NOTE ON THE HARDY-LANDAU SUMMATION FORMULA

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Broadly speaking, the Hardy-Landau summation formula* is given by

$$\sum_{n=0}^{\infty} r(n)f(n) = \sum_{n=0}^{\infty} r(n)g(n) ,$$

where $r(n)$ is the number of integer solutions of the Diophantine equation $x^2 + y^2 = n$, and $f(x)$ and $g(x)$ are transforms with respect to the Watson kernel $\pi J_0(2\pi\sqrt{x})$, that is:

$$g(x) = \pi \int_0^{\infty} f(t)J_0(2\pi\sqrt{xt})dt$$

and

$$f(x) = \pi \int_0^{\infty} g(t)J_0(2\pi\sqrt{xt})dt .$$

It is the purpose of this note to show that, by means of chain transforms, the Hardy-Landau formula can be derived using kernels simpler than $\pi J_0(2\pi\sqrt{x})$.

DEFINITION. A function $f(x)$ is said to belong to the class $G^2(o,\infty)$ if

* One of the earliest versions of this summation formula appears in Landau [3] (Theorem 559, p. 274).

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(i) $f(x) = -\int_{x}^{\infty} f'(t)dt$,

and

(ii) $xf'(x)$ belongs to $L^2(o, \infty)$.

The class $G^2(o, \infty)$ is a subclass of $L^2(o, \infty)$ (see Miller [4], Theorem 2). Also, if $f(x) \in G^2(o, \infty)$, it is not difficult to show that $x^{-1}f(x^{-1}) \in G^2(o, \infty)$.

LEMMA. If $f(x) \in G^2(o, \infty)$, then there exists $g(x) \in G^2(o, \infty)$ such that

$$g(x) = 2 \int_{o}^{\to\infty} f(t) \cos 2\pi xt \, dt \qquad (x > 0)$$

and

$$f(x) = 2 \int_{o}^{\to\infty} g(t) \cos 2\pi xt \, dt \qquad (x > 0) .$$

A similar result holds for the kernel $\sin \frac{1}{2}\pi x$.

Proof. Miller [4], Theorem 1.

The following is our main result.

THEOREM 1. Let $f(x)$ be a function belonging to $G^2(o, \infty)$, and define $\phi(x) \in G^2(o, \infty)$ by the equation

$$(1) \qquad \phi(x) = 2 \int_{o}^{\to\infty} f(t) \cos 2\pi xt \, dt \qquad (x > 0) .$$

Let

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$$(2) \qquad g(x) = 2 \int_0^{\to\infty} t^{-1} \phi(t^{-1}) \sin \tfrac{1}{2}\pi x t \, dt \qquad (x > 0) .$$

Then

$$\lim_{N\to\infty} \{ \sum_{n=1}^{N} r(n)f(n) - \pi \int_0^N f(t)dt\} = \lim_{N\to\infty} \{ \sum_{n=1}^{N} r(n)g(n) - \pi \int_0^N g(t)dt\} .$$

Proof. By the lemma, $\phi(x) \in G^2(o, \infty)$, so $x^{-1}\phi(x^{-1}) \in G^2(o, \infty)$ in accordance with the remark following the definition of the class $G^2(o, \infty)$. Therefore, it follows from the lemma and equation (2) that $g(x) \in G^2(o, \infty)$.

Denote by $\mathcal{K}_1(s)$, $\mathcal{K}_2(s)$ $(s = \frac{1}{2} + it)$ the Mellin transforms of $2\cos 2\pi x$, $\sin \frac{1}{2}\pi x$ respectively. Then

$$\mathcal{K}_1(s) = 2(2\pi)^{-s}\Gamma(s) \cos \tfrac{1}{2}s\pi ,$$

$$\mathcal{K}_2(s) = (2/\pi)^{s}\Gamma(s) \sin \tfrac{1}{2}\, s\pi ,$$

and

$$\mathcal{K}_1(s)\mathcal{K}_2(s) = \frac{\pi^{1-2s}\Gamma(s)}{\Gamma(1-s)} = \mathcal{K}_3(s) .$$

But $\mathcal{K}_3(s)$ is just the Mellin transform of $\pi J_0(2\pi\sqrt{x})$. Therefore, appealing to results of Fox [1], we can conclude that

$$\int_0^x f(t)\, dt = \int_0^\infty g(t) \sqrt{x/t}\, J_1(2\pi\sqrt{xt})\, dt$$

and

$$\int_0^x g(t)\, dt = \int_0^\infty f(t) \sqrt{x/t}\, J_1(2\pi\sqrt{xt})\, dt .$$

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Finally, putting $a_n = r(n)$ and $\beta = 1$ in Theorem 2 of Guinand [2], we get $R_o(x) = \pi x$, and the following form of the Hardy-Landau formula results:

THEOREM 2. If $f(x)$ is an integral and $f(x)$ and $xf'(x)$ belong to $L^2(o, \infty)$, then

$$\lim_{N\to\infty} \left\{ \sum_{n=1}^{N} r(n)f(n) - \pi \int_o^N f(t)dt \right\} = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} r(n)g(n) - \pi \int_o^N g(t)dt \right\} ,$$

where

$$\int_o^x g(y)dy = \int_o^\infty f(y) \sqrt{x/y}\, J_1 (2\pi \sqrt{xy})dy ,$$

and $g(x)$ is chosen to be the integral of its derivative.

Combining these results, we obtain Theorem 1.

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