenvalues satisfy the orthogonality relation of the true eigenfunctions. The method of proof normally used in that case is also valid more generally.

THEOREM 2. *Let L_1, L_2 be symmetric linear operators in a real Hilbert space of functions. Then if Λ_1, Λ_2 are distinct Ritz approximate eigenvalues of* (1) *and W_1, W_2 are the corresponding Ritz approximate eigenfunctions,* $(L_1W_1, W_2) = (L_2W_1, W_2) = 0$.

PROOF. For $k = 1$ and 2, $W_k = \sum_{j=1}^{n} x_{kj}\xi_j$ where the coordinate functions ξ_j are any linearly independent functions in the domains of L_1 and L_2 and the scalars x_{kj} satisfy (7) with $\Lambda = \Lambda_k$, $x = x_k = \{x_{k1}, \cdots, x_{kn}\}$, $p_{ij} = (L_1\xi_i, \xi_j)$, and $q_{ij} = (L_2\xi_i, \xi_j)$. Now

$$\begin{aligned}(L_2W_1, W_2) &= (\sum_{i=1}^{n} x_{1i}L_2\xi_i, \sum_{j=1}^{n} x_{2j}\xi_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_{1i}x_{2j}q_{ij} \\ &= x_2^{\tau}Qx_1 \quad \text{(where } x^{\tau} \text{ denotes the transpose of } x) \\ &= 0 \ \text{ since } \Lambda_1 \neq \Lambda_2 \text{ and } P \text{ and } Q \text{ are symmetric.}\end{aligned}$$

Similarly $(L_1W_1, W_2) = 0$.

4. A simpler equation

Let H be the Hilbert space of real square summable 2-vector valued functions $\{u, v\}$ defined on the interval $[0, 1]$, in which the inner product is defined by

$$(10) \qquad (\{u_1, v_1\}, \{u_2, v_2\}) = \int_0^1 (u_1u_2 + v_1v_2)$$

where in (10) *and throughout this section* the abbreviated notation $\int f$ is used for $\int f(x)dx$.

Define the operators B_1, B_2 on those sufficiently differentiable functions in H satisfying

$$(11) \qquad v(0) = v(1) = 0$$

by

$$B_1\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}a_1u + a_2v - a_1a_4v' \\ [a_4(a_1u+a_2v-a_1a_4v')]' + a_2u + a_3v - a_2a_4v'\end{pmatrix}$$

$$B_2\{u, v\} = \{a_5u, a_6v\}$$

where the a_i are continuously differentiable, non-vanishing, real-valued functions defined on $[0, 1]$. It follows from (10) and (11) that

$$\left(B_1\begin{pmatrix}u_1\\v_1\end{pmatrix}, \begin{pmatrix}u_2\\v_2\end{pmatrix}\right) = \int_0^1 \left[a_1\left(u_1 + \frac{a_2}{a_1}v_1 - a_4v_1'\right)\left(u_2 + \frac{a_2}{a_1}v_2 - a_4v_2'\right) + \left(a_3 - \frac{a_2^2}{a_1}\right)v_1v_2\right]$$

and $$(B_2\{u_1, v_1\}, \{u_2, v_2\}) = \int_0^1 (a_5 u_1 u_2 + a_6 v_1 v_2).$$

These equations show that B_1 and B_2 are symmetric and that if, almost everywhere on (0, 1),

(12a) $$a_3 > a_2^2/a_1 > 0$$

then B_1 is positive, while B_2 is positive if, almost everywhere on (0, 1),

(12b) $$a_5 > 0, \; a_6 > 0.$$

In the rest of this section it is assumed that (12) is satisfied since in the simplest case the operators A_1, A_2 of section 2 are positive and one purpose of the present investigation is to shed further light on the problem considered there. Other cases are mentioned briefly in section 5.

In exactly the same manner in which (4) was derived from (3), the eigenvalue problem

(13) $$B_1\{u, v\} = \lambda B_2\{u, v\}$$

may be rewritten as

(14) $$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -b_1 - b_6 & b_5 - b_3/\lambda \\ b_2 - \lambda b_4 & b_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where

$$\begin{aligned} b_1 &= a_2/a_1 a_4, \quad & b_2 &= 1/a_4, \quad & b_3 &= (a_1 a_3 - a_2^2)/a_1 a_4 a_5, \\ b_4 &= a_5/a_1 a_4, \quad & b_5 &= a_6/a_4 a_5, \quad & b_6 &= (a_4 a_5)'/a_4 a_5. \end{aligned}$$

The non-linear occurrence of λ in (14) and its linear occurrence in (13) are reminiscent of (4) and (3).

If a_1/a_5 $(= b_2/b_4)$ is a constant, (14) shows that a_1/a_5 is an eigenvalue of (13) with corresponding eigenfunction $\{u_0, v_0\}$ where

$$u_0 = c(a_4 a_5)^{-1} \exp\left(-\int b_1\right), \quad v_0 \equiv 0,$$

and c is a constant. Denote this eigenvalue by λ_0. Substitution shows that $\{u_0, v_0\}$ is the only eigenfunction with $v \equiv 0$.

Application of the Ritz method to the solution of (13) involves extremising $\lambda = (B_1\{u, v\}, \{u, v\})/(B_2\{u, v\}, \{u, v\})$ where $\{u, v\}$ is restricted to a finite dimensional subspace of H. The coordinate functions (basis of the subspace) usually chosen are of the form $\{\xi_i, 0\}$, $\{0, \eta_j\}$ so that it is assumed that

(15) $$u = \sum_{i=1}^{n} c_i \xi_i, \quad v = \sum_{j=1}^{m} c_j^* \eta_j$$

where the c_i, c_j^* are constants and the $\{\xi_i, \eta_j\}$ are in the domain of B_1 and

https://doi.org/10.1017/S144678870000762X Published online by Cambridge University Press

hence in the domain of B_2. The coordinate functions used in [3], [14], [15] and [16] are of this type.

THEOREM 3. *If a_1/a_5 is constant, application of the Ritz method to* (13) *using* (15) *with $n > m$ will yield at least $n-m$ linearly independent Ritz approximate eigenfunctions in which $v \equiv 0$. The Ritz approximate eigenvalue corresponding to each of these will be a_1/a_5, the exact value of λ_0.*

PROOF. The matrix equation yielded by the Ritz method will be of the form (7) where in this case it is easily shown that

$$P = \begin{pmatrix} a_1(0)P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix}, \quad Q = \begin{pmatrix} a_5(0)P_1 & 0_{nm} \\ 0_{mn} & P_4 \end{pmatrix} \tag{16}$$

where P_1 is an $n \times n$ matrix, P_3 and P_4 are $m \times m$, P_2 is $n \times m$ and 0_{mn}, 0_{nm} are zero matrices. If $\Lambda = a_1/a_5$, then $x^T = \{y, 0\}$, where y is a $1 \times n$ vector and 0 the $1 \times m$ zero vector, will be a solution of (7) if and only if $P_2^T y^T = 0$. Since P_2^T is $m \times n$ and $n > m$ this will have a solution space of dimension at least $n-m$. The result follows. Alternatively the multiplicity of λ_0 could be established by noting that det $(P - \Lambda Q)$ has a factor $(a_1 - a_5\Lambda)^{n-m}$.

The remainder of this paper will be concerned with the case when the a_i are all constant. In this case (14) is very simple as $b_6 = 0$ and the b_i are all constant. To distinguish this case, (13) with all the a_i constant will be labelled (13a).

The exact solution of (13a) may easily be found. Let

$$d(\lambda) = b_1^2 + b_2 b_5 + b_3 b_4 - \lambda b_4 b_5 - b_2 b_3/\lambda.$$

Since a_1/a_5 is constant it has already been shown that λ_0 is an eigenvalue with unique corresponding eigenfunction $\{u_0, v_0\}$ and that no other eigenfunctions have $v \equiv 0$. For any given λ, the solutions of (13a), if any, may be found by the usual methods for homogeneous linear differential equations with constant coefficients. It is readily seen that there is no eigenvalue λ (except perhaps λ_0) for which $d(\lambda) = 0$ as (11) shows that this would require $v \equiv 0$. Hence all remaining eigenfunctions must be of the form

$$\{u(x), v(x)\} = \{c_1, c_2\} \exp\left(\sqrt{(d(\lambda))}x\right) + \{c_3, c_4\} \exp\left(-\sqrt{(d(\lambda))}x\right)$$

where the c_i are constants. Boundary conditions (11) then show that the only solutions with v not identically zero have $v(x) = c \sin\left(\sqrt{(-d(\lambda))}x\right)$, where c is a constant and that this requires $d(\lambda) = -n^2\pi^2$ where n is a positive integer, so that all eigenvalues except λ_0 must satisfy

$$\lambda^2 b_4 b_5 - \lambda(b_1^2 + b_2 b_5 + b_3 b_4 + n^2\pi^2) + b_2 b_3 = 0 \tag{17}$$

so that

https://doi.org/10.1017/S144678870000762X Published online by Cambridge University Press

$$\lambda = (2b_4b_5)^{-1}[b_1^2+b_2b_5+b_3b_4+n^2\pi^2\pm((b_1^2+b_2b_5+b_3b_4+n^2\pi^2)^2 -4b_2b_3b_4b_5)^{\frac{1}{2}}].$$

For each n, denote the values of λ corresponding to the $+$sign and the $-$sign above by λ_{n+} and λ_{n-} respectively. It follows from (12) that for all n

$$0 < \lambda_{(n+1)-} < \lambda_{n-} < \lambda_{n+} < \lambda_{(n+1)+}$$

and

$$\lambda_{n+} \sim n^2\pi^2(b_4b_5)^{-1} \to \infty, \quad \lambda_{n-} \sim b_2b_3(n^2\pi^2)^{-1} \to +0 \quad \text{as } n \to \infty. \tag{18}$$

Substitution in (13a) shows that all $\lambda_{n\pm}$ thus defined are eigenvalues of (13a) and that the eigenfunctions corresponding to λ_{n+} and λ_{n-} are $\{u_{n+}, v_{n+}\}$ and $\{u_{n-}, v_{n-}\}$ respectively where $v_{n\pm}(x) = c \sin n\pi x$ and

$$\begin{aligned}(b_2-b_4\lambda_{n\pm})u_{n\pm}(x) &= v'_{n\pm}(x)-b_1v_{n\pm}(x)\\ &= c(n\pi \cos n\pi x - b_1 \sin n\pi x).\end{aligned} \tag{19}$$

The above expressions and (17) show that

$$\lambda_{n+}\lambda_{n-} = b_2b_3/b_4b_5 = (a_1a_3-a_2^2)/a_5a_6, \tag{20}$$

$$(a_1-a_5\lambda_{n+})u_{n+} = (a_1-a_5\lambda_{n-})u_{n-};\ v_{n+} = v_{n-}. \tag{21}$$

Since (12) implies

$$(b_1^2+b_2b_5+b_3b_4+n^2\pi^2)^2-4b_2b_3b_4b_5 > (b_1^2-b_2b_5+b_3b_4+n^2\pi^2)^2$$

and $b_4b_5 > 0$, it follows that

$$\lambda_{n-} < \lambda_0 < \lambda_{n+}.$$

Although (12a) ensures that B_1 is positive, putting $a_5 = a_6 = 1$ in (18) shows that it is not positive definite. The $+$ and $-$families of eigenvalues and eigenfunctions and the single eigenvalue and eigenfunction in between them are in many ways similar to Cowling's p-modes, g-modes and f-mode mentioned in Section 2. While $v_0 \equiv 0$, u_0 has no zero in $[0, 1]$. Both u_{n+} and u_{n-} have exactly n zeros in $(0, 1)$ and do not vanish at either 0 or 1 while v_{n+} and v_{n-} have exactly $n-1$ zeros in $(0, 1)$ but also vanish at both 0 and 1. The positions of the zeros of $u_{n\pm}$ depend on b_1 but they are separated by those of $v_{n\pm}$ (which is independent of the b_i).

Direct calculation shows that if the Ritz method is applied to (13a) with the space of allowable functions given by

$$u(x) = c_1 \sin n\pi x + c_2 \cos n\pi x, \quad v(x) = c_3 \sin n\pi x \tag{22}$$

where the c_i are the variational parameters, then the Ritz approximate eigenvalues will be exactly λ_{n-}, λ_{n+} and λ_0, and that the Ritz approximate eigenfunction corresponding to $\lambda_{n\pm}$ will be exactly $\{u_{n\pm}, v_{n\pm}\}$. The boundary conditions and the fact that the a_i are constant make (22) quite a logical choice. However it can hardly be expected that the results yielded by this choice will be representative of the accuracy obtainable by the Ritz method in general. (See Theorems 1 and 3.)

Polynomials are a favourite choice of coordinate functions with the Ritz method and were one of the types of coordinate functions used in [3]. Since the eigenfunctions of (13a) are not polynomials, solutions of (13a) obtained using polynomial coordinate functions should give a truer picture of the reliability of the Ritz method for this problem than (22). It would then seem useful to consider the application of the Ritz method to (13a) using polynomial coordinate functions. The simplest choice of polynomial coordinate functions satisfying (11) is given by (15) where in this case

$$\xi_i(x) = x^{i-1}, \quad \eta_i(x) = x^i(1-x). \tag{23}$$

THEOREM 4. *If $n-m \geqq 2$ when the Ritz method is applied to* (13a) *using* (15) *and* (23) *then, apart from the multiplicity of λ_0, the Ritz approximate eigenvalues and eigenfunctions will be independent of n, depending on m only. In this case the Ritz approximate eigenvalues other than λ_0 will occur in pairs satisfying* (20) *and the corresponding Ritz approximate eigenfunctions will satisfy* (21).

PROOF. With this choice of coordinate functions, direct calculation shows that P and Q in the matrix equation (7) obtained by the Ritz method are of the form (16) where on this occasion $P_k = (p_{kij})$, $k = 1, \cdots, 4$ and

$$p_{1ij} = \frac{1}{i+j-1},$$

$$p_{2ij} = \frac{-a_1a_4j}{i+j-1} + \frac{a_1a_4(j+1)+a_2}{i+j} - \frac{a_2}{i+j+1},$$

$$p_{3ij} = \frac{2a_1a_4^2ij}{(i+j-1)(i+j)(i+j+1)} + \frac{2a_3}{(i+j+1)(i+j+2)(i+j+3)},$$

$$p_{4ij} = \frac{2a_6}{(i+j+1)(i+j+2)(i+j+3)}.$$

Clearly

$$p_{2ij} = -a_1a_4jp_{1ij} + (a_1a_4(j+1)+a_2)\,p_{1,i,j+1} - a_2p_{1,i,j+2}$$

so that, if $\Lambda \neq a_1/a_5$, column operations on $(P-\Lambda Q)$ can reduce the first n elements of the last m columns to zero. These column operations are equivalent to post multiplication by the matrix $R(\Lambda)$ given by

$$R(\Lambda) = \begin{pmatrix} I_n & R_1(\Lambda) \\ 0_{mn} & I_m \end{pmatrix}$$

https://doi.org/10.1017/S144678870000762X Published online by Cambridge University Press

where $(a_1 - a_5 \Lambda) R_1(\Lambda) = R_2 = (r_{ij})$ and, for $1 \leqq i \leqq m$, $r_{ii} = ia_1 a_4$, $r_{i+1,i} = -(i+1)a_1 a_4 - a_2$, $r_{i+2,i} = a_2$, and $r_{ij} = 0$ for $|i-j-1| > 1$. Now

$$(P - \Lambda Q) R(\Lambda) = \begin{pmatrix} (a_1 - a_5 \Lambda) P_1 & 0_{nm} \\ P_2^T & P_5(\Lambda) \end{pmatrix}$$

where $(a_1 - a_5 \Lambda) P_5(\Lambda) = P_6(\Lambda) = (p_{6ij}(\Lambda))$ and

$$p_{6ij}(\Lambda) = \frac{-2\Lambda ij a_1 a_4^2 a_5}{(i+j-1)(i+j)(i+j+1)} + \frac{2[a_1 a_3 - a_2^2 - (a_1 a_6 + a_3 a_5)\Lambda + a_5 a_6 \Lambda^2]}{(i+j+1)(i+j+2)(i+j+3)}. \tag{24}$$

Since P_1 is the Gram matrix of a linearly independent set of functions it is non-singular (in fact positive definite) and hence the characteristic equation $\det (P - \Lambda Q) = 0$, which determines the Ritz approximate eigenvalues, may be written

$$(a_1 - a_5 \Lambda)^{n-m} \det P_6(\Lambda) = 0. \tag{25}$$

The left hand side of (25) is identically equal to $(\det (P - \Lambda Q))/\det P_1$ for $\Lambda \neq a_1/a_5$ and hence, by continuity, for all Λ. The factor $(a_1 - a_5 \Lambda)^{n-m}$ gives the roots predicted by Theorem 3 and $\det P_6(\Lambda)$, a polynomial of degree $2m$ and independent of n, gives the others. Clearly $P_6((a_1 a_3 - a_2^2)/\Lambda a_5 a_6)$ is a scalar multiple of $P_6(\Lambda)$, so that $(a_1 a_3 - a_2^2)/\Lambda a_5 a_6$ is a zero of $\det P_6$ if and only if Λ is also. Hence the Ritz approximate eigenvalues occur in pairs satisfying (20).

The equation $(P - \Lambda Q)x = 0$ may be thought of as

$$(P - \Lambda Q) R(\Lambda) (R^{-1}(\Lambda) x) = 0. \tag{26}$$

Let x_1, x_2 be column vectors representing the first n and the last m elements respectively of x and let x_3, x_4 be the corresponding parts of $R^{-1}(\Lambda)x$. Then, for $\Lambda \neq a_1/a_5$, (26) may be written

$$P_1 x_3 = 0, \quad P_2^T x_3 + P_5(\Lambda) x_4 = 0. \tag{27}$$

For each eigenvalue Λ of (25) this has a non-zero solution and since P_1 is nonsingular, this solution must have $x_3 = 0$. From the form of R it is obvious that

$$R^{-1}(\Lambda) = \begin{pmatrix} I_n & -R_1(\Lambda) \\ 0_{mn} & I_m \end{pmatrix}$$

so that $x_2 = x_4$ and

$$x_1 - R_1(\Lambda) x_2 = x_3 = 0. \tag{28}$$

https://doi.org/10.1017/S144678870000762X Published online by Cambridge University Press

If $n-m > 2$ the $n-m-2$ bottom rows of R_1, and hence the last $n-m-2$ elements of x_1 are zero. Also x_4 and the first $m+2$ rows of R_1, and hence the first $m+2$ elements of x_1, are independent of n. Hence the Ritz approximate eigenfunctions depend only on m. Since $R_2 = (a_1-a_5\Lambda)R_1(\Lambda)$ is constant, and since (24) and (27) show that the same x_4 is obtained by the eigenvalues Λ and $(a_1a_3-a_2^2)/\Lambda a_5 a_6$ it follows from (15), (23), (28) and the definition of x_1 and x_2 that the corresponding Ritz approximate eigenfunctions satisfy (21). This completes the proof.

Theorem 4 shows that for each m, the Ritz approximate eigenvalues in this case will consist of $n-m$ eigenvalues equal to λ_0 together with m eigenvalues greater than or equal to $\Lambda_0 = (b_2b_3/b_4b_5)^{\frac{1}{2}}$ and m less than or equal to Λ_0. That is the Ritz approximate eigenvalues obtained in this case may be labelled λ_0 ($n-m$ times), $\Lambda_{1\pm}(m)$, $\Lambda_{2\pm}(m), \cdots, \Lambda_{m\pm}(m)$ where

$$\Lambda_{m-}(m) \leqq \cdots \leqq \Lambda_{2-}(m) \leqq \Lambda_{1-}(m) \leqq \Lambda_0 \leqq \Lambda_{1+}(m) \leqq \cdots \leqq \Lambda_{m+}(m).$$

Denote the Ritz approximate eigenfunction corresponding to $\Lambda_{k\pm}(m)$ by $\{U_{k\pm}(m), V_{k\pm}(m)\}$.

Theorem 4 shows that $\Lambda_{k-}(m)$ and $\{U_{k-}(m), V_{k-}(m)\}$ will be good approximations of λ_{k-} and $\{u_{k-}, v_{k-}\}$ *if and only if* $\Lambda_{k+}(m)$ *and* $\{U_{k+}(m), V_{k+}(m)\}$ *are good approximations of* λ_{k+} *and* $\{u_{k+}, v_{k+}\}$. This case contrasts with those results in [3] where p-modes were detected but not g-modes. However Theorem 4 gives no indication of whether $\Lambda_{k\pm}(m)$ and $\{U_{k\pm}(m), V_{k\pm}(m)\}$ will in fact be approximations of $\lambda_{k\pm}$ and $\{u_{k\pm}, v_{k\pm}\}$ in any meaningful sense, or whether they will be spurious like some of the modes discovered in [3]. To settle this question it is necessary to consider whether $\lim_{m\to\infty}\Lambda_{k\pm}(m)$ and $\lim_{m\to\infty}\{U_{k\pm}(m), V_{k\pm}(m)\}$ exist, and if so whether these limits are equal, or nearly equal, to $\lambda_{k\pm}$ and $\{u_{k\pm}, v_{k\pm}\}$, and how rapidly the limits are approached.

The methods normally used to discuss the convergence of the Ritz method (as in [1]) are not immediately applicable here as they use the fact that in standard problems the kth eigenvalue of (1) is the minimum of $(L_1w, w)/(L_2w, w)$ when w is subject to the restriction that it is orthogonal to the eigenfunctions corresponding to the $k-1$ lowest eigenvalues. In the present problem not only is there no minimum (or maximum) eigenvalue but also there are functions (not eigenfunctions) which are orthogonal to $\{u_0, v_0\}$ and yet give the same value of the ratio

$$(B_1\{u, v\}, \{u, v\})/(B_2\{u, v\}, \{u, v\})$$

as $\{u_0, v_0\}$. An example is

$$\{u(x), v(x)\} = \{\cos(2\pi nx+c)\exp b_1x, 0\}$$

where c is a constant and n an integer.

It is planned to consider the question of convergence in this and related problems in a later paper. In the meantime it may be noted that $\{U_{k\pm}(m), V_{k\pm}(m)\}$ resembles $\{u_{k\pm}, v_{k\pm}\}$ in several important ways (Theorem 5 and corollaries) and that for small m, $\Lambda_{k\pm}(m)$ and $\{U_{k\pm}(m), V_{k\pm}(m)\}$ are in fact good approximations of $\lambda_{k\pm}$ and $\{u_{k\pm}, v_{k\pm}\}$ (Theorem 6 and the succeeding paragraphs).

Define U_{km+}, U_{km-} and V_{km} to be the extensions of $U_{k+}(m)$, $U_{k-}(m)$ and $V_{k+}(m)$ $(= V_{k-}(m))$ to the whole real line.

THEOREM 5. (i) $(b_2-b_4\Lambda_{k\pm}(m))U_{km\pm} = V'_{km}-b_1V_{km}$.
(ii) *If* $\Lambda_{i+}(m) > \Lambda_{j+}(m)$, *then*

$$\int_0^1 V_{im}V_{jm} = \int_0^1 U_{im\pm}U_{jm\pm} = 0.$$

PROOF. (i) If

$$V_{km}(x) = x(1-x)\sum_{i=0}^{m-1} c_i x^i,$$

(28) shows that

$$\begin{aligned}(b_2-b_4\Lambda_{k\pm}(m))U_{km\pm}(x) &= \sum_{i=1}^{m}[ic_{i-1}x^{i-1}-(b_1+i+1)c_{i-1}x^i+b_1c_{i-1}x^{i+1}]\\ &= V'_{km}(x)-b_1V_{km}(x).\end{aligned}$$

(ii) Since $b_4 \neq 0$ and $\Lambda_{i+}(m) > \Lambda_{j+}(m) \geqq \Lambda_{j-}(m) > \Lambda_{i-}(m)$ it follows that $b_2-b_4\Lambda_{i+}(m) \neq b_2-b_4\Lambda_{i-}(m)$. By Theorem 2,

$$\int_0^1 (a_5U_{im+}U_{jm+}+a_6V_{im}V_{jm}) = 0 = \int_0^1 (a_5U_{im-}U_{jm-}+a_6V_{im}V_{jm}).$$

Combining these relations, using

$$(b_2-b_4\Lambda_{i+}(m))U_{im+} = (b_2-b_4\Lambda_{i-}(m))U_{im-},$$

gives the result.

COROLLARY 1. *If the zeros of* V_{km} *are all real, then between each pair of consecutive distinct zeros of* V_{km} *there is exactly one real zero of* $U_{km\pm}$.

PROOF. Since V_{km} is a polynomial with only real zeros there is exactly one real zero of V'_{km} between each pair of consecutive distinct zeros of V_{km}. Hence by (i) the sign of $U_{km\pm}$ immediately after a zero of V_{km} is opposite to its sign immediately before the next zero of V_{km}. Hence there is an odd number of real zeros of $U_{km\pm}$ between any two consecutive distinct zeros of V_{km}. Moreover at a zero of multiplicity r of V_{km}, $U_{km\pm}$ has a zero of multiplicity exactly $r-1$. Since $U_{km\pm}$ has the same number of zeros as V_{km} and the zeros of V_{km} are all real the result follows.

COROLLARY 2. *If* $\Lambda_{i+}(m) > \Lambda_{j+}(m)$ *then* $\int_0^1 V'_{im}V'_{jm} = 0$.

PROOF. Since $\int_0^1 (V'_{im}V_{jm}+V_{im}V'_{jm}) = 0$, the result follows by combining results (i) and (ii) of the Theorem.

THEOREM 6. *For all non-zero constants a_i satisfying* (12),

$$\lambda_{1+} < \Lambda_{1+}(3) < \Lambda_{1+}(2) = \Lambda_{1+}(1) < 1.014\lambda_{1+},$$
$$\lambda_{2+} < \Lambda_{2+}(3) = \Lambda_{2+}(2) < 1.064\,\lambda_{2+},$$
$$\lambda_{3+} < \Lambda_{3+}(3) < 1.15\lambda_{3+} \quad \textit{and} \quad \Lambda_{1+}(3) < 1.000015\lambda_{1+}.$$

PROOF. Define $f(\lambda)$ by

$$f(\lambda) = (a_1a_3-a_2^2-(a_1a_6+a_3a_5)\lambda+a_5a_6\lambda^2)/a_1a_4^2a_5.$$

Equation (17), which determines the eigenvalues $\lambda_{k\pm}$ may be written $f(\lambda_{k\pm}) = k^2\pi^2\lambda_{k\pm}$. Direct calculation shows that the equation $\det P_6(\lambda) = 0$, which determines the $\Lambda_{k\pm}(m)$, may be written as $f(\lambda)-10\,\lambda = 0$ when $m = 1$, as $(f(\lambda)-10\lambda)(f(\lambda)-42\lambda) = 0$ when $m = 2$, and as

$$(f(\lambda)-(56-4\sqrt{(133)})\lambda)(f(\lambda)-42\lambda)(f(\lambda)-(56+4\sqrt{(133)})\lambda) = 0$$

when $m = 3$. Since

$$0 < \pi^2 < 56-4\sqrt{(133)} < 10 < 1.014\,\pi^2,$$
$$4\pi^2 < 42 < 1.064(4\,\pi^2),$$
$$9\pi^2 < 56+4\sqrt{(133)} < 1.15(9\,\pi^2)$$

and
$$56-4\sqrt{(133)} < 1.000015\,\pi^2,$$

the result follows from the following lemma.

LEMMA. *If the a_i are non-zero and satisfy* (12), *and $K > 0$ the roots of $f(\lambda) = K\lambda$ are real, distinct and positive. If $K_2 > K_1 > 0$, then*

$$K_1\lambda^*(K_1) < K_1\lambda^*(K_2) < K_2\lambda^*(K_1)$$

where $\lambda^*(K)$ is the greater root of $f(\lambda) = K\lambda$.

PROOF. If the constants $c_1, c_2, c_3, c_4, c_2^2-4c_1c_4$ are all strictly positive, the roots of $c_1y^2-(c_2+c_3)y+c_4 = 0$ are real distinct and positive. Let $y(c_3)$ be the greater of these roots. Direct solution shows that if $\delta > 0$, then $y(c_3+\delta) > y(c_3)$. Also

$$y(c_3+\delta)-y(c_3)$$
$$= \frac{1}{2c_1}\left[\delta+(c_3+\delta)\left(1+\frac{2c_2}{c_3+\delta}+\frac{c_2^2-4c_1c_4}{(c_3+\delta)^2}\right)^{\frac{1}{2}} - c_3\left(1+\frac{2c_2}{c_3}+\frac{c_2^2-4c_1c_4}{c_3^2}\right)^{\frac{1}{2}}\right]$$
$$< \frac{\delta}{2c_1}\left[1+\left(1+\frac{2c_2}{c_3}+\frac{c_2^2-4c_1c_4}{c_3^2}\right)^{\frac{1}{2}}\right] < \frac{\delta}{c_3}\,y(c_3).$$

https://doi.org/10.1017/S144678870000762X Published online by Cambridge University Press

Since

$$a_5 a_6,\ a_1 a_6 + a_3 a_5,\ a_1 a_4^2 a_5,\ a_1 a_3 - a_2^2 \text{ and } (a_1 a_6 + a_3 a_5)^2 - 4 a_5 a_6 (a_1 a_3 - a_2^2)$$

are all strictly positive, the result follows.

The bounds given by Theorem 6 are uniform bounds. Finer bounds can be found for specific values of the a_i. Since, by Theorem 4,

$$\Lambda_{k+}(m)\Lambda_{k-}(m) = \lambda_{k+}\lambda_{k-},$$

bounds for the $\Lambda_{k-}(m)$ follow immediately from Theorem 6. These initial results exhibit a convergence similar to that encountered in standard problems. Note that for $m \leqq 3$, $\Lambda_{k-}(m) < \lambda_{k-} < \lambda_{k+} < \Lambda_{k+}(m)$. This is similar to the result in [14].

Since the value of $f(\lambda)/\lambda$ at the zeros of det $P_6(\lambda)$ in Theorem 4 is independent of the a_i, it follows that the eigenvectors x_4 of $P_6(\lambda)$, and hence the $V_{k\pm}(m)$, are also independent of the a_i, as are also the $v_{k\pm}$. Direct calculation shows that x_4 corresponding to $\Lambda_{2\pm}(3)$ is $\{1, -2, 0\}$. Hence $V_{2\pm}(3)$, like $v_{2\pm}$, has a zero at $\frac{1}{2}$ and no other zeros in $(0, 1)$. Moreover $V_{2\pm}(2) = V_{2\pm}(3)$. Also with $m = 3$, x_4 corresponding to the eigenvalue given by $f(\lambda) = (56 \pm 4\sqrt{(133)})\lambda$ is $\{5 \pm 3\sqrt{(133)}/7, -23 \mp 2\sqrt{(133)}, 23 \pm 2\sqrt{(133)}\}$. This shows that $V_{3\pm}(3)$ and $V_{1\pm}(3)$ have the same number of zeros in $(0, 1)$ as $v_{3\pm}$ and $v_{1\pm}$ respectively. The zeros of $V_{3\pm}(3)$ in $(0, 1)$ are the roots of $(23+2\sqrt{(133)})(y^2-y)+5+3\sqrt{(133)}/7 = 0$, which have the approximate values 0.32 and 0.68. The zeros of $v_{3\pm}$ in $(0, 1)$ are $\frac{1}{3}$, $\frac{2}{3}$. Similarly direct calculation shows that $V_{1\pm}(1) = V_{1\pm}(2)$ and that these functions, like $v_{1\pm}$ and $V_{1\pm}(3)$, have no zeros in $(0, 1)$.

It follows from Theorem 5 and Corollary 1 that for $m \leqq 3$, the zeros of $U_{k\pm}(m)$ follow a pattern similar to those of $u_{k\pm}$. Another point of resemblance between the two sets of vectors easily shown by direct calculation for $m \leqq 3$ is that $|U_{km\pm}(0)| = |U_{km\pm}(1)|$.

5. Concluding remarks

The results of the last section indicate that there exist cases not covered by the classical theory in which useful results may be obtained by the Ritz method. It remains to find general criteria of the reliability of the method in particular circumstances. Although, as noted in section 3, the results of [16] will not be representative of the effectiveness of application of the Ritz method to (2) for general models, and as noted in section 2, even [14] and [15] leave the efficacy of the method in doubt, it does seem that the results of [3] may not be typical.

Rounding errors, which affect all computer calculations such as those in [3], [14] and [15], were not considered in sections 3 and 4. The matrix

P_1 arising in Theorem 4 is the ill-conditioned Hilbert matrix [17] and some moderately ill-conditioned matrices were obtained in the calculations described in [3], as will always happen if the coordinate functions are not strongly minimal [18]. It is possible that the effect of rounding errors on the solution of (2) could be different when different stellar models are used, especially as (6) will approximate the true eigenfunctions of (2) more closely for some stellar models than for others. It is planned to consider the question of rounding errors in a later paper.

In section 2 it was suggested that the fact that the model used in [3], unlike those used in [14], [15] and [16], contains a core throughout which $A \equiv 0$, might be partly responsible for the difference in results. Comparison of equations (2) and (13a) shows that the condition $a_3 > a_2^2/a_1$ in (12) may be regarded as analogous to the condition $Adp/dr > 0$ with (2), while the conditions $a_1 > 0$, $a_5 > 0$, $a_6 > 0$ correspond to the conditions $\Gamma_1 p > 0$ and $\rho > 0$. Since $\Gamma_1 p$, ρ and $-dp/dr$ (unlike $-A$) are strictly positive, except perhaps at the boundaries, in *all* the stellar models considered here (and indeed in any realistic stellar model), it would seem of interest to see how many of the results of the last section continue to hold when the requirement $a_3 > a_2^2/a_1$ is relaxed, while the requirement that a_1, a_5 and a_6 be positive and all the a_i non-zero is still imposed.

If $a_3 < a_2^2/a_1$, (14) and (17) still hold and in this case the λ_{n-} eigenvalues of (13a) are all negative. (Very similar results have been obtained for IA and IB, considered in section 2, for stellar models with $A > 0$ throughout [5, p. 514]. More complicated results have been obtained [8] when A changes sign.) Theorems 3, 4 and 5 also still hold, and provided $a_1 a_6 + a_3 a_5 > 0$, Theorem 6 will hold also.

When $a_1 a_3 = a_2^2$, all functions $\{u, v\}$ in the domain of B_1, satisfying $u = a_4 v' - (a_2/a_1)v$ are eigenfunctions of (13a). The corresponding eigenvalue, which has infinite multiplicity, is zero. For the non-zero eigenvalues, many of the results of section 4, including equations (14), (17), (19), (25) and (28) and Theorems 3, 5(i) and 6, still hold.

The analogy between (13a) with $a_1 a_3 = a_2^2$ and IB with the convective model discussed in section 2 is manifest. With the stellar model used [3], all sufficiently differentiable functions satisfying the boundary conditions with ψ vanishing throughout the envelope (where $A \neq 0$) but not throughout the core, and with $\chi' = \psi' + (p'/\Gamma_1 p)\psi$ throughout the star, are eigenfunctions of (2) and hence of IB. The corresponding eigenvalue is zero, and again has infinite multiplicity. Results of Lebovitz [19] show that these functions are also eigenfunctions of IA for this model. Results of [11] (where the zero eigenvalues are not mentioned) show that with this model IB has, as well as the zero eigenvalues, the normal p-modes, f-mode and g-modes, although the eigenvalues obtained there for g-modes were small compared with those obtained by others for models where $A < 0$ throughout.

The relatively complicated spectrum when A vanishes in parts of the star could help explain why the method was less successful in [3] than in [14] and [15], although in practice rounding errors have introduced some (very small) non-zero values of A into the core of the model used in [3]. This should make the model resemble a real star still more closely [8]. However the spurious eigenfunctions bear no resemblance to the true eigenfunctions corresponding to zero eigenvalues.

References

[1] S. G. Mikhlin, *Variational methods in mathematical physics* (Pergamon, 1964).
[2] R. Courant and D. Hilbert, *Methods of mathematical physics*. Vol. 1 (Interscience, 1953).
[3] A. L. Andrew, 'Higher modes of non-radial oscillations of stars by variational methods', *Aust. J. Phys.* 20 (1967), 363—368.
[4] S. Chandrasekhar, 'A general variational principle governing the radial and the non-radial oscillations of gaseous masses', *Astrophys. J.* 139 (1964), 664—674.
[5] P. Ledoux and Th. Walraven, 'Variable stars', in S. Flügge, *Handbuch der Physik*, Vol. 51, (Springer, Berlin, 1958), 353—604.
[6] N. R. Lebovitz, 'On Schwarzschild's criterion for the stability of gaseous masses', *Astrophys. J.* 142 (1965), 229—242.
[7] T. G. Cowling, 'The non-radial oscillations of polytropic stars', *Monthly Notices Roy. Astronom. Soc.* 101 (1941) 367—375.
[8] P. Ledoux and P. Smeyers 'Sur le spectre des oscillations non radiales d'un modèle stellaire' *C. R. Acad. Sci. Paris, Sér. B* 262 (1966), 841—844.
[9] J. W. Owen, 'The non-radial oscillations of centrally condensed stars', *Monthly Notices Roy. Astronom. Soc.* 117 (1957), 384—392.
[10] R. Van der Borght, 'The evolution of massive stars initially composed of pure hydrogen', *Aust. J. Phys.* 17 (1964), 165—174.
[11] Wan Fook Sun and R. Van der Borght, 'Note on Cowling's method in the theory of non-radial oscillations of massive stars', *Aust. J. Phys.* 19 (1966), 467—470.
[12] A. L. Andrew, 'A note on the Ritz method with an application to overtone stellar pulsation theory', *J. Aust. Math. Soc.* 8 (1968), 275—286.
[13] S. Chandrasekhar and N. R. Lebovitz, 'Non-radial oscillations of gaseous masses', *Astrophys. J.* 140 (1964), 1517—1528.
[14] H. Robe and L. Brandt, 'Sur les oscillations non radiales des polytropes', *Ann. Astrophys.* 29 (1966), 517—524.
[15] J. L. Tassoul, 'Sur l'instabilité convective d'une masse gazeuse inhomogène', *Ann. Astrophys.* 30 (1967), 363—370.
[16] H. Robe, 'Sur les oscillations non radiales d'une sphère', *Acad. Roy. Belg. Bull. Cl. Sci.* 51 (1965), 598—603.
[17] M. Marcus, 'Basic theorems in matrix theory', *Nat. Bur. Standards Appl. Math. Ser.* 57 (1960).
[18] S. G. Mihlin (Mikhlin), 'The stability of the Ritz method', *Soviet Math. Dokl.* 1 (1960), 1230—1233.
[19] N. R. Lebovitz, 'On the onset of convective instability', *Astrophys. J.* 142 (1965), 1257—1260.
[20] A. L. Andrew, 'Non-radial oscillations of massive stars by variational methods', *Acad. Roy. Belg. Bull. Cl. Sci.* 54 (1968), 1046—1055).

La Trobe University
Melbourne