

Modular Reduction in Abstract Polytopes

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

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Abstract. The paper studies modular reduction techniques for abstract regular and chiral polytopes, with two purposes in mind: first, to survey the literature about modular reduction in polytopes; and second, to apply modular reduction, with moduli given by primes in $\mathbb{Z}[\tau]$ (with τ the golden ratio), to construct new regular 4-polytopes of hyperbolic types $\{3, 5, 3\}$ and $\{5, 3, 5\}$ with automorphism groups given by finite orthogonal groups.

1 Introduction

Polytopes and their symmetry have inspired mathematicians since antiquity. In the past three decades, a modern abstract theory of polytopes has emerged featuring an attractive interplay of mathematical areas, including geometry, combinatorics, group theory, graph theory and topology [20]. Abstract polytopes share many properties with ordinary convex polytopes but a priori are not embedded in the geometry of an ambient space.

The present paper studies modular reduction techniques for regular and chiral abstract polytopes. Modular reduction has proved to be a powerful tool in the construction and analysis of new classes of polytopes. Our paper serves two purposes: first, to survey the literature about modular reduction techniques in polytopes; and second, to apply modular reduction, with moduli given by primes in the ring of integers $\mathbb{Z}[\tau]$ of the quadratic number field $\mathbb{Q}(\sqrt{5})$, to construct new infinite classes of regular 4-polytopes of hyperbolic types $\{3, 5, 3\}$ and $\{5, 3, 5\}$ with automorphism groups given by orthogonal groups over finite fields.

Last, but not least, as a birthday greeting, we wish to acknowledge the many contributions by our friend and colleague Ted Bisztriczky in convexity, polytope theory and combinatorial geometry (see also [2]).

2 Basic Notions and Methods

For general background material on abstract polytopes we refer the reader to [20, Chapters 2,3]. Here we just review some basic terminology.

An (*abstract*) *polytope of rank n* , or an *n -polytope*, is a partially ordered set \mathcal{P} with a strictly monotone rank function having range $\{-1, 0, \dots, n\}$. An element of rank j is

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called a j -face of \mathcal{P} , and a face of rank 0, 1 or $n - 1$ is a *vertex*, *edge* or *facet*, respectively. The maximal chains, or *flags*, of \mathcal{P} all contain exactly $n + 2$ faces, including a unique least face F_{-1} (of rank -1) and a unique greatest face F_n (of rank n). Two flags are said to be *adjacent* (i -*adjacent*) if they differ in a single face (only their i -face, respectively). Then \mathcal{P} is required to be *strongly flag-connected* [20, Chapter 2]. Finally, \mathcal{P} has the following homogeneity property: whenever $F \leq G$, with F a $(j - 1)$ -face and G a $(j + 1)$ -face for some j , then there are exactly two j -faces H with $F \leq H \leq G$.

Whenever $F \leq G$ are faces of ranks $j \leq k$ in \mathcal{P} , the *section*

$$G/F := \{H \in \mathcal{P} \mid F \leq H \leq G\}$$

is thus a $(k - j - 1)$ -polytope in its own right. In particular, we can identify F with F/F_{-1} . Moreover, we call F_n/F the *co-face at F* , or the *vertex-figure at F* if F is a vertex.

Our interest is primarily in regular or chiral polytopes. A polytope \mathcal{P} is *regular* if its *automorphism group* $\Gamma(\mathcal{P})$ is transitive on the flags of \mathcal{P} , and \mathcal{P} is *chiral* if $\Gamma(\mathcal{P})$ has two flag orbits such that adjacent flags are always in distinct orbits.

For a regular polytope \mathcal{P} , the group $\Gamma(\mathcal{P})$ is generated by n involutions $\rho_0, \dots, \rho_{n-1}$, where ρ_i maps a fixed, or *base*, flag Φ to the flag Φ^i , i -adjacent to Φ . These generators satisfy (at least) the standard Coxeter-type relations

$$(2.1) \quad (\rho_i \rho_j)^{p_{ij}} = \epsilon \quad \text{for } i, j = 0, \dots, n - 1,$$

where $p_{ii} = 1$, $p_{ji} = p_{ij} =: p_{i+1}$ if $j = i + 1$, and $p_{ij} = 2$ otherwise; thus the underlying Coxeter diagram is a string diagram. Note that $p_j := p_{j-1, j} \in \{2, 3, \dots, \infty\}$ for $j = 1, \dots, n - 1$. These numbers determine the (*Schläfli*) *type* $\{p_1, \dots, p_{n-1}\}$ of \mathcal{P} . Moreover, the following *intersection condition* holds:

$$(2.2) \quad \langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \quad (I, J \subseteq \{0, 1, \dots, n - 1\}).$$

The rotations $\sigma_i := \rho_i \rho_{i-1}$ ($i = 1, \dots, n - 1$) generate the *rotation subgroup* $\Gamma^+(\mathcal{P})$ of $\Gamma(\mathcal{P})$, which is of index at most 2. We call \mathcal{P} *directly regular* if this index is 2.

A group $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ whose generators satisfy (2.1) and (2.2), is called a *string C-group*; here, the “C” stands for “Coxeter”, though not every C-group is a Coxeter group. These string C-groups are precisely the automorphism groups of regular polytopes, since, in a natural way, such a polytope can be uniquely reconstructed from Γ (see [20, §2E]). Therefore, we often identify a regular polytope with its automorphism (string C-) group.

The group $\Gamma(\mathcal{P})$ of a chiral polytope \mathcal{P} is generated by $n - 1$ elements $\sigma_1, \dots, \sigma_{n-1}$, which again are associated with a base flag $\Phi = \{F_1, F_0, \dots, F_n\}$, such that σ_i fixes all the faces in $\Phi \setminus \{F_{i-1}, F_i\}$ and cyclically permutes (“rotates”) consecutive i -faces of \mathcal{P} in the (polygonal) section F_{i+1}/F_{i-2} of rank 2. By replacing a generator by its inverse if need be, we can further require that if F'_i denotes the i -face of \mathcal{P} with $F_{i-1} < F'_i < F_{i+1}$ and $F'_i \neq F_i$, then $\sigma_i(F'_i) = F_i$. The resulting distinguished generators $\sigma_1, \dots, \sigma_{n-1}$ of $\Gamma(\mathcal{P})$ then satisfy relations

$$(2.3) \quad \sigma_i^{p_i} = (\sigma_j \sigma_{j+1} \dots \sigma_k)^2 = \epsilon \quad \text{for } i, j, k = 1, \dots, n - 1, \text{ with } j < k,$$

where again the numbers p_i determine the type $\{p_1, \dots, p_{n-1}\}$ of \mathcal{P} . Moreover, $\Gamma(\mathcal{P})$ and its generators satisfy a certain intersection condition resembling that for C-groups. Conversely, if Γ is a group generated by $\sigma_1, \dots, \sigma_{n-1}$ such that the relations (2.3) and the new intersection condition hold, then Γ is the group of a chiral polytope or the rotation subgroup for a directly regular polytope; here the polytope is regular if and only if Γ admits an involutory automorphism ρ such that $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2 \sigma_2$, and $\rho(\sigma_j) = \sigma_j$ for $j \geq 2$. For a chiral polytope \mathcal{P} , the two flag orbits yield two sets of generators σ_i which are not conjugate in $\Gamma(\mathcal{P})$; thus a chiral polytope occurs in two *enantiomorphic* (mirror image) forms.

We now describe the basic idea of modular reduction. We begin with a linear group G over a ring \mathbb{D} , choose an ideal J of \mathbb{D} , and try to construct a polytope from the quotient group of G obtained by viewing G as a linear group over \mathbb{D}/J . For the latter step the main obstruction is typically the intersection condition for the resulting quotient group of G . In most applications, G itself is already a string C-group (often a Coxeter group), \mathbb{D} is the ring of integers in an algebraic number field, and J is an ideal of \mathbb{D} .

More precisely, let \mathbb{D} be a commutative ring with identity 1, let V be a free module over \mathbb{D} of rank n with basis b_0, \dots, b_{n-1} , and let G be a subgroup of the general linear group $GL_n(\mathbb{D})$ over \mathbb{D} , whose elements we may view as invertible linear transformations (module isomorphisms) of V . Now let J be an ideal of \mathbb{D} . Then the natural ring epimorphism $\mathbb{D} \rightarrow \mathbb{D}/J$ defined by $a \rightarrow a + J$ induces a group epimorphism $G \rightarrow G^J$, where G^J is a subgroup of $GL_n(\mathbb{D}/J)$, the *modular reduction* of G ; in other words, we obtain G^J by simply viewing the matrix entries of the elements in G as entries in \mathbb{D}/J . Then G^J naturally acts on the free module V^J over \mathbb{D}/J of rank n with basis b_0^J, \dots, b_{n-1}^J . We often abuse notation by referring to the modular images of objects by the same name (such as V or b_0, \dots, b_{n-1} , etc.), that is, we drop the superscript J .

Modular reduction is, of course, a natural idea and has been used in various ways to construct maps and polytopes [27, 28, 34]. Here we begin by applying the idea to crystallographic Coxeter groups.

3 Polytopes from Crystallographic Coxeter Groups

Let Γ be an abstract *string* Coxeter group with generators $\rho_0, \dots, \rho_{n-1}$ and presentation as in (2.1), where again $p_{ii} = 1$ and $p_{ij} = 2$ for $|i - j| \geq 2$. Let V be real n -space, with basis $\alpha = \{a_0, \dots, a_{n-1}\}$ and symmetric bilinear form $x \cdot y$ defined by

$$(3.1) \quad a_i \cdot a_j := -2 \cos \frac{\pi}{p_{ij}}, \quad 0 \leq i, j \leq n - 1.$$

Let $R: \Gamma \rightarrow G$ be the (faithful) standard representation of Γ in V , where $G = \langle r_0, \dots, r_{n-1} \rangle$ is the isometric reflection group on V generated by the reflections with roots a_i (see [14, §5.3–5.4]); thus, $r_i(x) = x - (x \cdot a_i)a_i$, ($i = 0, \dots, n - 1$). Now let $m = 2m'$, where m' is the lowest common multiple of all p_{ij} which are finite. Let ξ be a primitive m -th root of unity, and let $\mathbb{D} := \mathbb{Z}[\xi]$. Then, with respect to the basis α of V , the reflections r_j are represented by matrices in $GL_n(\mathbb{D})$ so that we may view G as a subgroup of $GL_n(\mathbb{D})$. (By [9, Theorem 21.13], \mathbb{D} is the ring of integers in the

algebraic number field $\mathbb{Q}(\xi)$; and \mathbb{D} has (finite) rank $\phi(m)$ as a \mathbb{Z} -module.) Now we can reduce G modulo p for any prime p , here allowing $p = 2$ [9, Chapter XII]. More precisely, suppose that p is a rational prime and that J is a maximal ideal in \mathbb{D} with $p\mathbb{D} \subseteq J \subset \mathbb{D}$. Then $\mathbb{K} := \mathbb{D}/J$ is a finite field of characteristic p , and reduction mod p of G is achieved by applying the natural epimorphism $\mathbb{D} \rightarrow \mathbb{K}$ to the matrix entries of the elements in G . This then defines a representation $\kappa: G \rightarrow GL_n(\mathbb{K})$ with image group $G^p := G^J = \kappa(G)$. (This construction is essentially independent of the choice of J .) Note that κ is faithful when G is finite and $p \nmid |G|$ (in fact, κ often is faithful even when p divides $|G|$). In any case, $\ker \kappa$ is a p -subgroup of G .

The modular reduction technique is considerably more straightforward for crystallographic Coxeter groups, meaning that G (or Γ) leaves invariant some lattice in V . A (string) Coxeter group G is known to be crystallographic if and only if $p_{ij} = 2, 3, 4, 6$ or ∞ for all $i \neq j$. In fact, even allowing non-string diagrams, it is also true that G is crystallographic if and only if there is a basic system $\beta = \{b_0, \dots, b_{n-1}\}$, with $b_i := t_i a_i$ for certain $t_i > 0$, such that $m_{ij} := -t_i^{-1}(a_i \cdot a_j) t_j \in \mathbb{Z}$ for $0 \leq i, j \leq n - 1$. Then, for the rescaled roots b_i , we have

$$(3.2) \quad r_i(b_j) = b_j + m_{ij}b_i,$$

so that the generators r_i are represented by integral matrices with respect to the basis β , and the corresponding root lattice $\bigoplus_i \mathbb{Z}b_j$ actually is G -invariant. Thus we may take $\mathbb{D} = \mathbb{Z}$, and G can be reduced modulo any integer $d \geq 2$. We focus primarily on the case when p is an odd prime, but address some questions regarding composite moduli as well. For a composite modulus $d \geq 2$, the reduced group G^d generally does not “split” according to the prime factorization of d , and its structure depends more heavily on the diagram $\Delta(G)$ used in the reduction process (two diagrams which modulo a prime are equivalent, may not be equivalent modulo a composite modulus). We now describe these diagrams.

The various possible basic systems $\{t_i a_i\}$ for a crystallographic Coxeter group G can be represented by a diagram $\Delta(G)$ (see [5, p. 415] or [21]): for $0 \leq i, j \leq n - 1$, node i is labelled $2t_i^2 (= b_i^2)$; and distinct nodes $i \neq j$ are joined by $\lambda_{ij} := \min\{m_{ij}, m_{ji}\}$ unlabelled branches. Note that $\lambda_{ij} = \lambda_{ji} = 0, 1$, or 2 , so the underlying graph is essentially that of the underlying Coxeter diagram $\Delta_c(G)$ of G , except that a mark $p_{ij} = \infty$ is indicated by a doubled branch in the case that $m_{ij} = m_{ji} = 2$. In Table 1 we display the possible subdiagrams corresponding to the dihedral subgroups $\langle r_i, r_j \rangle$. For simplicity we have replaced the node labels $2t_i^2, 2t_j^2$ by s, t or s, ks ($k = 1, 2, 3, 4$), as appropriate. Note here that $m_{ij}m_{ji} = 4 \cos^2(\pi/p_{ij})$, so that if $m_{ij} \geq m_{ji} \geq 1$, we have $t_j^2/t_i^2 = m_{ij}/m_{ji} = 1, 2, 3, 4$ (or 1) for $p_{ij} = 3, 4, 6, \infty$, respectively.

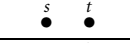
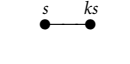
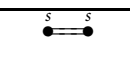
Nodes i, j	Parameters	λ_{ij}
	$p_{ij} = 2$	0
	$p_{ij} = 3, 4, 6, \infty$ ($k = 1, 2, 3, 4$)	1
	$p_{ij} = \infty$ ($k = 1$)	2

Table 1. Basic Systems for the Crystallographic Dihedral Groups $\langle r_i, r_j \rangle$.

The Gram matrix $B = [b_{ij}] := [b_i \cdot b_j]$ is easily computed from $\Delta(G)$, since $b_{ii} = 2t_i^2$ is simply the label attached to node i , and $b_{ij} = -\lambda_{ij} \max\{b_{ii}, b_{jj}\}/2$ for $i \neq j$. Moreover, $m_{ij} = \lambda_{ij} \max\{1, b_{jj}/b_{ii}\}$ for $i \neq j$.

For a connected Coxeter diagram $\Delta_c(G)$, the corresponding group G has to scale only finitely many basic systems β . Any such system is represented by an essentially unique diagram $\Delta(G)$, in which node labels form a set of relatively prime integers. For two such basic systems β, β' , we can convert from $\Delta(G)$ to $\Delta'(G)$ by consecutively adjusting the labels and branches of various pairs of adjacent nodes by operations of the following kind: inverting the ratio of the node labels, doubling a single branch, and balancing its labels, or converting a double branch to a single branch with ratio 4, if the corresponding branch in $\Delta_c(G)$ is marked ∞ . Following these adjustments on pairs of nodes, we may finally have to rescale the entire set of labels to obtain a set of relatively prime integers.

Then we may reduce G modulo an odd prime p to obtain a subgroup G^p of $GL_n(\mathbb{Z}_p)$ generated by the modular images of the r_i 's. We call the prime p *generic* for G if $p \geq 5$, or $p = 3$ but no branch of $\Delta_c(G)$ is marked 6. In the generic case, no node label of $\Delta(G)$ vanishes mod p and a change in the underlying basic system for G has the effect of merely conjugating G^p in $GL_n(\mathbb{Z}_p)$. On the other hand, in the non-generic case, the group G^p may depend essentially on the actual diagram $\Delta(G)$ used in the reduction mod p . For any odd prime p , we always find that $G^p = \langle r_0, \dots, r_{n-1} \rangle$ is a subgroup of the orthogonal group $O(\mathbb{Z}_p^n)$ of isometries for the (possibly singular) symmetric bilinear form $x \cdot y$, the latter being defined on \mathbb{Z}_p^n by means of (the modular image of) the Gram matrix B ; in particular, r_i is the orthogonal reflection with *root* b_i if $b_i^2 \neq 0$.

Now recall from [22, Theorem 3.1] that an irreducible group G^p of the above sort, generated by $n \geq 3$ reflections, must necessarily be one of the following:

- an orthogonal group $O(n, p, \epsilon) = O(V)$ or $O_j(n, p, \epsilon) = O_j(V)$, excluding the cases $O_1(3, 3, 0)$, $O_2(3, 5, 0)$, $O_2(5, 3, 0)$ (supposing for these three that $\text{disc}(V) \sim 1$), and also excluding the case $O_j(4, 3, -1)$;
- the reduction mod p of one of the finite linear Coxeter groups of type A_n ($p \nmid n + 1$), B_n, D_n, E_6 ($p \neq 3$), E_7, E_8, F_4, H_3 or H_4 .

We shall say in these two cases that G^p is of *orthogonal* or *spherical type*, respectively, although there is some overlap for small primes. (The notation $a \sim b$ means that $a = t^2b$ for some t in the field.) Our description in [22, Theorem 3.1] rests on the classification of the finite irreducible reflection groups over any field, obtained in Zalesskiĭ and Serežkin [36] (see also [16, 31, 32, 35]). For a non-singular space V , recall that $\epsilon = 0$ if n is odd, and $\epsilon = 1$ or -1 if n is even and the Witt index of V is $n/2$ or $(n/2) - 1$. When n is even, $\text{disc}(V) \sim (-1)^{n/2}$ if $\epsilon = 1$, and $\text{disc}(V) \sim (-1)^{n/2}\gamma$ with γ a non-square, if $\epsilon = -1$. Moreover, $O_1(V)$ and $O_2(V)$ are the subgroups of $O(V)$ generated by the two distinct conjugacy classes of reflections, each characterized by the quadratic character of the spinor norm of its reflections, namely 1 or γ , respectively.

The above analysis sets up the stage for the construction of finite regular polytopes from crystallographic string Coxeter groups G . Clearly, the generators r_i of the

reduced group G^p satisfy (at least) the Coxeter-type relations inherited from G . In most cases it also is more or less straightforward to determine the overall structure of G^p by appealing to the above description of possible reflection groups. However, the main challenge is to determine when G^p has the intersection property (2.2) for its standard subgroups. Here the outcome is often quite unpredictable, with the result depending on the underlying group G , the diagram $\Delta(G)$ used in the reduction, and the prime p . The reduced group G^p may turn out to be a C-group for all primes, or for all primes in certain congruence classes, or for only finitely many primes (for example, only for $p = 3$), or for no prime at all. Thus there is no theorem that covers all cases simultaneously.

Here we describe a quite general theorem (see [24, Theorem 2.3]) which, in the three basic cases considered, typically enables us to determine (often after additional considerations) whether or not the reduced group G^p is a C-group. For any $k, l \in \{0, \dots, n-1\}$ we let $G_k^p := \langle r_j \mid j \neq k \rangle$ and $G_{k,l}^p := \langle r_j \mid j \neq k, l \rangle$. Then G^p is a string C-group if and only if G_0^p and G_{n-1}^p are string C-groups and $G_0^p \cap G_{n-1}^p = G_{0,n-1}^p$. (This enables an inductive attack on the problem.) Similarly we let V_k and $V_{k,l}$ denote the subspace of $V = \mathbb{Z}_p^n$ spanned by the vectors b_j with $j \neq k$ or $j \neq k, l$, respectively. For a singular subspace W of a non-singular space V , we also let $\widehat{O}(W)$ denote the subgroup of $O(W)$ consisting of those isometries which act trivially on the radical of W .

Theorem 3.1 *Let $G = \langle r_0, \dots, r_{n-1} \rangle$ be a crystallographic linear Coxeter group with string diagram. Suppose that $n \geq 3$, that the prime p is generic for G , and that there is a square among the labels of the nodes $1, \dots, n-2$ of the diagram $\Delta(G)$ (this can be achieved by readjusting the node labels). For various subspaces W of V we identify $O(W)$, $\widehat{O}(W)$, etc. with suitable subgroups of the pointwise stabilizer of W^\perp in $O(V)$.*

(i) *Let the subspaces V_0, V_{n-1} and $V_{0,n-1}$ be non-singular, and let G_0^p, G_{n-1}^p be of orthogonal type.*

(a) *Then $G_0^p \cap G_{n-1}^p$ acts trivially on $V_{0,n-1}^\perp$, and*

$$O_1(V_{0,n-1}) \leq G_0^p \cap G_{n-1}^p \leq O(V_{0,n-1}).$$

(b) *If $G_0^p = O(V_0)$ and $G_{n-1}^p = O(V_{n-1})$, then $G_0^p \cap G_{n-1}^p = O(V_{0,n-1})$.*

(c) *If either $G_0^p = O_1(V_0)$ or $G_{n-1}^p = O_1(V_{n-1})$, then $G_0^p \cap G_{n-1}^p = O_1(V_{0,n-1})$.*

(ii) *Let V, V_0, V_{n-1} be non-singular, let $V_{0,n-1}$ be singular (so that $n \geq 4$), and let G_0^p, G_{n-1}^p be of orthogonal type.*

(a) *Then $G_0^p \cap G_{n-1}^p$ acts trivially on $V_{0,n-1}^\perp$, and*

$$\widehat{O}_1(V_{0,n-1}) \leq G_0^p \cap G_{n-1}^p \leq \widehat{O}(V_{0,n-1}).$$

(b) *If $G_0^p = O(V_0)$ and $G_{n-1}^p = O(V_{n-1})$, then $\widehat{O}(V_{0,n-1}) = G_0^p \cap G_{n-1}^p$.*

(c) *If either $G_0^p = O_1(V_0)$ or $G_{n-1}^p = O_1(V_{n-1})$, then $\widehat{O}_1(V_{0,n-1}) = G_0^p \cap G_{n-1}^p$.*

(iii) *Suppose $V, V_{0,n-1}$ are non-singular while at least one of the subspaces V_0, V_{n-1} is singular. Also suppose that $G_{0,n-1}^p$ is of orthogonal type, with $G^p = O_1(V)$ when $G_{0,n-1}^p = O_1(V_{0,n-1})$. Then $G_0^p \cap G_{n-1}^p = G_{0,n-1}^p$.*

We briefly discuss the modular polytopes associated with some interesting classes of crystallographic string Coxeter groups. Recall that $[p_1, p_2, \dots, p_{n-1}]$ denotes the Coxeter group with a string Coxeter diagram on n nodes and branches labelled p_1, p_2, \dots, p_{n-1} . They are the automorphism groups of the universal regular polytopes $\{p_1, p_2, \dots, p_{d-1}\}$ (see [20, §3D]).

3.1 Groups in Which G has Spherical or Euclidean Type

The modular reduction G^p of any spherical or euclidean (crystallographic) group G is a string C-group for any prime $p \geq 3$, and $G^p \cong G$ if G is spherical. There are four kinds of (connected) spherical string diagrams (up to duality), namely A_n, B_n, F_4 and $I_2(6)$ (dihedral of order 12). The corresponding modular polytopes are isomorphic to the n -simplex, n -cube, 24-cell and hexagon, respectively, and admit “modular realizations” in the finite space \mathbb{Z}_p^n . For the euclidean groups $[4, 3^{n-3}, 4]$ ($n \geq 3$), $[3, 4, 3, 3]$ and $[\infty]$, we obtain the regular toroids $\{4, 3^{n-3}, 4\}_{(p, 0^{n-2})}$ of rank n and $\{3, 4, 3, 3\}_{(p, 0, 0, 0)}$ of rank 5, and the regular p -gon $\{p\}$.

For a spherical group G , we have $G^d \simeq G$ for any modulus $d \geq 3$, and sometimes even for $d = 2$ (here depending on choice of diagram). For a euclidean group G and $d \geq 3$ (again, sometimes for $d = 2$ as well), the reduced group G^d is the group of a regular toroid of rank n , but now its type vector (suffix) depends on $\Delta(G)$ (in particular, on the parity of n) and, as well, on the parity of d ; the details are quite involved (see [25]).

3.2 Groups of Ranks 3 or 4

The groups of ranks 1 or 2 are subsumed by our discussion in the previous paragraph. For rank 3, any group G^p is a string C-group. For example, the hyperbolic group $G = [3, \infty]$ with diagram

$$\overset{1}{\bullet} \text{ --- } \overset{1}{\bullet} \text{ --- } \overset{4}{\bullet} \quad (\text{disc}(V) = -1),$$

yields a regular map of type $\{3, p\}$. For $p \geq 5$ we find that $G^p = O_1(3, p, 0) \simeq PSL_2(\mathbb{Z}_p) \rtimes C_2$ of order $p(p^2 - 1)$. When $p = 3, 5$, or 7 , respectively, we obtain the regular tetrahedron $\{3, 3\}$, icosahedron $\{3, 5\}$ and the Klein polyhedron $\{3, 7\}_8$. This construction redescribes the family of regular maps discussed (and generalized) in [18, 19].

The situation changes drastically for groups G of higher ranks, with obstructions already occurring in rank 4. We shall not attempt to fully summarize the findings of [23], which settle all groups $G = [k, l, m]$ of rank 4 (employing results such as Theorem 3.1). Suffice it here to mention some possible scenarios illustrating that the outcome is often unpredictable.

One possible scenario is that G^p is a string C-group for all primes $p \geq 3$; this happens, for instance, if the subgroup $[k, l]$ or $[l, m]$ is spherical, if $l = \infty$, or if $[k, l]$ or $[l, m]$ is euclidean and $l = 4$ or 6 . For example, from $[3, 3, \infty]$ we obtain a regular 4-polytope of type $\{3, 3, p\}$ whose vertex-figures are isomorphic to the maps of type $\{3, p\}$ derived from $[3, \infty]$; its group G^p is isomorphic to S_5 if $p = 3$, $O_1(4, 3, 1)$

if $p \equiv 1 \pmod{4}$, and $O_1(4, 3, -1)$ if $p \equiv 3 \pmod{4}$, $p \neq 3$. When $p = 3, 5$, or 7 , respectively, this gives the 4-simplex $\{3, 3, 3\}$, the 600-cell $\{3, 3, 5\}$, and the universal 4-polytope $\{\{3, 3\}, \{3, 7\}_8\}$ first described in [26].

On the other extreme, there are groups G for which G^p is a C-group for only finitely many primes p . For example, for $G = [6, 3, 6]$ only G^3 is a C-group. More important examples arise when the four subspaces $V, V_0, V_3, V_{0,3}$ are all non-singular. For example, when $G = [\infty, 3, \infty]$ the reduced group G^p is a C-group only for $p = 3, 5$, or 7 ; from G^3 and G^5 we obtain the 4-simplex $\{3, 3, 3\}$ and the regular star-polytope $\{5, 3, \frac{5}{2}\}$, and from G^7 a (non-universal) 4-polytope with facets and vertex-figures given by a dual pair of Klein maps $\{7, 3\}_8$ and $\{3, 7\}_8$.

Another possible scenario is that G^p is a C-group only for certain congruence classes of primes. If $G = [6, 3, \infty]$, then G^p is a C-group if and only if $p = 3$ or $p \equiv \pm 5 \pmod{12}$.

3.3 Groups of Higher Ranks

The large number of crystallographic Coxeter groups G of ranks $n \geq 5$ makes it difficult to fully enumerate the regular polytopes obtained by our method.

For the group $[4, 3, 4, 3]$ of rank 5 we obtain regular 5-polytopes with toroids $\{4, 3, 4\}_{(p,0,0)}$ as facets and 24-cells $\{3, 4, 3\}$ as vertex-figures; their group G^p is given by $O_1(5, p, 0)$ if $p \equiv \pm 1 \pmod{8}$ and $O(5, p, 0)$ if $p \equiv \pm 3 \pmod{8}$. These are examples of *locally toroidal* regular polytopes, meaning that their facets and vertex-figures are spherical or toroidal (but not all spherical). Such polytopes have not been fully classified (see [20, Chapter 12]). The three closely related groups $[3, 4, 3, 3, 3]$, $[3, 3, 4, 3, 3]$, and $[4, 3, 3, 4, 3]$ of rank 6 similarly give rise to locally toroidal 6-polytopes with groups $O_1(6, p, +1)$ if $p \equiv \pm 1 \pmod{8}$ and $O(6, p, +1)$ if $p \equiv \pm 3 \pmod{8}$. Each polytope is covered by the respective universal polytope

$$\begin{aligned} & \{\{3, 4, 3, 3\}_{(p,0,0,0)}, \{4, 3, 3, 3\}\}, \\ & \{\{4, 3, 3, 4\}_{(p,0,0,0)}, \{3, 3, 4, 3\}_{(p,0,0,0)}\}, \\ & \{\{3, 3, 4, 3\}_{(p,0,0,0)}, \{3, 4, 3, 3\}_{(p,0,0,0)}\}, \end{aligned}$$

which has been conjectured to exist for all primes $p \geq 3$ and to be infinite for $p > 3$ (see [20, Chapter 12]). For $p = 3$ (and $p = 2$, but this is outside our discussion), these three universal polytopes are known to be finite. The first two have group $\mathbb{Z}_3 \rtimes O(6, 3, +1)$, and the last has group $(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \rtimes O(6, 3, +1)$ (see [24]).

More generally we can reduce the four groups G modulo any integer $d \geq 2$ to obtain other kinds of locally toroidal regular polytopes of ranks 5 or 6, now with type vectors for facets and vertex-figures depending on $\Delta(G)$ and its subdiagrams for facets and vertex-figures, as well as on the parity of d (see [25]). In particular, this establishes Conjecture [20, 12C2] concerning the existence of locally toroidal regular 6-polytopes of type $\{3, 4, 3, 3, 3\}$, saying that the universal regular 6-polytopes $\{\{3, 4, 3, 3\}_{(d,0,0,0)}, \{4, 3, 3, 3\}\}$ and $\{\{3, 4, 3, 3\}_{(d,d,0,0)}, \{4, 3, 3, 3\}\}$ exist for all $d \geq 2$; this then settles the existence of the first kind of 6-polytopes mentioned in the previous paragraph. The corresponding conjectures [20, 12D3, 12E3] for the

locally toroidal regular 6-polytopes of types $\{3, 3, 4, 3, 3\}$ and $\{4, 3, 3, 4, 3\}$ are still open.

Another interesting special class consists of the 3-infinity groups $G = [p_1, \dots, p_{n-1}]$, for which all periods $p_j \in \{3, \infty\}$. Typically then we have an alternating string of 3's and ∞ 's, and the outcome depends on the nature of the string. For the prime $p = 3$ we always have a C-group, namely S_{n+1} . If $G = [3^k, \infty^l]$ with $k + l = n - 1$, then for all primes $p \geq 3$, the group G^p is a string C-group. We have seen examples with $n = 3$ or 4. For $G = [3, \infty^l, 3]$, with $n = l + 3, l \geq 1$, we have a string C-group, except possibly when $p = 7$ and $l \geq 4$, with $l \equiv 1 \pmod 3$. On the other hand, if $G = [3^k, \infty^l, 3^m]$, with $k > 1$ or $m > 1$, and $l \geq 1$, then G^p is a string C-group for all but finitely many primes p . However, G^p very often is not a C-group. For example, suppose that G has a string subgroup of the form $[\dots, \infty, 3^k, \infty, \dots]$, with $k \geq 1$, and that $p \geq 5$. Then G^p is not a string C-group, except possibly when $p = 5$ or 7 and $k \leq 1$ for all such string subgroups $[\dots, \infty, 3^k, \infty, \dots]$.

4 Polytopes from Hyperbolic Coxeter Groups

Following [30] we begin with the symmetry group $\Gamma = [r, s, t]$ of a regular honeycomb $\{r, s, t\}$ in hyperbolic 3-space \mathbb{H}^3 and consider a faithful representation of Γ as a group of complex Möbius transformations. Recall here that the absolute of \mathbb{H}^3 can be identified with the complex inversive plane (compare the Poincaré half-space model of \mathbb{H}^3), and that every group of hyperbolic isometries is isomorphic to a group of Möbius transformations over \mathbb{C} . In particular, a plane reflection in \mathbb{H}^3 corresponds to an inversion in a circle (or line) determined as the “intersection” of the mirror plane in \mathbb{H}^3 with the absolute. Under this correspondence the generating plane reflections ρ_0, \dots, ρ_3 of the hyperbolic reflection group Γ become inversions in circles, again denoted by ρ_0, \dots, ρ_3 , cutting one another at the same angles as the corresponding reflection planes in \mathbb{H}^3 . The group Γ is one of ten possible groups, namely

$$[4, 4, 3], [4, 4, 4], [6, 3, 3], [6, 3, 4], [6, 3, 5], [6, 3, 6],$$

$$[3, 6, 3], [3, 5, 3], [5, 3, 4], [5, 3, 5];$$

see [33] for a complete list of the generating inversions for these groups.

Next recall that Möbius transformations may conveniently be represented (uniquely up to scalar multiplication) by 2×2 matrices, namely

$$\frac{az + b}{cz + d} \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \frac{a\bar{z} + b}{c\bar{z} + d} \longleftrightarrow \# \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with appropriate interpretations for multiplication of such matrices. We then obtain matrices r_0, r_1, r_2, r_3 and s_1, s_2, s_3 , respectively, for the generators $\rho_0, \rho_1, \rho_2, \rho_3$ of Γ and $\sigma_1, \sigma_2, \sigma_3$ of Γ^+ . Let G denote the group of complex 2×2 matrices generated by s_1, s_2, s_3 . When considered modulo scalars, G (or rather the corresponding projective group PG) is isomorphic to Γ^+ , the rotation group of $\{r, s, t\}$. It turns out that the matrices in G all have entries in a certain subring \mathbb{D} of \mathbb{C} depending on $[r, s, t]$. Thus

modular reduction applies. We choose appropriate ideals J in \mathbb{D} and consider the matrices s_1, s_2, s_3 over the quotient ring \mathbb{D}/J , again modulo scalars (determined by a suitable subgroup of the center). Then under certain conditions, the resulting projective group G^J (say) is either the rotation subgroup for a directly regular polytope, or the full automorphism group of a chiral polytope.

We illustrate the method for the hyperbolic group $[4, 4, 3]$. When $\Gamma = [4, 4, 3]$ is viewed as a group of complex Möbius transformations, the generators may be taken as

$$\rho_0(z) = \bar{z}, \quad \rho_1(z) = i\bar{z}, \quad \rho_2(z) = 1 - \bar{z}, \quad \rho_3(z) = 1/\bar{z}.$$

Then the rotation subgroup Γ^+ of Γ is generated by

$$\sigma_1 = \rho_0\rho_1 = -iz, \quad \sigma_2 = \rho_1\rho_2 = -iz + i, \quad \sigma_3 = \rho_2\rho_3 = 1 - 1/z,$$

and consists only of proper Möbius transformations (not involving complex conjugation). Here the matrices s_1, s_2, s_3 for $\sigma_1, \sigma_2, \sigma_3$ are given by

$$s_1 = \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -i & i \\ 0 & 1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},$$

and G is the group of all invertible 2×2 matrices over the Gaussian integers $\mathbb{D} = \mathbb{Z}[i]$ (with determinants $\pm 1, \pm i$). In particular, $[4, 4, 3]^+ \cong PSL_2(\mathbb{Z}[i]) \rtimes C_2$, where the first factor is also known as the Gaussian modular group or Picard group. Since we wish to obtain toroidal facets of type $\{4, 4\}_{(b,c)}^+$, we must consider the imposition of the extra relation $(\sigma_1^{-1}\sigma_2)^b(\sigma_1\sigma_2^{-1})^c = 1$ on the generators of Γ^+ (see [8]). Note here that the corresponding matrix product in G is

$$(s_1^{-1}s_2)^b(s_1s_2^{-1})^c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^b \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}^c = \begin{bmatrix} 1 & -(b+ci) \\ 0 & 1 \end{bmatrix}.$$

Thus the ideals J of $\mathbb{Z}[i]$ must be chosen in such a way that $b + ci = 0$ in $\mathbb{Z}[i]/J$.

One natural choice of ideal is $J = m\mathbb{Z}[i]$, where $m \geq 3$ is an integer in \mathbb{Z} . This choice of ideal typically produces directly regular polytopes of type

$$\{\{4, 4\}_{(m,0)}, \{4, 3\}\}$$

whose rotation subgroup is $PSL_2(\mathbb{Z}_m[i])$ or a closely related group, with the exact structure determined by the prime factorization of m in \mathbb{Z} (see [30, p. 238]). For example, if $m \equiv 3 \pmod 4$ is a prime, we obtain $PSL_2(\mathbb{Z}_m[i]) \cong PSL_2(m^2)$.

A more interesting choice of ideal arises from a solution of the equation $x^2 = -1 \pmod m$, with m as above. Let $m = 2^e p_1^{e_1} \cdots p_k^{e_k}$ be the prime factorization in \mathbb{Z} . Then the equation is solvable if and only if $e = 0, 1$ and $p_j \equiv 1 \pmod 4$ for each j . If $i \in \mathbb{Z}_m$ is such that $i^2 = -1 \pmod m$, then there exists a unique pair of positive integers b, c such that $m = b^2 + c^2$, $(b, c) = 1$, and $b = -ic \pmod m$. We now take $J = (b + ci)\mathbb{Z}[i]$, which is the kernel of the ring epimorphism $\mathbb{Z}[i] \rightarrow \mathbb{Z}_m$ that maps the complex number $x + yi$ to the element $x_m + y_m i$ in \mathbb{Z}_m , where $x_m \equiv x$ and

$y_m \equiv y \pmod m$. This choice of ideal “destroys” the reflections in the overlying reflection group and typically yields chiral polytopes of type $\{\{4, 4\}_{(b,c)}, \{4, 3\}\}$ whose group is $PSL_2(\mathbb{Z}_m)$ or a closely related group, with the exact structure again determined by the prime factorization of m . For example, if $m \equiv 1 \pmod 8$ is a prime, then the group is $PSL_2(m)$. Note that the structure of the polytope also depends on the solution \hat{i} . For example, when $m = 65$, the solution $\hat{i} = 8$ gives facets $\{4, 4\}_{(1,8)}$, while $\hat{i} = 18$ leads to facets $\{4, 4\}_{(4,7)}$.

With similar techniques we can construct a host of regular or chiral 4-polytopes of types $\{4, 4, 4\}$, $\{4, 4, 4\}$, $\{6, 3, 3\}$, $\{6, 3, 4\}$, $\{6, 3, 5\}$, $\{6, 3, 6\}$ or $\{3, 6, 3\}$ (see [29, 30]). However, the underlying ring \mathbb{D} will depend on the Schläfli symbol. The Gaussian integers $\mathbb{Z}[i]$ and Eisenstein integers $\mathbb{Z}[\omega]$ (with ω a cube root of unity) suffice in all cases except $\{6, 3, 4\}$ and $\{6, 3, 5\}$ (this is based on subgroup relationships between the rotation subgroups for the various types). For $\{6, 3, 4\}$ and $\{6, 3, 5\}$, respectively, we can work over the ring $\mathbb{Z}[\omega, \sqrt{2}]$ and $\mathbb{Z}[\omega, \tau]$ (with τ the golden ratio; see the next section).

Chiral polytopes also exist in ranks larger than 4, but explicit constructions of finite examples (or rank 5) were only discovered quite recently [4]. Modular reduction is an effective method to produce examples in rank 4. In addition to the symmetry groups of 3-dimensional hyperbolic honeycombs, several other discrete hyperbolic groups admit representations as groups of linear fractional transformations over other rings of complex or quaternionic integers.¹ Although the arithmetic involved is likely to be considerably more complicated than in the rank 4 case, there is a good chance that the reduction method will carry over to produce examples of finite chiral polytopes of rank larger than 4.

5 The Groups $[3, 5, 3]$ and $[5, 3, 5]$

In Section 3 we observed that any crystallographic string Coxeter group G has rotational periods $p_j \in \{2, 3, 4, 6, \infty\}$ and can be represented faithfully as a matrix group over the domain \mathbb{Z} . Here we widen the discussion a little by allowing the period $p_j = 5$. Keeping (3.1) in mind, we note that $2 \cos \frac{\pi}{5} = \tau$, where the golden ratio $\tau = (1 + \sqrt{5})/2$ is the positive root of $\tau^2 = \tau + 1$. We therefore move to the larger coefficient domain $\mathbb{D} := \mathbb{Z}[\tau] = \{a + b\tau : a, b \in \mathbb{Z}\}$, and soon find that we need only add the subdiagram



to those already listed in Table 1 in order for the Cartan integers m_{ij} of (3.2) to be in \mathbb{D} for all i, j . This subdiagram, say on nodes i, j , does indeed define the non-crystallographic dihedral group $\langle r_i, r_j \rangle$ with order 10 and period $p_{ij} = 5$. (In other notation, this is the group $H_2 \simeq I_2(5)$.) Naturally, we must now allow rescaling of nodes by any “integer” $s \in \mathbb{D}$ or its inverse. Furthermore, referring back to (3.2), we find that $m_{ij} = \tau^2 \in \mathbb{D}$, so that G is represented as a matrix group over \mathbb{D} through its action on the \mathbb{D} -module $\bigoplus_j \mathbb{D}b_j$.

¹Described in a forthcoming book by Norman Johnson entitled *Geometries and Transformations*.

Let us now summarize the key arithmetic properties of the domain \mathbb{D} . (We refer to [10] for a detailed account of this, and to [3] for a deeper discussion of “ \mathbb{D} -lattices” and the related finite Coxeter groups H_k , $k = 2, 3, 4$.) First of all, we recall that \mathbb{D} is the ring of algebraic integers in the field $\mathbb{Q}(\sqrt{5})$. The non-trivial field automorphism mapping $\sqrt{5} \mapsto -\sqrt{5}$ induces a ring automorphism $\prime: \mathbb{D} \rightarrow \mathbb{D}$, which in this section we shall call *conjugation*. Thus $(a + b\tau)^\prime = (a + b) - b\tau$. In particular, $\tau^\prime = 1 - \tau = -\tau^{-1}$. Recall that $z = a + b\tau$ has *norm* $N(z) := zz^\prime = a^2 + ab - b^2$. We note that \mathbb{D} is a Euclidean domain through a division algorithm based on $|N(z)|$.

The set of *units* in \mathbb{D} is $\{\pm\tau^n : n \in \mathbb{Z}\} = \{u \in \mathbb{D} : N(u) = \pm 1\}$. Recall that integers $z, w \in \mathbb{D}$ are *associates* if $z = uw$ for some unit u . Up to associates, the primes $\pi \in \mathbb{D}$ can be classified as follows:

- the prime $\pi = \sqrt{5} = 2\tau - 1$, which is self-conjugate (up to associates: $\pi^\prime = -\pi$);
- rational primes $\pi = p \equiv \pm 2 \pmod{5}$, also self-conjugate;
- primes $\pi = a + b\tau$, for which $|N(\pi)|$ equals a rational prime $q \equiv \pm 1 \pmod{5}$. In this case, the conjugate prime $\pi^\prime = (a + b) - b\tau$ is not an associate of π .

Let us now turn to the group $G = [3, 5, 3]$, here acting as an orthogonal group on real 4-space V (in contrast to the conformal representation on \mathbb{H}^3 mentioned in the previous section). Since τ^2 is a unit, there is essentially only one choice of diagram, namely

$$\Delta(G) = \bullet \text{ --- } \bullet \text{ --- } \tau^2 \text{ --- } \tau^2 \text{ .}$$

The discriminant is $\text{disc}(V) = -\frac{1}{16}(2 + 5\tau) \sim -(2 + 5\tau)$, where the prime $\delta := -(2 + 5\tau)$ has norm -11 .

Now consider any prime $\pi \in \mathbb{D}$. Our goals are to show that $G^\pi = \langle r_0, r_1, r_2, r_3 \rangle^\pi$ is a string C -group and to determine its structure, then say a little about the corresponding polytope $\mathcal{P}^\pi := \mathcal{P}(G^\pi)$.

In fact, we can almost immediately apply a suitable generalization of [22, Theorem 4.2]. First note that the subgroup $G_3^\pi = \langle r_0, r_1, r_2 \rangle^\pi$ is obviously some quotient of the spherical group $[3, 5] \simeq H_3$. Now it is easy to check that after reduction modulo any prime π , even for associates of 2, the reflections r_j still have period 2. Next, we consider the isometry

$$(5.1) \quad z := (r_0 r_1 r_2)^5 = \begin{bmatrix} -1 & 0 & 0 & \tau^4 \\ 0 & -1 & 0 & 2\tau^4 \\ 0 & 0 & -1 & 3\tau^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in G . Since τ^4 is a unit, this means that $r_0 r_1 r_2$ still has period 10 in G^π . Thus $\langle r_0, r_1, r_2 \rangle^\pi \simeq [3, 5]$ and dually $\langle r_1, r_2, r_3 \rangle^\pi \simeq [5, 3]$. Consulting the proof of [22, Theorem 4.2], we see that we need only show that the orbit of $\mu_0 := [1, 0, 0, 0]$ under the right action of the matrix group $\langle r_0, r_1, r_2 \rangle$ has the same size modulo π as in characteristic 0, namely 12. This is routinely verified, so we have proved most of the following proposition.

Proposition 5.1 *Let $G = [3, 5, 3]$. For any prime $\pi \in \mathbb{D}$, the group*

$$G^\pi = \langle r_0, r_1, r_2, r_3 \rangle^\pi$$

is a finite string C-group. The corresponding finite regular polytope \mathcal{P}^π is self-dual and has icosahedral facets $\{3, 5\}$ and dodecahedral vertex figures $\{5, 3\}$.

Proof To verify self-duality we define $g \in GL(V)$ by

$$g: [b_0, b_1, b_2, b_3] \mapsto [\tau^{-1}b_3, \tau^{-1}b_2, \tau b_1, \tau b_0].$$

Then $g^2 = 1$, $gr_0g = r_3$, and $gr_1g = r_2$. (See [20, 2E12].) ■

A more detailed description of G^π must depend on the nature of the prime π . (Of course, our results are typically unaffected by replacing π by any associate $\pm\tau^m\pi$.) In all cases the underlying finite field $\mathbb{K} := \mathbb{D}/(\pi)$ has order $|N(\pi)|$, so that G^π acts as an orthogonal group on the 4-dimensional vector space V over \mathbb{K} preserving the modular image of the bilinear form for G .

Case 1: $\pi = 2$. Here an easy calculation using GAP confirms that G^π is the orthogonal group $O(4, 2^2, -1)$ with Witt index 1 over $\mathbb{K} = GF(2^2)$. Since $|G^2| = 8160$, the polytope \mathcal{P}^2 has 68 vertices and 68 icosahedral facets.

Henceforth we suppose that π is not an associate of 2. To work with such primes, we need a generalization of the rational Legendre symbol $(p|q)$. Thus for any $\alpha \in \mathbb{D}$ and prime π we set

$$(\alpha|\pi)_{\mathbb{D}} := \begin{cases} +1 & \text{if } \alpha \text{ is a quadratic residue (mod } \pi), \\ -1 & \text{otherwise,} \end{cases}$$

(cf. [10, Ch. VIII]). We are mainly interested in computing $\epsilon := (\delta|\pi)_{\mathbb{D}}$, where $\delta = -(2 + 5\tau)$ is the discriminant. Since every label in $\Delta(G)$ is square, we conclude that G^π is a subgroup of $O_1(4, |N(\pi)|, \epsilon)$, so long as δ and π are relatively prime. Indeed, G^π will almost always equal such an orthogonal group.

Case 2: $\pi = \sqrt{5} = 2\tau - 1$. Here $|\pi\pi'| = 5 \equiv 0 \pmod{\pi}$, so that the discriminant $\delta = -(2 + 5\tau) \equiv 3 \pmod{\pi}$, which is non-square in $\mathbb{K} = GF(5)$. Thus $\epsilon = -1$ and $G^\pi = O_1(4, 5, -1)$ has order 15600. In fact, the polytope $\mathcal{P}^{\sqrt{5}}$ is isomorphic to that obtained in [23, p. 347] through reduction mod 5 of the crystallographic group $[3, \infty, 3]$.

Case 3: π is an associate of an odd rational prime $p \equiv \pm 2 \pmod{5}$. Since $\mathbb{K} = GF(p^2)$, $G^\pi = G^p$ is, for suitable ϵ , a subgroup of $O_1(4, p^2, \epsilon)$, whose order we recall is $p^4(p^4 - \epsilon)(p^4 - 1)$. Consulting [22, Theorem 3.1], we see that $G^p = O_1(4, p^2, \epsilon)$ so long as we can rule out two remote alternatives.

First of all, it is conceivable that $G^p \simeq H_4 = [3, 3, 5]$. But here it is easy to check directly that H_4 cannot be generated by reflections r_j satisfying the Coxeter-type relations inherited from $[3, 5, 3]$, let alone the independent relations induced by reduction modulo p .

Secondly, we must show that G^p is not isomorphic to some orthogonal group $O_1(4, p, \eta)$, $\eta = \pm 1$, over the subfield $GF(p)$. If this were so, then Theorem 3.1 in [22] would actually imply that G^p is similar to $O_1(4, p, \eta)$ under extension of scalars. More precisely, if \mathbb{L} is an algebraic closure of $\mathbb{K} = GF(p^2)$, then there would exist some $g \in GL(V_{\mathbb{L}})$ with $gG^p g^{-1} = O_1(4, p, \eta)$. Using (3.2), we compute with respect to the new basis $\{c_i\} = \{g(b_i)\}$ for $V_{\mathbb{L}}$. Thus the reflection $\tilde{r}_i := gr_i g^{-1}$ satisfies

$$\tilde{r}_i(c_j) = g(b_j + m_{ij}b_i) = c_j + m_{ij}c_i.$$

We conclude that the field of definition for G^p must always contain the subfield generated by the Cartan integers m_{ij} . In our case, $m_{12} = \tau^2 \notin GF(p)$, so that $gG^p g^{-1}$ cannot possibly be a group $O_1(4, p, \eta)$.

Having shown that $G^p = O_1(4, p^2, \epsilon)$, we next determine ϵ . From [10, Theorem 8.5(a)] we have

$$(\delta|\pi)_{\mathbb{D}} = (N(\delta)|p) = (-11|p) = (-1|p)(11|p) = (p|11),$$

by (rational) quadratic reciprocity. Since the non-zero squares (mod 11) are 1, 3, 4, 5, 9, we conclude that

$$\epsilon := \begin{cases} +1 & \text{if } p \equiv 3, 12, 23, 27, 37, 38, 42, 47, 48, 53 \pmod{55}, \\ -1 & \text{if } p \equiv 2, 7, 8, 13, 17, 18, 28, 32, 43, 52 \pmod{55}. \end{cases}$$

Case 4: $\pi = a + b\tau$, where $N(\pi) = a^2 + ab - b^2 = q$, where the rational prime $q \equiv \pm 1 \pmod{5}$; however, π is not an associate of $\delta = -(2 + 5\tau)$.

We now have $\mathbb{K} = GF(q)$. An even easier appeal to [22, Theorem 3.1] gives $G^\pi = O_1(4, q, \epsilon)$. We need only determine $\epsilon = (\delta|\pi)_{\mathbb{D}}$. Since $a + b\tau \equiv 0 \pmod{\pi}$, we may suppose $\tau = -a/b \in \mathbb{K}$. Thus

$$\delta = -(2 + 5\tau) \sim -b^2(2 + 5\tau) \equiv 5ab - 2b^2 \pmod{\pi}.$$

By [10, Theorem 8.5(a)], we obtain

$$\epsilon = (\delta|\pi)_{\mathbb{D}} = ((5ab - 2b^2)|q) = (b|q)((5a - 2b)|q).$$

Using the rational Legendre symbol, we can thus compute ϵ for any prime $\pi = a + b\tau$.

It is possible to say when π and its conjugate π' give opposite ϵ 's, so that the corresponding orthogonal spaces have, in some order, Witt indices 1 and 2. This happens if and only if q is a square mod 11, since

$$(\delta|\pi)_{\mathbb{D}} (\delta|\pi')_{\mathbb{D}} = ((b(5a - 2b)(-b)(5a + 7b))|q) = (-11|q) = (q|11).$$

One notable instance here is $\pi = \delta' = -7 + 5\tau$, which is relatively prime to the discriminant δ . We have $G^{\delta'} = O_1(4, 11, -1)$, which is of order 1771440.

Case 5: $\pi = \delta = -(2 + 5\tau)$. This is the only case in which the orthogonal space V is singular. Now $\mathbb{K} = GF(11)$ and $\tau = -2/5 = 4$. We find that $\text{rad}(V)$ is spanned by

$c = 7b_0 + 3b_1 + 2b_2 + b_3$, and that $V = \text{rad}(V) \perp V_3$, where V_3 is the non-singular subspace spanned by b_0, b_1, b_2 . It is then not hard to see that

$$O(V) \simeq \check{V}_3 \rtimes (\mathbb{K}^* \times O(V_3)),$$

where $\mathbb{K}^* \simeq GL(\text{rad}(V))$ and \check{V}_3 is dual to V_3 . We observe that the abelian group $\check{V}_3 \simeq \mathbb{K}^3$ consists of all transvections $r(x) = x + \varphi(x)c$, where $\varphi \in \check{V}_3$ (with \check{V}_3 viewed as a subspace of \check{V} fixing c). Now since every r_j fixes c , G^δ must be a subgroup of the pointwise stabilizer of $\text{rad}(V)$. In fact, another calculation with GAP confirms that $G^\delta = \widehat{O}_1(V) \simeq \check{V}_3 \rtimes O_1(V_3)$, which has order $11^3 \cdot 11 \cdot (11^2 - 1) = 1756920$. Now consider the isometry $z \in G$ defined in (5.1). It is easy to check that $z(c) \equiv c \pmod{\delta}$, so that $z = 1_{\text{rad}(V)} \perp -1_{V_3} \in \widehat{O}_1(V)$ acts as the *central inversion* in the group $O_1(V_3)$ for the icosahedral facet. Thus G^δ has a normal subgroup A isomorphic to $\check{V}_3 \rtimes \langle z \rangle$ and so of order $2 \cdot 11^3$. Using $O_1(3, 11, 0) \simeq PSL_2(11) \rtimes C_2$ (see [1, Theorem 5.20]), we conclude that $\overline{G} := G^\delta/A \simeq PSL_2(11)$, of order 660. Remarkably, \overline{G} is also a string C -group. The resulting polytope is the 11-cell independently discovered by Coxeter [7] and Grünbaum [11]. Indeed, both $r_0r_1r_2$ and $r_1r_2r_3$ have period 5 in the quotient (see (5.1)), and $\mathcal{P}(\overline{G}) = \{\{3, 5\}_5, \{5, 3\}_5\}$ is the universal 4-polytope with hemi-icosahedral facets and hemi-dodecahedral vertex-figures.

This finishes our investigation of the group $[3, 5, 3]$. Evidently a somewhat similar analysis is possible for the group $H = [5, 3, 5]$ with diagram

$$\Delta(H) = \bullet \text{---} \tau^2 \bullet \text{---} \tau^2 \bullet \text{---} \bullet$$

and corresponding discriminant $\frac{-1}{16}(3 + 7\tau) \sim -(3 + 7\tau) =: \lambda$. Since $N(\lambda) = -19$, we see that λ is also prime. We note only that the group H^λ for the singular space V again has an interesting quotient. In fact, $\overline{H} \simeq PSL_2(19)$ is the automorphism group for the universal regular polytope $\mathcal{P}(\overline{H}) = \{\{5, 3\}_5, \{3, 5\}_5\}$, with hemi-dodecahedral facets and hemi-icosahedral vertex-figures. This is the 57-cell described by Coxeter [6].

With the exception of the 11-cell and 57-cell, the polytopes described here can be viewed as regular tessellations on hyperbolic 3-manifolds (see [20, 6]). Moreover, the two exceptions are the only regular polytopes of rank 4 (or higher) with automorphism group isomorphic to $PSL_2(r)$ for some prime power r (see [17]). For related work see also [12, 13, 15].

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