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Abstract

The saturation theorem of Knutson and Tao concerns the nonvanishing of Littlewood–Richardson coefficients. In combination with work of Klyachko, it implies Horn's conjecture about eigenvalues of sums of Hermitian matrices. This eigenvalue problem has a generalization to *majorized* sums of Hermitian matrices, due to S. Friedland. We further illustrate the common features between these two eigenvalue problems and their connection to Schubert calculus of Grassmannians. Our main result gives a Schubert calculus interpretation of Friedland's problem, via *equivariant* cohomology of Grassmannians. In particular, we prove a saturation theorem for this setting. Our arguments employ the aforementioned work together with recent work of H. Thomas and A. Yong.

1. Introduction and the main results

1.1 Eigenvalue problems of A. Horn and of S. Friedland

The eigenvalue problem for Hermitian matrices asks how imposing the condition $A+B=C$ on three $r\times r$ Hermitian matrices constrains their eigenvalues λ, μ, and ν, written as weakly decreasing vectors of real numbers. This problem was considered in the 19th century, and has reappeared in various guises since. A general survey is given in [Ful00b]; here we mention a few highlights of the story. Building on observations of H. Weyl, K. Fan, and others, A. Horn recursively defined a list of inequalities on triples $(\lambda,\mu,\nu)\in\mathbb{R}^{3r}$, and conjectured that these give a complete solution to the eigenvalue problem [Hor62]. The fact that these inequalities (or others that turn out to be equivalent) are necessary has been proved by several authors, including B. Totaro [Tot94] and A. Klyachko [Kly98]. Klyachko also established that his list of inequalities is sufficient, giving the first solution to the eigenvalue problem.

In fact, he showed more: the same inequalities give an asymptotic solution to the problem of which Littlewood–Richardson coefficients $c_{\lambda,\mu}^{\nu}$ are nonzero. More precisely, suppose λ,μ,ν are partitions with at most r parts. Klyachko showed that if $c_{\lambda,\mu}^{\nu}\neq 0$, then $(\lambda,\mu,\nu)\in\mathbb{Z}_{\geqslant 0}^{3r}$ satisfies his inequalities; conversely, if $(\lambda,\mu,\nu)\in\mathbb{Z}_{\geqslant 0}^{3r}$ satisfy his inequalities then $c_{N\lambda,N\mu}^{N\nu}\neq 0$ for some $N\in\mathbb{N}$. (Here, $N\lambda$ is the partition with each part of λ stretched by a factor of N.) Sharpening this last statement, Knutson–Tao [KT99] established the *saturation theorem*: $c_{\lambda,\mu}^{\nu}\neq 0$ if and only if $c_{N\lambda,N\mu}^{N\nu}\neq 0$. Combined with [Kly98], it follows that Klyachko's solution agrees with Horn's conjectured solution.

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The Littlewood–Richardson coefficients are structure constants for multiplication of Schur polynomials. Therefore, they can be alternatively interpreted as tensor product multiplicities in the representation theory of GL_n, or as intersection multiplicities in the Schubert calculus of Grassmannians. Indeed, [KT99] adopts the former viewpoint, providing conjectural extensions to other Lie groups. Subsequent work includes [BK10, KM08, Kum10, Res10, Sam12]; see also the references therein.

The main goal of this paper is to provide further evidence of the naturality of the connection of Horn's problem to Schubert calculus. We demonstrate how the connection persists for the following extension of this eigenvalue problem. Recall that a Hermitian matrix M *majorizes* another Hermitian matrix M' if $M - M'$ is positive semidefinite (its eigenvalues are all nonnegative). In this case, we write $M \geqslant M'$. Friedland [Fri00] considered the following question.

Which eigenvalues (λ, μ, ν) *can occur if* $A + B \geqslant C$*?*

His solution is in terms of linear inequalities, which includes Klyachko's inequalities, a trace inequality and some additional inequalities. Later, Fulton [Ful00a] proved the additional inequalities are unnecessary. See follow-up work by Buch [Buc06] and by Chindris [Chi06] (who extends the work of Derksen–Weyman [DW00]).

Our finding is that the solution to S. Friedland's problem also governs the *equivariant* Schubert calculus of Grassmannians. This connection parallels the Horn problem's connection to classical Schubert calculus, but separates the problem from GL_n-representation theory.

Let $C_{\lambda,\mu}^{\nu}$ be the equivariant Schubert structure coefficient (defined in § 1.2). The analogy with the earlier results is illustrated by the following theorem.

Theorem 1.1 (Equivariant saturation). *The coefficient* $C_{\lambda,\mu}^{\nu} \neq 0$ *if and only if* $C_{N\lambda,N\mu}^{N\nu} \neq 0$ *for any* $N \in \mathbb{N}$.

When $|\lambda| + |\mu| = |\nu|$ then $C_{\lambda,\mu}^{\nu} = c_{\lambda,\mu}^{\nu}$. Hence Theorem 1.1 actually generalizes the saturation theorem. That said, our proofs rely on the classical Horn inequalities and so do not provide an independent proof of the earlier results. In addition, we use the recent combinatorial rule for $C_{\lambda,\mu}^{\nu}$ developed by Thomas and the third author [TY12].[1]

1.2 Equivariant cohomology of Grassmannians

Let $\mathrm{Gr}_r(\mathbb{C}^n)$ denote the Grassmannian of r-dimensional subspaces $V \subseteq \mathbb{C}^n$. This space comes with an action of the torus $T = (\mathbb{C}^*)^n$ (induced from the action of T on $\mathbb{C}^n$). Therefore, it makes sense to discuss the *equivariant cohomology ring* $H_T^*\mathrm{Gr}_r(\mathbb{C}^n)$. This ring is an algebra over $\mathbb{Z}[t_1, \ldots, t_n]$. (A more complete exposition of equivariant cohomology may be found in, e.g., [Ful07].)

As a $\mathbb{Z}[t_1, \ldots, t_n]$-module, $H_T^*\mathrm{Gr}_r(\mathbb{C}^n)$ has a basis of *Schubert classes*. To define these, fix the flag of subspaces

$$F_\bullet : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n,$$

where F_i is the span of the standard basis vectors $e_n,\ e_{n-1}, \ldots, e_{n+1-i}$. For each Young diagram λ inside the $r \times (n-r)$ rectangle, which we denote by Λ, there is a corresponding *Schubert variety*, defined by

$$X_\lambda := \{V \subseteq \mathbb{C}^n \mid \dim(V \cap F_{n-r+i-\lambda_i}) \geqslant i,\ \text{for } 1 \leqslant i \leqslant r\}.$$

[1] The easy direction of (equivariant) saturation, $C_{\lambda,\mu}^{\nu} \neq 0 \Rightarrow C_{N\lambda,N\mu}^{N\nu} \neq 0$, can be proved directly by using this rule (or others). However, as in the classical situation, it is the converse that is nonobvious.

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Since X_λ is invariant under the action of T, and has codimension $2|\lambda|$, it determines a class $[X_\lambda]$ in $H_T^{2|\lambda|}\mathrm{Gr}_r(\mathbb{C}^n)$. As λ varies over all Young diagrams inside Λ, the classes $[X_\lambda]$ form a basis for $H_T^*\mathrm{Gr}_r(\mathbb{C}^n)$ over $\mathbb{Z}[t_1,\dots,t_n]$. Therefore, in $H_T^*\mathrm{Gr}_r(\mathbb{C}^n)$ we have

$$[X_\lambda]\cdot[X_\mu]=\sum_{\nu\subseteq\Lambda}C_{\lambda,\mu}^{\nu}[X_\nu],$$

where the coefficients $C_{\lambda,\mu}^{\nu}\in\mathbb{Z}[t_1,\dots,t_n]$ are the *equivariant Schubert structure coefficients*. By homogeneity, $C_{\lambda,\mu}^{\nu}$ is a polynomial of degree $|\lambda|+|\mu|-|\nu|$. In particular, this coefficient is zero unless $|\lambda|+|\mu|\geqslant|\nu|$.

The polynomials $C_{\lambda,\mu}^{\nu}$ depend on the parameters r and n, but our notation drops this dependency, with the following justification. First, we already fixed r. Next, the standard embedding $\iota\colon \mathrm{Gr}_r(\mathbb{C}^n)\hookrightarrow\mathrm{Gr}_r(\mathbb{C}^{n+1})$ induces a map $\iota^*\colon H_T^*\mathrm{Gr}_r(\mathbb{C}^{n+1})\to H_T^*\mathrm{Gr}_r(\mathbb{C}^n)$. Using superscripts to indicate where a subvariety lives, we have $\iota^{-1}X_\lambda^{(n+1)}=X_\lambda^{(n)}$, and therefore $\iota^*[X_\lambda^{(n+1)}]=[X_\lambda^{(n)}]$. Let us write H_T^* for the graded inverse limit of these equivariant cohomology rings, so it is an algebra over $\mathbb{Z}[t_1,t_2,\dots]$. Write $\hat\sigma_\lambda\in H_T^*$ for the stable limit of the Schubert classes $[X_\lambda^{(n)}]$. The same structure constants $C_{\lambda,\mu}^{\nu}$ describe $\hat\sigma_\lambda\cdot\hat\sigma_\mu$ in this limit, so we can work in that limit without reference to n.

1.3 Inequalities for $C_{\lambda,\mu}^{\nu}$ and for eigenvalues

We will deduce Theorem 1.1 from inequalities describing the nonvanishing of $C_{\lambda,\mu}^{\nu}$. Let $[r]:=\{1,2,\dots,r\}$. For any

$$I=\{i_1<i_2<\cdots<i_d\}\subseteq[r]$$

define the partition

$$\tau(I):=(i_d-d\geqslant\cdots\geqslant i_2-2\geqslant i_1-1).$$

This defines a bijection between subsets of $[r]$ of cardinality d and partitions whose Young diagrams are contained in a $d\times(r-d)$ rectangle. The following theorem combines the main results of [Kly98, KT99].

THEOREM 1.2 [Kly98, KT99]. *Let λ,μ,ν be partitions with at most r parts such that*

$$|\lambda|+|\mu|=|\nu|. \tag{1}$$

The following are equivalent:

(i) $c_{\lambda,\mu}^{\nu}\neq0$;

(ii) *for every $d<r$, and every triple of subsets $I,J,K\subseteq[r]$ of cardinality d such that $c_{\tau(I),\tau(J)}^{\tau(K)}\neq 0$, we have*

$$\sum_{i\in I}\lambda_i+\sum_{j\in J}\mu_j\geqslant\sum_{k\in K}\nu_k; \tag{2}$$

(iii) *there exist $r\times r$ Hermitian matrices A,B,C with eigenvalues λ,μ,ν such that $A+B=C$.*

We are now ready to state our main result, which is a generalization of Theorem 1.2.

THEOREM 1.3. *Let λ,μ,ν be partitions with at most r parts such that*

$$|\lambda|+|\mu|\geqslant|\nu|\quad\textit{and}\quad\max\{\lambda_i,\mu_i\}\leqslant\nu_i\quad\textit{for all } i\leqslant r. \tag{3}$$

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The following are equivalent:

(i) $C^{\nu}_{\lambda,\mu} \neq 0$;

(ii) *for every* $d < r$, *and every triple of subsets* $I, J, K \subseteq [r]$ *of cardinality* d *such that* $c^{\tau(K)}_{\tau(I),\tau(J)} \neq 0$, *we have*

$$\sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j \geqslant \sum_{k\in K} \nu_k;$$

(iii) *there exist* $r \times r$ *Hermitian matrices* A, B, C *with eigenvalues* λ, μ, ν *such that* $A + B \geqslant C$.

Theorem 1.3 asserts that the main recursive inequalities (ii) controlling nonvanishing of $C^{\nu}_{\lambda,\mu}$ are just Horn's inequalities (2). The only difference between the governing inequalities lies in (1) versus (3). Notice that the second condition in (3) is unnecessary in Theorem 1.2 since it is already implied by (1) combined with (2).

In fact, we will use Theorem 1.2 to prove Theorem 1.3. Moreover, the equivalence of conditions (ii) and (iii) is immediate from [Fri00, Ful00a]. Since the inequalities of Theorem 1.3 are homogeneous, the equivalence of (i) and (ii) immediately implies Theorem 1.1.

1.4 Further comparisons to the literature

The proof of the saturation theorem given in [KT99] is combinatorial, employing their *honeycomb model* for $c^{\nu}_{\lambda,\mu}$. In contrast, Belkale first geometrically proves the equivalence '(i) $\Leftrightarrow$ (ii)' of Theorem 1.2, and then deduces the saturation theorem as an easy consequence [Bel08]. By comparison, our main tool is again a new combinatorial model for $C^{\nu}_{\lambda,\mu}$; we similarly deduce (equivariant) saturation from the eigenvalue inequalities.

It seems plausible to give geometric proofs of our theorems, along the lines of [Bel08], using the equivariant moving lemma of the first author [And07]. This approach is especially pertinent where one does not have good combinatorial control of the equivariant Schubert coefficients, e.g., in the case of minuscule G/P, cf. [PS09]. (Belkale and Kumar [BK06] also consider the vanishing problem for classical Schubert structure constants associated to more general G/P.)

Schubert calculus on Grassmannians has two other basic extensions that have been extensively studied: quantum and K-theoretic Schubert calculus. It is therefore natural to ask if and how the Horn problem may extend in each of these directions.

The first of these was studied by Belkale [Bel08], who established a relationship between an eigenvalue problem for products of unitary matrices and analogues of the saturation and Horn theorems for quantum cohomology of Grassmannians. (A combined quantum-equivariant extension is plausible, and investigating this seems worthwhile, but we have not yet undertaken such an investigation.)

For the second, let $k^{\nu}_{\lambda,\mu}$ denote the K-theoretic structure constant with respect to the basis structure sheaves of Schubert varieties. The 'easy' implication $k^{\nu}_{\lambda,\mu} \neq 0 \implies k^{N\nu}_{N\lambda,N\mu} \neq 0$ is false in general. (However, it is not known if the converse is true.) For example, [Buc02, § 7] notes that $k^{(2,1)}_{(1),(1)} = -1$ but $k^{(4,2)}_{(2),(2)} = 0$. One can also check that the same partitions give a counterexample for saturation in T-equivariant K-theory, as well. Moreover, consider structure constants $\widetilde{k}^{\nu}_{\lambda,\mu}$ for the multiplication of the dual basis in K-theory. Using the rule of [TY10, Theorem 1.6], one checks that $\widetilde{k}^{N(2,1)}_{N(1),N(1)}$ is nonzero for $N = 1, 2, 3$ but zero for $N = 4$.

Summarizing, this paper addresses the remaining basic extension of Schubert calculus where a complete analogue of the saturation theorem exists. The result linking Friedland's problem to equivariant Schubert calculus gives further evidence towards the thesis that Schubert

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calculus is a natural perspective for Horn's problem. That said, some room for clarification of this thesis remains: on one hand, the polynomials $C_{\lambda,\mu}^{\nu}$ also have representation theoretic interpretations [MS99]; on the other hand, saturation fails in K-theoretic Schubert calculus (in three forms). Finding deeper connections and explanations for these phenomena seems an interesting possibility for future work.

1.5 Organization

In § 2, we give the proof of Theorem 1.3, assuming a fact about the equivariant coefficients (Proposition 2.1) that we use in our inductive proof. This in turn is proved in § 3, after a review of the combinatorial rule of [TY12].

2. Proof of Theorem 1.3

As remarked above, we only need to show that parts (i) and (ii) of Theorem 1.3 are equivalent. We run an induction on the degree $p = |\lambda| + |\mu| - |\nu|$, simultaneously with an induction on r. To do this, we identify two key nonvanishing criteria in the following proposition.

PROPOSITION 2.1. *Assume $C_{\lambda,\mu}^{\nu} \neq 0$. Then:*

(A) *$C_{\lambda,\mu^{\uparrow}}^{\nu} \neq 0$ for any $\mu \subset \mu^{\uparrow} \subseteq \nu$;*

(B) *if $|\nu| < |\lambda| + |\mu|$, then for any s such that $|\nu| - |\lambda| \leqslant s < |\mu|$, there is a $\mu^{\downarrow} \subset \mu$ with $|\mu^{\downarrow}| = s$ and $C_{\lambda,\mu^{\downarrow}}^{\nu} \neq 0$ (in particular, taking $s = |\nu| - |\lambda|$, we have $c_{\lambda,\mu^{\downarrow}}^{\nu} \neq 0$).*

We postpone the proof to the next section.

Proof of Theorem 1.3, (i) $\Rightarrow$ (ii). If $C_{\lambda,\mu}^{\nu} \neq 0$, then by Proposition 2.1(B), we can find $\lambda^{\downarrow} \subseteq \lambda$ such that $|\lambda^{\downarrow}| + |\mu| = |\nu|$ and $c_{\lambda^{\downarrow},\mu}^{\nu} \neq 0$. By Theorem 1.2, for any triple (I, J, K) such that $c_{\tau(I),\tau(J)}^{\tau(K)} \neq 0$, we have

$$\sum_{i\in I} \lambda_i^{\downarrow} + \sum_{i\in J} \mu_j \geqslant \sum_{k\in K} \nu_k.$$

Since $\sum_{i\in I} \lambda_i \geqslant \sum_{i\in I} \lambda_i^{\downarrow}$, (i) implies (ii), as desired. □

Recall the bijection between d-subsets $I \subseteq [r]$ and partitions $\lambda = \tau(I)$ in the $d \times (r-d)$ rectangle, and write $\sigma_I = \sigma_{\tau(I)}$ for the corresponding Schubert class in the *ordinary* cohomology ring $H^*\mathrm{Gr}_d(\mathbb{C}^r)$. Define $I^{\vee} \subseteq [r]$ as

$$I^{\vee} := \{r+1-i_d < \cdots < r+1-i_2 < r+1-i_1\};$$

this is the subset associated to the shape $\lambda^{\vee}$ which is defined by taking the complement of λ in $d \times (r-d)$ and rotating by 180°.

We need an alternative characterization of Theorem 1.3(ii).

LEMMA 2.2. *Let λ, μ, ν be partitions as in Theorem 1.3. Then λ, μ, ν satisfy condition (ii) of Theorem 1.3 if and only if for any triple (I, J, K) such that $\sigma_I\sigma_J\sigma_{K^{\vee}} \neq 0$ in the ordinary cohomology ring $H^*\mathrm{Gr}_d(\mathbb{C}^r)$, we have*

$$\sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j \geqslant \sum_{k\in K} \nu_k.$$

Proof. If $c_{\tau(I),\tau(J)}^{\tau(K)} \neq 0$, then $\sigma_I\sigma_J\sigma_{K^{\vee}} \neq 0$, so the inequalities of the lemma include those of Theorem 1.3(ii), which proves the 'if' statement. For the 'only if' statement, we first recall a well-known fact (with proof, for completeness).

Claim 2.3. *If $\sigma_\alpha\sigma_\beta\sigma_{\gamma^\vee} \neq 0$ then there exists $\widetilde{\gamma}$ such that $\widetilde{\gamma} \subseteq \gamma$ and $c_{\alpha,\beta}^{\widetilde{\gamma}} \neq 0$.*

Proof of Claim 2.3. We proceed by induction on

$$\Delta = |\alpha| + |\beta| + |\gamma^\vee|.$$

If $\Delta = d(r-d)$, then we can take $\widetilde{\gamma} = \gamma$. If $\Delta < d(r-d)$, then the product $\sigma_\alpha\sigma_\beta\sigma_{\gamma^\vee}$ does not lie in the top degree of $H^*\mathrm{Gr}_d(\mathbb{C}^r)$. Thus $\sigma_\alpha\sigma_\beta\sigma_{\gamma^\vee} \neq 0$ implies

$$\sigma_{(1)}\sigma_\alpha\sigma_\beta\sigma_{\gamma^\vee} \neq 0.$$

In particular, we can choose $\gamma^\uparrow$ such that $\gamma^\vee \subseteq \gamma^\uparrow$, $|\gamma^\uparrow| = |\gamma^\vee| + 1$ and $\sigma_\alpha\sigma_\beta\sigma_{\gamma^\uparrow} \neq 0$. By induction, we can choose $\widetilde{\gamma}^\vee$ such that $\gamma^\uparrow \subseteq \widetilde{\gamma}^\vee$ and $c_{\alpha,\beta}^{\widetilde{\gamma}} \neq 0$. But then $\gamma^\vee \subseteq \gamma^\uparrow \subseteq \widetilde{\gamma}^\vee$, and hence $\widetilde{\gamma} \subseteq \gamma$. This proves the claim. $\square$

Now, if $\sigma_I\sigma_J\sigma_{K^\vee} \neq 0$, then by the claim there exists $\widetilde{K}$ such that $\tau(\widetilde{K}) \subseteq \tau(K)$ and $c_{\tau(I),\tau(J)}^{\tau(\widetilde{K})} \neq 0$. Thus, the condition of Theorem 1.3(ii) implies

$$\sum_{i\in I}\lambda_i + \sum_{j\in J}\mu_j \geqslant \sum_{k\in\widetilde{K}}\nu_k,$$

and since $\tau(\widetilde{K}) \subseteq \tau(K)$, we have

$$\sum_{k\in\widetilde{K}}\nu_k \geqslant \sum_{k\in K}\nu_k.$$

Combining these two inequalities yields the inequality of the lemma. $\square$

Lemma 2.2 allows us to replace the inequalities of Theorem 1.3(ii) by a larger set of inequalities. That is, we instead use inequalities corresponding to (I, J, K) from the sets

$$S_d^r := \{(I, J, K) \subseteq [r]^3 \mid |I| = |J| = |K| = d \text{ and } \sigma_I\sigma_J\sigma_{K^\vee} \neq 0 \text{ in } H^*\mathrm{Gr}_d(\mathbb{C}^r)\},$$

for $d < r$. This larger class of inequalities allows us to perform the induction.

We will first need a result from [Ful00a]. Let $I = \{i_1 < i_2 < \cdots < i_d\}$ be a subset of $[r]$ of cardinality d and let F be a subset of $[d]$ of cardinality x. Define

$$I_F := \{i_f | f \in F\}.$$

If F is a subset of $[r-d]$ of cardinality y, then define

$$I_F^+ := I \cup (I^c)_F$$

where I^c denotes the complement of I in $[r]$.

Proposition 2.4 [Ful00a, Proposition 1]. *Let $(I, J, K) \in S_d^r$.*

(i) *If $(F, G, H) \in S_x^d$, then $(I_F, J_G, K_H) \in S_x^r$.*

(ii) *If $(F, G, H) \in S_y^{r-d}$, then $(I_F^+, J_G^+, K_H^+) \in S_{d+y}^r$.*

For any partitions λ and α, define $\phi(\lambda, \alpha)$ to be the partition with parts

$$\lambda_1, \ldots, \lambda_d, \alpha_1, \ldots, \alpha_d$$

arranged in weakly decreasing order.

Lemma 2.5. *Let λ, μ, ν and α, β, γ be partitions such that $C_{\lambda,\mu}^\nu \neq 0$ and $C_{\alpha,\beta}^\gamma \neq 0$. Then $C_{\phi(\lambda,\alpha),\phi(\mu,\beta)}^{\phi(\nu,\gamma)} \neq 0$.*

Proof. Since $C^{\nu}_{\lambda,\mu} \neq 0$ and $C^{\gamma}_{\alpha,\beta} \neq 0$, by Proposition 2.1(B), there exist partitions $\lambda^{\downarrow} \subseteq \lambda$ and $\beta^{\downarrow} \subseteq \beta$ such that $c^{\nu}_{\lambda^{\downarrow},\mu} \neq 0$ and $c^{\gamma}_{\alpha,\beta^{\downarrow}} \neq 0$. By Theorem 1.2(iii), there exist Hermitian matrices $A_1 + B_1 = C_1$ and $A_2 + B_2 = C_2$ such that A_i, B_i, C_i have eigenvalues $\lambda^{\downarrow}, \mu, \nu$ and $\alpha, \beta^{\downarrow}, \gamma$ respectively. Then

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

are Hermitian matrices with eigenvalues $\phi(\lambda^{\downarrow}, \alpha), \phi(\mu, \beta^{\downarrow}), \phi(\nu, \gamma)$ respectively.

Hence, $c^{\phi(\nu,\gamma)}_{\phi(\lambda^{\downarrow},\alpha),\phi(\mu,\beta^{\downarrow})} \neq 0$. We have

$$\phi(\lambda^{\downarrow}, \alpha) \subseteq \phi(\lambda, \alpha) \subseteq \phi(\nu, \gamma) \quad \text{and} \quad \phi(\mu, \beta^{\downarrow}) \subseteq \phi(\mu, \beta) \subseteq \phi(\nu, \gamma).$$

Thus, by Proposition 2.1(A), we conclude $C^{\phi(\nu,\gamma)}_{\phi(\lambda,\alpha),\phi(\mu,\beta)} \neq 0$. ◻

Remark 2.6. The converse of Lemma 2.5 does not hold. For example, we have $c^{(3,2,1,1)}_{(3),(2,1,1)} \neq 0$, but $c^{(2,1,1)}_{(3),(1)} = c^{(3)}_{\emptyset,(2,1)} = 0$.

We will need the following lemma.

Lemma 2.7. *Let μ, ν be partitions with at most r parts such that $\mu \subseteq \nu$, and let I, J be subsets of $[r]$ of cardinality d. If $\tau(I) \subseteq \tau(J)$, then $\mu_{I^c} \subseteq \nu_{J^c}$.*

Proof. Let $I = (i_1 < \cdots < i_d)$ and $J = (j_1 < \cdots < j_d)$. Since $\tau(I) \subseteq \tau(J)$, we have that $i_k \leqslant j_k$ for all $k \in [d]$. This implies that if

$$I^c = (i'_1 < \cdots < i'_{r-d}) \quad \text{and} \quad J^c = (j'_1 < \cdots < j'_{r-d}),$$

then $i'_k \geqslant j'_k$ for all $k \in [r-d]$. Now $\mu_{i'_k} \leqslant \nu_{j'_k}$, since $\mu \subseteq \nu$. ◻

Proof of Theorem 1.3. (ii) ⇒ (i): Let $p := |\lambda| + |\mu| - |\nu|$. We prove the converse by a double induction on p and r. If $p = 0$, then (ii) implies (i) by Theorem 1.2. The second induction is on r. In particular, if $r = 1$, then $C^{\nu}_{\lambda,\mu} \neq 0$ if and only if $\lambda_1 + \mu_1 \geqslant \nu_1$. These are the base cases of our induction.

Now assume $p > 0$ and $r > 1$. Suppose that (λ, μ, ν) satisfies the inequalities given by triples of subsets in S^r_d for all $d < r$. In order to reach a contradiction, suppose $C^{\nu}_{\lambda,\mu} = 0$.

Since $p > 0$, we can assume that $|\lambda| \geqslant 1$. Remove any box from λ, to obtain a subpartition $\lambda^{\downarrow} \subseteq \lambda$ with $|\lambda| = |\lambda^{\downarrow}| + 1$. By Proposition 2.1(A) and condition (3) of Theorem 1.3, we know $C^{\nu}_{\lambda^{\downarrow},\mu} = 0$. By induction on p, since $|\lambda^{\downarrow}| + |\mu| - |\nu| = p - 1$, we can choose an inequality $(I, J, K) \in S^r_d$ satisfied by (λ, μ, ν) but not by $(\lambda^{\downarrow}, \mu, \nu)$. It is therefore true that

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j = \sum_{k \in K} \nu_k. \tag{4}$$

Let $\lambda_I := (\lambda_{i_1} \geqslant \cdots \geqslant \lambda_{i_d})$ and similarly define μ_J and ν_K. Note that (4) is the statement $|\lambda_I| + |\mu_J| = |\nu_K|$. By assumption, (λ, μ, ν) satisfies (I, J, K). If $(F, G, H) \in S^d_x$ (for any $x < d$) then by Proposition 2.4(i), (λ, μ, ν) also satisfies $(I_F, I_G, K_H) \in S^r_x$. This is the same as saying that $(\lambda_I, \mu_J, \nu_K)$ satisfies the inequalities $(F, G, H) \in S^d_x$ for all $x < d$. Thus, by Theorem 1.2, $c^{\nu_K}_{\lambda_I,\mu_J} \neq 0$.

Consider the partitions $\lambda_{I^c}, \mu_{J^c}, \nu_{K^c}$, where I^c, J^c, K^c are the complements of I, J, K in $[r]$. By Claim 2.3, $\tau(I) \subseteq \tau(K)$, so by Lemma 2.7, λ_{I^c} is a subpartition of ν_{K^c}. Similarly, we have that μ_{J^c} is a subpartition of ν_{K^c}. Now, again using our assumption that (λ, μ, ν) satisfies the inequalities $(I, J, K) \in S_d^r$, it follows from Proposition 2.4(ii) and (4) that $(\lambda_{I^c}, \mu_{J^c}, \nu_{K^c})$ satisfies the inequalities $(F, G, H) \in S_y^{r-d}$ for $y < r - d$. Since $r - d < r$, we have by induction that $C_{\lambda_{I^c},\mu_{J^c}}^{\nu_{K^c}} \neq 0$.

Since $c_{\lambda_I,\mu_J}^{\nu_K} \neq 0$ and $C_{\lambda_{I^c},\mu_{J^c}}^{\nu_{K^c}} \neq 0$, Lemma 2.5 implies that $C_{\lambda,\mu}^{\nu} \neq 0$, which contradicts our original assumption. This completes the proof. $\square$

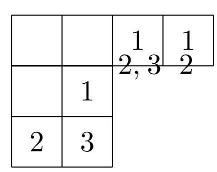
3. Proof of Proposition 2.1

3.1 A Littlewood–Richardson rule via edge-labeled tableaux

There are several combinatorial rules for computing the equivariant structure constants $C_{\lambda,\mu}^{\nu}$; see, e.g., [MS99] and [KT03] for some early ones (the latter being the first one to manifest the 'Graham-positivity' of the polynomials $C_{\lambda,\mu}^{\nu}$). However, in order to prove Proposition 2.1 we use a more recent rule of [TY12].

Consider Young diagrams λ, μ, ν inside Λ, with $\lambda, \mu \subseteq \nu$. An *equivariant Young tableau* of shape ν/λ and content μ is a filling of the boxes of the skew shape ν/λ, and labeling of some of the edges, by integers $1, 2, \ldots, |\mu|$, where 1 appears μ_1 times, 2 appears μ_2 times, etc. The edges that can be labeled are the horizontal edges of boxes in ν/λ, as well as edges along the southern border of λ; several examples are given below. The tableau is *semistandard* if the box labels weakly increase along rows (left to right), and all labels strictly increase down columns. A single edge may be labeled by a *set* of integers, without repeats; the smallest of them must be strictly greater than the label of the box above, and the largest must be strictly less than the label of the box below.

Example 3.1. Below is an equivariant semistandard Young tableau on $(4, 2, 2)/(2, 1)$.



The content of this tableau is $(3, 3, 2)$.

Let $\mathtt{EqSSYT}(\nu/\lambda)$ be the set of all equivariant semistandard Young tableaux of shape ν/λ. A tableau $T \in \mathtt{EqSSYT}(\nu/\lambda)$ is *lattice* if, for every column c and every label ℓ, we have:

$$(\#\text{ labels } \ell\text{'s weakly right of column } c) \geqslant (\#\text{ labels } (\ell+1)\text{'s weakly right of column } c).$$

The lattice condition can also be phrased in terms of the *column reading word* $w(T)$, which is obtained by reading the columns from top to bottom, starting with the rightmost column and moving to the left. (In the example above, $w(T) = 1\,2\,1\,2\,3\,1\,3\,2$.) The tableau T is lattice if and only if $w(T)$ is a lattice word; that is, for each ℓ and each p, among the first p letters of $w(T)$, the number of labels ℓ that appear is at least the number of labels $\ell + 1$.

Given a tableau $T \in \mathtt{EqSSYT}(\nu/\lambda)$, a (box or edge) label ℓ is *too high* if it appears weakly above the upper edge of a box in row ℓ. In the above example, all edge labels are too high. (When there are no edge labels, the semistandard and lattice conditions imply no box label is too high, but in general the three conditions are independent.)

Each box in the $r \times (n - r)$ rectangle Λ has a *distance* from the lower-left box: using matrix coordinates for a box $\mathtt{x} = (i, j)$, we define

$$\mathrm{dist}(\mathtt{x}) = r + j - i.$$

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Now suppose an edge label ℓ lies on the bottom edge of a box $\mathtt{x}$ in row i. Let $\rho_\ell(\mathtt{x})$ be the number of times ℓ appears as a (box or edge) label strictly to the right of $\mathtt{x}$. We define

$$\mathtt{apfactor}(\ell, \mathtt{x}) = t_{\mathrm{dist}(\mathtt{x})} - t_{\mathrm{dist}(\mathtt{x})+i-\ell+1+\rho_\ell(\mathtt{x})}. \tag{5}$$

When the edge label is not too high, this is always of the form $t_p - t_q$, for $p < q$. (In particular, it is nonzero.) Finally, we define[2] the *weight* of $T \in \mathtt{EqSSYT}$ by

$$\mathtt{apwt}(T) = \prod \mathtt{apfactor}(\ell, \mathtt{x}), \tag{6}$$

the product being over all edge labels ℓ.

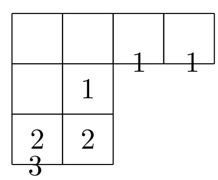
We can now state a combinatorial rule for equivariant Schubert calculus.

Theorem 3.2 [TY12, Theorem 3.1]. *We have $C^{\nu}_{\lambda,\mu} = \sum_T \mathtt{apwt}(T)$, where the sum is over all $T \in \mathtt{EqSSYT}(\nu/\lambda)$ of content μ that are lattice and have no label which is too high.*

This is a nonnegative rule: from the definition of the weights, complete cancellation is impossible, so the existence of one such T means that $C^{\nu}_{\lambda,\mu}$ is nonzero. In particular, we have the following corollary.

Corollary 3.3. *The coefficient $C^{\nu}_{\lambda,\mu}$ is nonzero if and only if there exists a tableau $T \in \mathtt{EqSSYT}(\nu/\lambda)$ of content μ which is lattice and has no label which is too high.*

Example 3.4. Consider the following lattice and semistandard tableau.



The associated $\mathtt{apwt}$ is $(t_1 - t_2)(t_4 - t_6)(t_5 - t_6)$; hence $C^{(4,2,2)}_{(4,1),(3,2,1)} \neq 0$.

3.2 Proof of Proposition 2.1

Our arguments for (A) and (B) are combinatorial, and both are based on Corollary 3.3.

Proof of (A). Let T be a witnessing tableau for $C^{\nu}_{\lambda,\mu} \neq 0$. That is, T is an (equivariant) semistandard tableau of shape ν/λ that is lattice, has content μ, and has no label that is too high. If $\mu = \nu$ then the desired assertion is trivial. Otherwise, by induction we quickly reduce to the case that $\mu^{\uparrow}/\mu$ is a single box. Suppose this additional box is a corner added to row ℓ of the shape of μ. Our goal is to construct $T^{\uparrow}$ by adding a single edge label ℓ to T so that $T^{\uparrow}$ witnesses $C^{\nu}_{\lambda,\mu^{\uparrow}} \neq 0$.

Procedure to obtain $T^{\uparrow}$: find the leftmost column c such that:

- c does not already have ℓ in the same column; and
- placing ℓ in that column as an edge label does not make that new ℓ too high.

Place ℓ in column c as an edge label. (This placement is uniquely determined.)

First we need to establish the following claim.

Claim 3.5. *The column c exists.*

[2] In [TY12], this is called the '*a priori* weight', to distinguish it from a weight arising from a sliding algorithm; hence the prefix '$\mathtt{ap}$'.

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Proof. To reach a contradiction, suppose otherwise. First assume there is a column d of T that does not have ℓ in it. We can take this column to be leftmost among all choices.

By our assumption for contradiction, we could not insert ℓ into column d because doing so would make it too high. Thus the edge we would put ℓ into (as forced by semistandardness) is the upper edge of the box in row ℓ, or higher. If there is a box label m in the box of row ℓ in that column, then $m > \ell$ (by assumption). But then m was too high in T, a contradiction. Hence it must be the case that the dth column of ν has at most $\ell - 1$ boxes. Since there were labels ℓ in each of the columns to the left, we conclude that the corner box $\mu^{\uparrow}/\mu$ must be in column d or to the right. This means the dth column of $\mu^{\uparrow}$ has at least ℓ boxes, which contradicts the assumption $\mu^{\uparrow} \subseteq \nu$. Finally, if column d does not exist, i.e., every column of T has an ℓ in it, then $\mu_\ell = \nu_\ell$ and thus $\mu^{\uparrow} \not\subseteq \nu$, which is a contradiction. □

Since we are adding ℓ to an edge, the assumptions imply that $T^{\uparrow}$ is semistandard (as the horizontal semistandard condition is vacuous here).

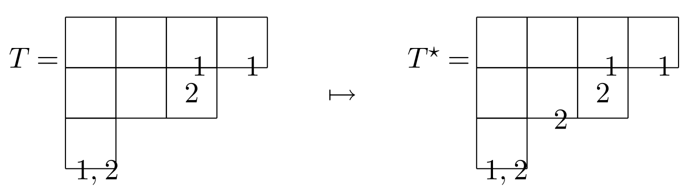
Claim 3.6. *The tableau $T^{\uparrow}$ is lattice.*

Proof. Suppose $T^{\uparrow}$ is not lattice. So there is a column d with strictly more labels ℓ than labels $(\ell - 1)$ in the region R consisting of columns weakly to the right. Notice that since T is lattice and we put an additional ℓ in column c, then column d must be weakly left of column c.

Before inserting the ℓ, the region R had an equal number of labels ℓ and $(\ell - 1)$. Since we could put an ℓ into column c and not be too high, we could put an edge label in each of the columns strictly left of column d, unless they all had ℓ in them. However, in that case, since T is lattice, those columns must each also contain $\ell - 1$. Thus, $\mu_{\ell-1} = \mu_\ell$. Hence we could not add a corner in row ℓ to obtain $\mu^{\uparrow}$, which is a contradiction.

This completes the proof of (A). □

Example 3.7. We illustrate the procedure below.



Above, T witnesses $C^{(4,3,1)}_{(4,2,1),(3,2)} \neq 0$ and $T^{\star}$ witnesses $C^{(4,3,1)}_{(4,2,1),(3,3)} \neq 0$.

Proof of (B). As in the proof of (A), let T be a witnessing tableau for $C^{\nu}_{\lambda,\mu} \neq 0$. We will modify T to obtain $T^{\star}$ that witnesses $C^{\nu}_{\lambda,\mu^{\star}} \neq 0$, where $\mu^{\star} \subset \mu$ and $|\mu/\mu^{\star}| = 1$. ($T^{\star}$ will have one fewer edge label than T.) The claim (B) follows by using this procedure to obtain a sequence of tableaux, each with one fewer edge label, until there are no edge labels.

Procedure to obtain $T^{\star}$: we introduce some temporary notation. For a word w, a position p, and a letter ℓ, let $N(w, p, \ell)$ be the number of occurrences of ℓ among the first p letters of w. Thus the lattice condition is that $N(w, p, \ell) \geqslant N(w, p, \ell + 1)$ for all ℓ and all p.

Now consider the (top to bottom, right to left) column reading word $w(T)$, and find the last occurrence of an edge label; say this label is ℓ, occurring in position $p^{(1)}$. Remove this label to obtain a new word $w^{(1)}$, and continue reading along the word, letter by letter. At any position $q \geqslant p^{(1)}$, and for any letter $k \neq \ell$, we have

$$N(w^{(1)}, q, k) = N(w(T), q, k) \quad \text{and} \quad N(w^{(1)}, q, \ell) = N(w(T), q, \ell) - 1.$$

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If at some position $p^{(2)} \geqslant p^{(1)}$, the lattice condition is violated in $w^{(1)}$, it must be because a letter $\ell+1$ appeared, causing

$$N(w^{(1)}, p^{(2)}, \ell+1) > N(w^{(1)}, p^{(2)}, \ell).$$

Fix this violation by replacing this problematic $\ell+1$ with ℓ, and call the resulting word $w^{(2)}$. Note that $w^{(2)}$ is lattice up to position $p^{(2)}$; moreover, for any $q \geqslant p^{(2)}$ and any $k \neq \ell+1$, we have

$$N(w^{(2)}, q, k) = N(w(T), q, k) \quad \text{and} \quad N(w^{(2)}, q, \ell+1) = N(w(T), q, \ell+1) - 1.$$

Continue in this way until the end of the word is reached, and call the result $w^\star$.

By construction, $w^\star$ is a lattice word. Furthermore, after the ℓ removed in the first step, the only letters in which $w^\star$ differs from $w(T)$ correspond to box labels. So there is no ambiguity in how to place these entries to create a tableau T^* of the same shape as T.

Let $\mu^\star$ be the content of $T^\star$. The argument is completed by Lemma 3.8 below. □

LEMMA 3.8. *The tableau $T^\star$ witnesses $C^{\nu}_{\lambda,\mu^\star} \neq 0$. In particular:*

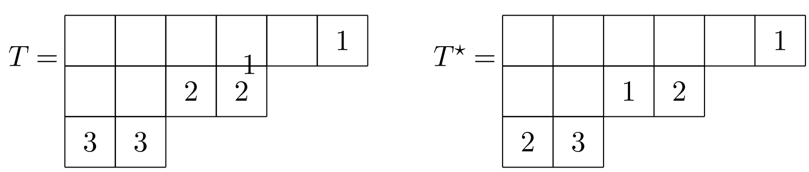
(i) *$T^\star$ is lattice;*

(ii) *$T^\star$ is semistandard;*

(iii) *no label of $T^\star$ is too high.*

Before proving the lemma, we illustrate the procedure with an example.

Example 3.9. Consider the tableau T.



The reading word is $w(T) = 1\,\underline{1}\,2\,2\,3\,3$, with the edge label underlined. Removing this letter, we have a violation of the lattice condition in the third position, which is fixed by

$$w^{(1)} = 1\,2\,2\cdots \rightsquigarrow w^{(2)} = 1\,2\,1\,\ldots.$$

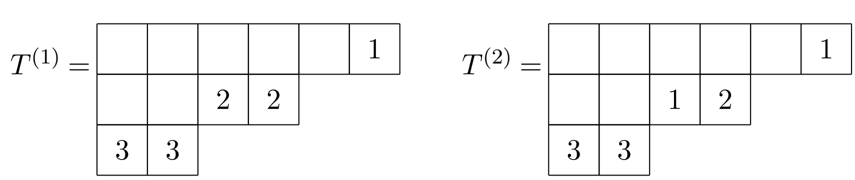
Continuing, another violation is at the fifth (and last) position, and we fix it as before by

$$w^{(2)} = 1\,2\,1\,3\,3 \rightsquigarrow w^{(3)} = 1\,2\,1\,3\,2.$$

The corresponding tableau $T^\star$ is shown above.

For use in the proof of Lemma 3.8, it will be convenient to let $T^{(i)}$ denote the tableau corresponding to an intermediate word $w^{(i)}$. This may not be a lattice tableau, but from the construction $T^{(i)}$ does satisfy the lattice condition with respect to labels $\ell+i$ and $\ell+i+1$.

Example 3.10. The intermediate tableaux for Example 3.9 are shown below.



Fixing $T^{(2)}$ gives $T^{(3)} = T^\star$, which was already given above.

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Proof of Lemma 3.8. As was already observed, (i) is true by construction. The claim (iii) is also easy to verify: we are given that no label of T is too high, and removing the initial edge label does not change this. Moreover, each step of our procedure only changes a box label to a label that is one smaller. If such a box label was not too high, replacing it by a smaller label will not change this either.

It remains to establish (ii), which we will do by induction on the steps of the procedure. To obtain $T^{(i+1)}$ from $T^{(i)}$, a box label $\ell+i$ is replaced by $\ell+i-1$. Let $c^{(i)}$ be the column where this replacement occurs (and let $c^{(0)}$ be the column of the edge label ℓ that was initially removed from T).

Claim 3.11. *Suppose $i \geqslant 1$. In column $c^{(i)}$ of $T^{(i)}$ there is an $\ell+i$ (by assumption), but there is no $\ell+i-1$.*

Proof. Were this not the case, the lattice condition would be violated by a label $\ell+i$ occurring earlier in the reading word $w^{(i)}$, but by construction we chose the first violation. □

Now assume $T^{(i)}$ is semistandard. In the following argument, it will help to refer to a diagram illustrating the replacement taking $T^{(i)}$ to $T^{(i+1)}$, locally:

$$T^{(i)} = \begin{array}{|c|c|c|}\hline \mathsf{a} & \mathsf{b} & \mathsf{c} \\ \hline \mathsf{d} & \ell+i & \mathsf{e} \\ \hline \mathsf{f} & \mathsf{g} & \mathsf{h} \\ \hline \end{array} \to \begin{array}{|c|c|c|}\hline \mathsf{a} & \mathsf{b} & \mathsf{c} \\ \hline \mathsf{d} & \ell+i-1 & \mathsf{e} \\ \hline \mathsf{f} & \mathsf{g} & \mathsf{h} \\ \hline \end{array} = T^{(i+1)}.$$

(If the second column in the above local diagram is $c^{(0)}$ there could be edge labels above the b or in the columns to the right. However, these labels do not affect the argument.)

Claim 3.12. *The tableau $T^{(i+1)}$ is semistandard.*

Proof. Clearly we have $\ell+i-1 \leqslant \mathsf{e}$ and $\ell+i-1 < \mathsf{g}$. That $\mathsf{b} < \ell+i-1$ is clear from the semistandardness of $T^{(i)}$ combined with Claim 3.11. It remains to show $\mathsf{d} \leqslant \ell+i-1$.

We have $\mathsf{d} \leqslant \ell+i$, so in order to reach a contradiction, let us assume $\mathsf{d} = \ell+i$. There are two cases. First, if $\mathsf{a} = \ell+i-1$, then by the semistandardness of $T^{(i)}$ we have $\mathsf{b} = \ell+i-1$, contradicting Claim 3.11. Second, if $\mathsf{a} < \ell+i-1$ (or if a does not exist), then it follows from Claim 3.11 that in $T^{(i-1)}$, the label $\mathsf{d} = \ell+i$ witnesses that $T^{(i-1)}$ is not lattice with respect to $\ell+i-1$ and $\ell+i$, a contradiction since we know $T^{(i-1)}$ is lattice with respect to these labels. Hence $\mathsf{d} < \ell+i$, and $T^{(i+1)}$ is semistandard. □

This completes the proof of the lemma. □

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David Anderson andersond@math.jussieu.fr
FSMP–Institut de Mathématiques de Jussieu, 75013 Paris, France

Current address: Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, Brasil

Edward Richmond erichmond@math.ubc.ca,ed.richmond@gmail.com
Department of Mathematics, University of British Columbia, UBC Room 121,
1984 Mathematics Road, Vancouver, BC, V6T 1Z2, Canada

Alexander Yong ayong@uiuc.edu
Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana,
IL 61801, USA