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Tractable Fields

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Abstract. A field F is said to be tractable when a condition described below on the simultaneous representation of quaternion algebras holds over F. It is shown that a global field F is tractable iff F has at most one dyadic place. Several other examples of tractable and nontractable fields are given.

Introduction

A classical result of Albert says that every central simple algebra A of degree 4 with involution τ decomposes into a tensor product of two quaternion algebras. However, in the next case, when A has degree 8, it is known by [ART] that A need not be a product of quaternion algebras (though Merkurjev's theorem shows that some size matrices over A is such a product). It is an interesting but very difficult question when A of degree 8 is decomposable, and even more so when A has a "stable" decomposition, *i.e.*, $A \cong Q_1 \otimes_F Q_2 \otimes_F Q_3$, where each Q_i is a quaternion algebra with $\tau(Q_i) = Q_i$. This is still unsettled even in the easiest case when A has a Henselian valuation and contains a 2-Kummer Galois extension field K of the center F of A with $[K : F] = 8$. In [C] and [CDD] a necessary condition for stable decomposability of A in this case was found. This condition was shown also to be sufficient iff F satisfies a condition called tractability, which was expressed in terms of the existence of solutions of certain systems of quadratic equations (see Section 1 for the precise statement). In this paper, we show that this condition can be expressed in a very suggestive way using quaternion algebras, and we investigate which fields satisfy this condition.

Let F be a field, $\operatorname{char}(F) \neq 2$. For $a, b \in F^* = F - \{0\}$, let $(a, b/F)$ denote the 4-dimensional quaternion algebra over F with F-base $1, i, j, k$, such that $i^2 = a$, $j^2 = b$, and $ij = -ji = k$. We call F a *tractable* field if, for every $a_1, a_2, a_3, b_1, b_2, b_3 \in F^* = F - \{0\}$, whenever

$$(*) \qquad (a_i, b_j/F) \text{ is split for all } j \neq i \quad \text{and} \quad (a_1, b_1/F) \cong (a_2, b_2/F) \cong (a_3, b_3/F),$$

then $(a_i, b_i/F)$ is split.

Thus, tractability expresses the fact that quaternion algebras display a specific behavior, which turns out to encapsulate significant information on arithmetical properties of the field. For example, we show in Prop. 3.2 that fields in which -1 is a sum of four squares but not of three squares are never tractable. In Section 2 we determine the local and the global fields which are tractable: We show in Cor. 2.3 that a local field is tractable iff it is not dyadic, and in Th. 2.10 that a global field is tractable iff it has at most one dyadic spot. (It was shown in [CDD] that $\mathbb{Q}$ is tractable.)

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The tractability of function fields raises intriguing questions which are addressed in Section 3. The problem is to relate the behavior of quaternion algebras over the function field to their behavior over the field of constants. When quaternion algebras satisfy good specialization properties, we can show that a purely transcendental extension of degree 1 of a tractable field is tractable. For example, every purely transcendental extension of $\mathbb{Q}$ is tractable. We describe in Prop. 3.10 and Note 3.12 exactly which function fields of genus 0 over $\mathbb{Q}$ are tractable. But we know very little about tractability of function fields of higher genus over $\mathbb{Q}$.

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1 Other Interpretations of Tractability, History, Norm Groups

The original definition of tractability was given in [CDD, Def. 1, p. 780] as follows: A field F ($\operatorname{char}(F) \neq 2$) is *tractable* iff for every $a_1, a_2, a_3 \in F^*$ with $[F(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : F] = 8$ and every $\delta \in F^*$ such that δ is in the image of the norm map from $F(\sqrt{a_2}, \sqrt{a_3})$ to F, if there is a nontrivial solution in F to each of the systems of equations

$$X_1^2 - a_1X_2^2 = \delta(X_3^2 - a_1a_2X_4^2) = X_5^2 - a_3X_6^2, \tag{1.1}$$

$$Y_1^2 - a_1Y_2^2 = \delta(Y_3^2 - a_1a_3Y_4^2) = Y_5^2 - a_2Y_6^2, \tag{1.2}$$

then there is a solution in F to the equation $Z_1^2 - a_1Z_2^2 = \delta$. We will show below (see Prop. 1.4) that this definition is equivalent to the one given in the Introduction.

The question of tractability arose in connection with the study of decomposability of algebras with involution over Henselian fields in [C] and [CDD]. If A is a central simple algebra over a field F ($\operatorname{char}(F) \neq 2$), and A has an involution τ with $\tau|_F = \mathrm{id}$ we say that A is *decomposable* if A is a tensor product of quaternion algebras. We say that A is *stably decomposable* if $A = Q_1 \otimes_F \cdots \otimes_F Q_k$, with each Q_i a quaternion subalgebra of A with $\tau(Q_i) = Q_i$. Thus, stable decomposability is a stronger condition than decomposability.

Suppose our field F has a Henselian valuation v. Let Γ_F be the value group of v (written additively); V_F the valuation ring; m_F the unique maximal ideal of V_F; $\overline{F} = V_F/\mathrm{m}_F$, the residue field; and $U_F = V_F - \mathrm{m}_F$, the group of valuation units. Assume that $\operatorname{char}(\overline{F}) \neq 2$. Now, let D be a division algebra with center F and $[D : F] < \infty$, such that D has an involution τ with $\tau|_F = \mathrm{id}$. Since v is Henselian, there is a unique extension of v to a valuation w of D (see [Schi, Th. 9, p. 53] or [W_2, Th.]). (Indeed, w can be defined by $w(d) = \frac{1}{n}v\big(\operatorname{Nrd}(d)\big)$ for $d \in D^*$, where $n = \sqrt{[D : F]}$ and Nrd denotes the reduced norm on D.) Let Γ_D be the value group; V_D the valuation ring; m_D the maximal ideal; $\overline{D}$ the residue division algebra; and U_D the group of units of the valuation w on D. By Draxl's Ostrowski-type theorem [D, Th. 2], as $\operatorname{char}(\overline{F}) \neq 2$ and $[D : F]$ is a 2-power as D has an involution, $[D : F] = |\Gamma_D : \Gamma_F| \cdot [\overline{D} : \overline{F}]$. We will assume that $\overline{D}$ is a field. Then, $\overline{D}$ is normal over $\overline{F}$, hence Galois as $\operatorname{char}(\overline{F}) \neq 2$; indeed, there is a canonical surjection $\Gamma_D/\Gamma_F \to \mathcal{G}al(\overline{D}/\overline{F})$, by [DK, p. 96] or [JW, Prop. 1.7]. Let $\exp(\Gamma_D/\Gamma_F)$ denote the exponent of the finite abelian group Γ_D/Γ_F, and let $\exp(D)$ denote the order of the class $[D]$ of D in the Brauer group

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$\mathrm{Br}(F)$. Then, $\exp(\Gamma_D/\Gamma_F) \mid \exp(D)$, by [PY, (3.19)] or [JW, Cor. 6.10] and $\exp(D) \mid 2$, as D has an involution. Consequently, $\mathcal{G}al(\overline{D}/\overline{F})$ is an elementary abelian 2-group, so $\overline{D}$ is a 2-Kummer extension of $\overline{F}$. Note also that by the uniqueness of the valuation w on D extending v, we must have $\tau(V_D) = V_D$ and $\tau(\mathrm{m}_D) = \mathrm{m}_D$ so τ induces a well-defined $\overline{F}$-automorphism of $\overline{D}$, which is denoted $\overline{\tau}$. It is quite possible that $\overline{\tau}$ is trivial, even when τ is not.

We will call the situation just described the "standard setup", *i.e.*, F has a Henselian valuation v with $\mathrm{char}(\overline{F}) \neq 2$, D a finite-dimensional F-central division algebra with an involution τ such that $\tau|_F = \mathrm{id}$, and $\overline{D}$ a field.

It was shown in [C, Th. 2.1], that in the standard setup, if D decomposes stably with respect to τ, then

(‡) there is a set of representatives T for Γ_D/Γ_F ($T \subseteq D^*$) such that, for each $t \in T$, $t\tau(t) \in F^*(1 + \mathrm{m}_D)$.

Assume now we have F, v, D, τ in the standard setup. The principal focus of [CDD] was to determine whether if (‡) holds, then D must decompose stably (with respect to τ). It was shown in [CDD, Th. 1, p. 768] that if $[\overline{D} : \overline{F}] \leq 4$ and $\overline{\tau} \neq id$, then D decomposes stably. Also, by [CDD, Th. 2, p. 769], regardless of the value of $[\overline{D} : \overline{F}]$, if $\overline{\tau} = \mathrm{id}$ and (‡) holds, then D decomposes stably. So, the first case not fully settled was when $[\overline{D} : \overline{F}] = 8$ and $\overline{\tau} \neq \mathrm{id}$. Here is where the condition of tractability arises, for the residue field $\overline{F}$. It was shown in [CDD, Th. 2, p. 780] that given a field F with Henselian valuation v with $\mathrm{char}(\overline{F}) \neq 2$, condition (‡) implies stable decomposability for all D, τ in the standard setup over F, v with $[\overline{D} : \overline{F}] = 8$ iff $\overline{F}$ is tractable. (It was also shown that this is equivalent to the stable decomposability of a certain family of "generic" division algebras D over F in the standard setup, with $[D : F] = 64$.) Examples of non-tractable fields yield examples of indecomposable division algebras of residue dimension 8 in the standard setup for which (‡) holds.

Henceforth, assume $\mathrm{char}(F) \neq 2$. The two formulations of tractability of F at first look rather different, but can be united by thinking in terms of norm groups. For $a \in F^*$, let

$$N_F(a) = \{c \in F^* \mid c = r^2 - as^2, \text{ for some } r, s \in F\}.$$

Note that $N_F(a)$ is a subgroup of F^*, since it is the image of $F(\sqrt{a})^*$ under the norm map from $F(\sqrt{a})$ to F. Clearly also, $F^{*2} \subseteq N_F(a)$. Similarly, for $a_1, a_2 \in F^*$, let $N_F(a_1, a_2)$ denote the image in F^* of the norm map from $F(\sqrt{a_1}, \sqrt{a_2})$ to F. The following lemma is well-known, see, *e.g.*, [EL_2, (2.13)].

Lemma 1.1 *For any $a_1, a_2 \in F^*$, we have $N_F(a_1) \cap N_F(a_2) = F^{*2} \cdot N_F(a_1, a_2)$.*

Quaternion algebras provide a convenient way of organizing facts about norm groups. We will need the following well-known properties of quaternion algebras (see, *e.g.*, [Sch_2, Section 11]). For $a_1, \ldots, a_n \in F^*$, let $\langle a_1, \ldots, a_n \rangle$ denote the diagonal quadratic form $a_1X_1^2 + \cdots + a_nX_n^2$.

Lemma 1.2 *For any $a, b, c, d \in F^*$,*

(i) $(a, b/F)$ is split iff $b \in N_F(a)$ iff $a \in N_F(b)$ iff the quadratic form $\langle 1, -a, -b \rangle$ is isotropic.

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(ii) $(a,b/F) \cong (c,d/F)$ *iff* $\langle 1,-a,-b,ab\rangle \cong \langle 1,-c,-d,cd\rangle$.
(iii) $(a,b/F) \cong (b,a/F) \cong (ac^2,b/F)$.
(iv) $(a,b/F) \otimes_F (a,c/F) \cong M_2\big((a,bc/F)\big)$.
(v) $(a,-a/F)$ *is split.*

Lemma 1.3 *If $a_1,\dots,b_3 \in F^*$ satisfy $(*)$ in the Introduction, but $(a_i,b_i/F)$ is not split, then a_1,a_2,a_3 are $\mathbb{Z}/2\mathbb{Z}$-independent in F^*/F^{*2}, as are b_1,b_2,b_3.*

Proof Suppose a_3 is $\mathbb{Z}/2\mathbb{Z}$-dependent on a_1,a_2 in F^*/F^{*2}. Then, modulo squares, a_3 equals $1,a_1,a_2$, or a_1a_2. Therefore, $(a_3,b_3/F)$ is split, as $(a_1,b_3/F)$ and $(a_2,b_3/F)$ are split by $(*)$, and $(a_1a_2,b_3/F) \sim (a_1,b_3/F) \otimes_F (a_2,b_3/F)$ in the Brauer group $\mathrm{Br}(F)$, so this is also split. This contradicts the hypothesis. All other cases of $\mathbb{Z}/2\mathbb{Z}$-dependence lead to the same contradiction. ∎

Proposition 1.4 *The definition of tractability of F in the Introduction is equivalent to the definition at the beginning of Section 1.*

Proof Observe first that to say the equations in (1.1) have a nontrivial solution in F is equivalent to

$$N_F(a_1) \cap \delta N_F(a_1a_2) \cap N_F(a_3) \neq \varnothing. \tag{1.3}$$

(For this, recall that $\delta N_F(a_1a_2) = \delta^{-1}N_F(a_1a_2)$, as $F^{*2} \subseteq N_F(a_1a_2)$.) Likewise, having a nontrivial solution of (1.2) in F is equivalent to

$$N_F(a_1) \cap \delta N_F(a_1a_3) \cap N_F(a_2) \neq \varnothing. \tag{1.4}$$

Now, suppose F is tractable, as defined in the Introduction. Take any $\delta \in N_F(a_2,a_3)$, and assume there are nontrivial solutions to (1.1) and (1.2) above. Then, as we just observed, there is $b_2 \in N_F(a_1) \cap \delta N_F(a_1a_2) \cap N_F(a_3)$ and $b_3 \in N_F(a_1) \cap \delta N_F(a_1a_3) \cap N_F(a_2)$. Since $\delta \in N_F(a_2,a_3) \subseteq N_F(a_2) \cap N_F(a_3)$, we have $(a_2,\delta/F)$ and $(a_3,\delta/F)$ are split. Further, the conditions on b_2 show that $(a_1,b_2/F)$, $(a_3,b_2/F)$, and $(a_1a_2,\delta b_2/F)$ are split. But, in the Brauer group $\mathrm{Br}(F)$,

$$\begin{aligned}(a_1a_2,\delta b_2/F) &\sim (a_1,\delta/F) \otimes_F (a_1,b_2/F) \otimes_F (a_2,\delta/F) \otimes_F (a_2,b_2/F)\\ &\sim (a_1,\delta/F) \otimes_F (a_2,b_2/F).\end{aligned} \tag{1.5}$$

Hence, $(a_1,\delta/F) \cong (a_2,b_2/F)$. Likewise, the conditions on b_3 show that $(a_1,b_3/F)$ and $(a_2,b_3/F)$ are split, and $(a_1,\delta/F) \cong (a_3,b_3/F)$. Thus, we have shown that a_1, a_2, a_3, δ, b_2, b_3 satisfy condition $(*)$ of the Introduction. Since F is assumed tractable, as in the Introduction, it follows that $(a_1,\delta/F)$ is split, hence $\delta \in N_F(a_1)$. This proves that F is tractable in the sense of Section 1.

Conversely, suppose F is tractable in the sense of Section 1, and take $a_1,\dots,b_3 \in F^*$ satisfying $(*)$. We need to show $(a_i,b_i/F)$ is split. This follows by Lemma 1.3 if a_1,a_2,a_3 are $\mathbb{Z}/2\mathbb{Z}$-linearly dependent in F^*/F^{*2}. So, we may assume a_1, a_2, a_3 are $\mathbb{Z}/2\mathbb{Z}$-linearly independent. Hence, $[F(\sqrt{a_1},\sqrt{a_2},\sqrt{a_3}) : F] = 8$. Since $(a_2,b_1/F)$ and $(a_3,b_1/F)$ are split,

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we have $b_1 \in N_F(a_2) \cap N_F(a_3)$. By Lemma 1.1, there is $c \in F^*$ with $b_1c^2 \in N_F(a_2, a_3)$. Let $\delta = b_1c^2$. Because $(a_1, b_2/F)$ is split, as is $(a_2, \delta/F) \cong (a_2, b_1/F)$, and because $(a_1, \delta/F) \cong (a_1, b_1/F)$, a calculation as in (1.5) shows that $(a_1a_2, \delta b_2/F)$ is split. Since we also have $(a_3, b_2/F)$ is split, this yields $b_2 \in N_F(a_1) \cap \delta N_F(a_1a_2) \cap N_F(a_3)$, showing (1.3) holds. Likewise, $b_3 \in N_F(a_1) \cap \delta N_F(a_1a_3) \cap N_F(a_2)$, showing that (1.4) holds. Thus, as we observed above, there are nontrivial solutions to the equations in (1.1) and (1.2). Hence, by the definition of tractability above in Section 1, we have $\delta \in N_F(a_1)$. Thus, as $\delta = b_1c^2$, we have $(a_1, b_1/F) \cong (a_1, \delta/F)$, which is split. Hence, F is tractable as defined in the Introduction. ■

2 Local and Global Fields

Throughout this section, all fields F are assumed to have characteristic different from 2.

We begin with some elementary observations about tractable fields, which are sufficient to show which local fields are tractable. This will then be used as we characterize which global fields are tractable. Observe first an immediate corollary of Lemma 1.3:

Corollary 2.1 *If F is a field with $|F^*/F^{*2}| \leq 4$, then F is tractable.*

Recall that F is called a *Hilbert field* if $F^* \neq F^{*2}$ and for every $a \in F^* - F^{*2}$, we have $|F^* : N_F(a)| = 2$. This terminology was introduced by Fröhlich in [F], where he showed (see [F, Th.]) that a Hilbert field has a unique nonsplit quaternion algebra. The condition that $|F^* : N_F(a)| = 2$ for $a \notin F^{*2}$ implies that if $(a, b/F)$ and $(a, c/F)$ are nonsplit, then they are isomorphic, so $(a, bc/F)$ must be split. It follows that the map $B\colon F^*/F^{*2} \times F^*/F^{*2} \to \mathrm{Br}(F)$ given by $(aF^{*2}, bF^{*2}) \mapsto [(a, b/F)]$ is bimultiplicative and nondegenerate, with image a group of order 2.

Proposition 2.2 *If F is a Hilbert field, then F is tractable iff $|F^*/F^{*2}| \leq 4$.*

Proof If $|F^*/F^{*2}| \leq 4$, Cor. 2.1 shows that F is tractable.

Before proving the converse, we note a consequence of the nondegeneracy of the pairing. Let $S = F^*/F^{*2}$, a $\mathbb{Z}/2\mathbb{Z}$-vector space, and view the pairing into $\mathrm{Br}(F)$ as a nondegenerate symmetric $\mathbb{Z}/2\mathbb{Z}$-bilinear map $B\colon S \times S \to \mathbb{Z}/2\mathbb{Z}$. This B induces a $\mathbb{Z}/2\mathbb{Z}$-linear map $\alpha\colon S \to S^* = \mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(S, \mathbb{Z}/2\mathbb{Z})$ given by $\alpha(s)(t) = B(s, t)$; the nondegeneracy of B is equivalent to the injectivity of α. For any $\mathbb{Z}/2\mathbb{Z}$-subspace W of S, let $\beta_W\colon S \to W^*$ be the composition of α followed by the canonical map $S^* \to W^* = \mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(W, \mathbb{Z}/2\mathbb{Z})$, *i.e.*, $\beta_W(s)(w) = B(s, w)$. Observe that if $\dim_{\mathbb{Z}/2\mathbb{Z}}(W) < \infty$, then β_W is surjective. For, if $\mathrm{im}(\beta_W) \subsetneqq W^*$ then there is a $\mathbb{Z}/2\mathbb{Z}$-subspace Y of W such that $\mathrm{im}(\beta_W) = \{w^* \in W^* \mid w^*(Y) = 0\}$ and $\dim_{\mathbb{Z}/2\mathbb{Z}}(Y) = \dim_{\mathbb{Z}/2\mathbb{Z}}(W^*) - \dim_{\mathbb{Z}/2\mathbb{Z}}\big(\mathrm{im}(\beta_W)\big)$. But then, $Y \subseteq \ker(\alpha)$, contradicting the injectivity of α. So, β_W must be surjective, yielding $|S : \ker(\beta_W)| = |W^*| = |W|$. Thus, for any $\mathbb{Z}/2\mathbb{Z}$-linearly independent $c_1, \ldots, c_n \in S$, the homomorphism $S \to (\mathbb{Z}/2\mathbb{Z})^n$ given by $e \mapsto \big(B(c_1, e), \ldots, B(c_n, e)\big)$ must be surjective, since its kernel, which is $\ker(\beta_{\mathbb{Z}/2\mathbb{Z}-\mathrm{span}(c_1,\ldots,c_n)})$, has index 2^n. Hence, there are $d_1, \ldots, d_n \in S$ with $B(c_i, d_j) = \delta_{ij}$ (Kronecker delta).

Now, suppose $|F^*/F^{*2}| \geq 8$. Let $a_1, a_2, a_3 \in F^*$ map to $\mathbb{Z}/2\mathbb{Z}$-linearly independent elements of F^*/F^{*2}. By the preceding paragraph, there exist $b_1, b_2, b_3 \in F^*$ with $(a_i, b_j/F)$

split iff $j \neq i$. In particular, each $(a_i, b_i/F)$ is the unique nonsplit quaternion algebra of F. Since $(*)$ thus holds for $a_1, \ldots, b_3$ with $(a_i, b_i/F)$ nonsplit, F is not tractable. ■

Corollary 2.3 *A local field F is tractable iff F is nondyadic.*

Proof By definition, a local field F is a field with a complete discrete valuation with finite residue field $\overline{F}$. Such an F is dyadic if $\operatorname{char}(\overline{F}) = 2$. As is well-known, if F is nondyadic, then $|F^*/F^{*2}| = 4$. But, if F is dyadic, then (since we are assuming $\operatorname{char}(F) \neq 2$) the 2-adic completion $\mathbb{Q}_2$ of $\mathbb{Q}$ embeds in F as a subfield of finite degree, and $|F^*/F^{*2}| = 2^{2+[F:\mathbb{Q}_2]} \geq 8$ (see [L, Cor. 2.30, p. 166]). Since a local field is a Hilbert field (see [OM, 63:15(ii)]), the corollary follows from Prop. 2.2. ■

Remark 2.4 In [K], Kaplansky defined a "generalized Hilbert field" F as one in which $F^* \neq F^{*2}$ and for each $c \in F^*$, we have $|F^* : N_F(c)| \leq 2$. Let $R_F = \bigcap_{c \in F^*} N_F(c) = \{d \in F^* \mid N_F(d) = F^*\}$, Kaplansky's radical of the generalized Hilbert field F. Clearly, R_F is a subgroup of F^*, with $F^{*2} \subseteq R_F \subseteq F^*$. Kaplansky showed that a generalized Hilbert field has a unique nonsplit quaternion algebra. Knowing this, the arguments given for Prop. 2.2 show that a generalized Hilbert field F is tractable iff $|F^*/R_F| \leq 4$.

Before proceeding to global fields, we observe another "local" result.

Proposition 2.5 *Let F be a field with Henselian valuation v, with residue field $\overline{F}$, such that* $\operatorname{char}(\overline{F}) \neq 2$. *Then F is tractable iff $\overline{F}$ is tractable.*

Proof Let V be the valuation ring of v, let $\mathfrak{m}$ be the maximal ideal of V, and let $U = V - \mathfrak{m}$, the group of valuation units; so $\overline{F} = V/\mathfrak{m}$. Let Γ be the value group of v (with group operation written additively), so v maps F^* onto Γ, with kernel U. For $c \in U$, we write $\overline{c}$ for the image of c in $\overline{F}^*$. Because v is Henselian and $\operatorname{char}(\overline{F}) \neq 2$, an easy application of Hensel's Lemma gives us the crucial property that $1 + \mathfrak{m} \subseteq F^{*2}$. Hence, for $c \in U$, we have $c \in F^{*2}$ iff $\overline{c} \in \overline{F}^{*2}$. It follows quickly from this, just as in the proof of Springer's Theorem for a complete discretely valued field, that for $c_1, \ldots, c_n, d_1, \ldots, d_n \in U$, $\langle c_1, \ldots, c_n\rangle \cong \langle d_1, \ldots, d_n\rangle$ as quadratic forms over F, iff $\langle \overline{c_1}, \ldots, \overline{c_n}\rangle \cong \langle \overline{d_1}, \ldots, \overline{d_n}\rangle$ as forms over $\overline{F}$; also, $\langle c_1, \ldots, c_n\rangle$ is isotropic iff $\langle \overline{c_1}, \ldots, \overline{c_n}\rangle$ is isotropic. Hence, $(c_1, c_2/F) \cong (d_1, d_2/F)$ iff $(\overline{c_1}, \overline{c_2}/\overline{F}) \cong (\overline{d_1}, \overline{d_2}/\overline{F})$.

Now, suppose $\overline{F}$ is not tractable. Then there exist $\tilde{a}_1, \ldots, \tilde{b}_3 \in \overline{F}^*$ satisfying $(*)$ over $\overline{F}$, but with $\left(\widetilde{a}_i, \widetilde{b}_i/\overline{F}\right)$ nonsplit. Choose inverse images $a_1, \ldots, b_3$ in U of $\widetilde{a}_1, \ldots, \widetilde{b}_3$. The observations in the previous paragraph show $a_1, \ldots, b_3$ satisfy $(*)$ but $(a_i, b_i/F)$ is not split. Thus, F is not tractable.

Before proving the converse, recall that if $c \in F^*$ satisfies $v(c) \notin 2\Gamma$, then $N_F(c) = F^{*2} \cup -cF^{*2}$. For, if $d = r^2 - cs^2$, then $v(r^2) \neq v(cs^2)$. If $v(r^2) < v(cs^2)$, then

$$d = r^2(1 - cs^2/r^2) \in F^{*2} \cdot (1 + \mathfrak{m}) = F^{*2}.$$

On the other hand, if $v(r^2) > v(cs^2)$, then

$$d = -cs^2(1 - r^2/cs^2) \in -cF^{*2} \cdot (1 + \mathfrak{m}) = -cF^{*2}.$$

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Now, suppose F is not tractable, and choose $a_1, \ldots, b_3 \in F^*$ satisfying $(*)$ with $(a_i, b_i/F)$ nonsplit. Suppose $v(a_1) \notin 2\Gamma$. Since $(a_1, b_2/F)$ is split, $b_2 \in N_F(a_1)$. Because $b_2 \notin F^{*2}$, as $(a_2, b_2/F)$ is not split, the preceding paragraph shows $b_2 \in -a_1 F^{*2}$. Likewise, $b_3 \in -a_1 F^{*2}$. This contradicts the $\mathbb{Z}/2\mathbb{Z}$-independence of b_2 and b_3 in F^*/F^{*2} (see Lemma 1.3). Hence, we must have $v(a_1) \in 2\Gamma$. Likewise, $v(a_2), v(a_3), v(b_1), v(b_2), v(b_3) \in 2\Gamma$. Therefore, we can change $a_1, \ldots, b_3$ by multiplying each by a suitable square without changing the quaternion algebras $(a_i, b_j/F)$, but so that now $a_1, \ldots, b_3 \in U$. Then, by the first paragraph of the proof, $\overline{a_1}, \ldots, \overline{b_3}$ satisfy $(*)$ over $\overline{F}$, but $\left(\overline{a_i}, \overline{b_i}/\overline{F}\right)$ is nonsplit, showing that $\overline{F}$ is not tractable. ∎

Remark 2.6 The only properties of the valuation used in proving Prop. 2.5 were that $\mathrm{char}(\overline{F}) \neq 2$ and $1 + \mathfrak{m} \subseteq F^{*2}$. A valuation v satisfying just these properties is said to be 2-Henselian (with $\mathrm{char}(\overline{F}) \neq 2$), because v has a unique extension to each Galois extension of F of degree a 2-power (see, *e.g.*, [W_1, Prop. 1.2]). So, Prop. 2.5 still holds if we replace "Henselian" by the weaker assumption "2-Henselian".

If F is a global field, let Ω_F denote the set of prime spots of F, and for $\mathfrak{p} \in \Omega_F$, let $F_\mathfrak{p}$ denote the corresponding complete field. So, either $F_\mathfrak{p}$ is a local field, in which case $\mathfrak{p}$ is said to be *finite*; or $F_\mathfrak{p} \cong \mathbb{R}$ and $\mathfrak{p}$ is called *real infinite*; or $F_\mathfrak{p} \cong \mathbb{C}$, and $\mathfrak{p}$ is *complex infinite*. If $F_\mathfrak{p}$ is a dyadic local field, we say that $\mathfrak{p}$ is dyadic. If A is any central simple algebra over F and $\mathfrak{p} \in \Omega_F$, then we write $A_\mathfrak{p}$ for $A \otimes_F F_\mathfrak{p}$. The following known results are key to our analysis of tractability for global fields. The first two are classical (see [OM, 66:4, 71:18, 71:19] for the number field case), the third a special case of a lemma given by Tate in [T], which he describes as "more-or-less well-known".

2.7 **Hasse-Minkowski Theorem (special case)** *Let F be a global field. For $a, b, c, d \in F^*$, $(a, b/F) \cong (c, d/F)$ iff $\left(a, b/F_\mathfrak{p}\right) \cong \left(c, d/F_\mathfrak{p}\right)$ for all $\mathfrak{p} \in \Omega_F$. In particular (taking $c = 1$), $(a, b/F)$ is split iff $\left(a, b/F_\mathfrak{p}\right)$ is split for all $\mathfrak{p} \in \Omega_F$.*

2.8 **Hilbert's Reciprocity Law** *Let F be a global field. For any $a, b \in F^*$, $\left(a, b/F_\mathfrak{p}\right)$ is split for all but a finite even number of $\mathfrak{p} \in \Omega_F$. Further, given any finite subset S of Ω_F with $|S|$ even, there is a unique quaternion algebra Q over F with $Q_\mathfrak{p}$ nonsplit iff $\mathfrak{p} \in S$.*

Lemma 2.9 **[T, Lemma 5.2]** *For any global field F, let $c_1, \ldots, c_k \in F^*$, and let $Q_1, \ldots, Q_k$ be quaternion algebras over F. Suppose that for each j, $1 \leq j \leq k$, and for each $\mathfrak{p} \in \Omega_F$, there is $d_{j,\mathfrak{p}} \in F_\mathfrak{p}^*$ such that $(Q_j)_\mathfrak{p} \cong \left(c_j, d_{j,\mathfrak{p}}/F_\mathfrak{p}\right)$. Then there exists $d \in F^*$ with $Q_j \cong (c_j, d/F)$, for each j.*

Theorem 2.10 *Let F be a global field. Then F is tractable iff F has at most one dyadic place.*

Proof Suppose F has at most one dyadic place. Take any $a_1, \ldots, b_3 \in F^*$ satisfying $(*)$ over F. For any $\mathfrak{p} \in \Omega_F$, $a_1, \ldots, b_3$ satisfy $(*)$ over $F_\mathfrak{p}$, as well. If $\mathfrak{p}$ is nondyadic, then $(a_i, b_i/F_\mathfrak{p})$ must be split, as $F_\mathfrak{p}$ is tractable, by Cor. 2.3 if $\mathfrak{p}$ is finite and Cor. 2.1 if $\mathfrak{p}$ is infinite. Since $(a_i, b_i/F_\mathfrak{p})$ is thus split for all but at most one $\mathfrak{p}$, Hilbert reciprocity shows that $(a_i, b_i/F_\mathfrak{p})$ is actually split for each $\mathfrak{p} \in \Omega_F$. Then, Hasse-Minkowski shows $(a_i, b_i/F)$ is split. Thus, F is tractable.

Now, suppose F has two dyadic places $\mathfrak{q}_1, \mathfrak{q}_2$. (It may have other dyadic places, as well.) For $j = 1, 2$, let Q_j be the nonsplit quaternion algebra over $F_{\mathfrak{q}_j}$, and let $a_{1,j}, \ldots, b_{3,j} \in F^*_{\mathfrak{q}_j}$ with $(a_{i,j}, b_{i,j}/F_{\mathfrak{q}_j}) \cong Q_j$ and $(a_{i,j}, b_{k,j}/F_{\mathfrak{q}_j})$ split for $k \neq i$. The $a_{i,j}, b_{i,j}$ exist by Cor. 2.3. By Hilbert Reciprocity there is a quaternion algebra Q over F with $Q_{\mathfrak{q}_j} \cong Q_j$ and $Q_{\mathfrak{p}}$ split for all $\mathfrak{p} \in \Omega_F - \{\mathfrak{q}_1, \mathfrak{q}_2\}$. By the (Weak) Approximation Theorem [We, p. 8], there are $a_1, a_2, a_3 \in F^*$ with $a_i \equiv a_{i,j} (\mathrm{mod} F^{*2}_{\mathfrak{q}_j})$ in $F^*_{\mathfrak{q}_j}$, for $j = 1, 2$.

This assures that $Q_{\mathfrak{q}_j} \cong Q_j \cong \big(a_1, b_{1,j}/F_{\mathfrak{q}_j}\big)$ for $j = 1, 2$. Also, for $\mathfrak{p} \in \Omega_F - \{\mathfrak{q}_1, \mathfrak{q}_2\}$, clearly $Q_{\mathfrak{p}} \cong M_2(F_{\mathfrak{p}}) \cong \big(a_1, 1/F_{\mathfrak{p}}\big)$. Furthermore, for all $\mathfrak{p} \in \Omega_F$, we have $M_2(F)_{\mathfrak{p}} \cong \big(a_2, 1/F_{\mathfrak{p}}\big) \cong \big(a_3, 1/F_{\mathfrak{p}}\big)$. It follows by Tate's Lemma 2.9 that there is $b_1 \in F^*$ with $(a_1, b_1/F) \cong Q$ while $(a_2, b_1/F)$ and $(a_3, b_1/F)$ are each split. Two more analogous applications of Tate's Lemma 2.9 yield $b_2, b_3 \in F^*$ with $(a_2, b_2/F) \cong Q$ while $(a_1, b_2/F)$ and $(a_3, b_2/F)$ are split; and $(a_3, b_3/F) \cong Q$ while $(a_1, b_3/F)$ and $(a_2, b_3/F)$ are split. Thus, $a_1, \ldots, b_3$ satisfy $(*)$ over F, while each $(a_i, b_i/F) \cong Q$, which is nonsplit. Hence, F is not tractable. ■

Examples 2.11 (a) If F is a global function field (*i.e.*, F is a global field with $\mathrm{char}(F) \neq 0$), then F is tractable, since it has no dyadic places.

(b) $\mathbb{Q}$ is tractable. This was originally proved in [CDD, Th. 7] by a direct (and also longer) argument.

(c) For $d \in \mathbb{Z}$ with d square-free, $d \neq 1$, we have $\mathbb{Q}(\sqrt{d})$ is tractable iff $d \not\equiv 1 \pmod 8$.

(d) If $F = \mathbb{Q}(c)$ with c algebraic over $\mathbb{Q}$, then F is tractable iff the minimal polynomial f of c over $\mathbb{Q}$ remains irreducible over the dyadic completion $\mathbb{Q}_2$ of $\mathbb{Q}$. For, the number of dyadic places of F equals the number of irreducible factors of f in $\mathbb{Q}_2[x]$.

3 Tractable and Nontractable Function Fields

We give here further examples of tractable and nontractable fields. In particular, we prove that for a global field F, the rational function field $F(x)$ is tractable iff F is tractable. We continue to assume that all fields have characteristic not 2.

Recall that the *level* $s(F)$ of a field F is n if -1 is expressible as a sum of n squares in F, but not as a sum of $n - 1$ squares. If there is no such n, then we set $s(F) = \infty$. It is known (see [L, p. 303]) that $s(F) = \infty$ iff F has an ordering. Also (see [L, Th. 2.2, p. 303]), if $s(F) = n < \infty$, then n is a power of 2 (a theorem of Pfister). For example, $s(\mathbb{Q}) = s(\mathbb{R}) = \infty$, $s\big(\mathbb{Q}(\sqrt{-3})\big) = s\big(\mathbb{Q}(\sqrt{-5})\big) = 2$, $s\big(\mathbb{Q}(\sqrt{-7})\big) = 4$, and if F is a global field, either $s(F) = \infty$ or $s(F) \leq 4$.

Remark 3.1 Note that for any field F, we have $s(F) = 4$ iff -1 is a sum of four squares in F and $(-1, -1/F)$ is nonsplit. For, $(-1, -1/F)$ is split iff $-1 \in N_F(-1)$ iff -1 is a sum of two squares in F.

Proposition 3.2 *Let F be a field of level* 4*. Then, F is not tractable.*

Proof Say $-1 = r^2 + s^2 + t^2 + u^2$, where $r, s, t, u \in F$. Let $c = r^2 + s^2$ and $d = t^2 + u^2$. Since $s(F) \geq 4$, we have $r, s, t, u, c, d \in F^*$. Now, $(-1, -1/F)$ is nonsplit (see Remark 3.1), but as $c, d \in N_F(-1)$, the quaternion algebras $(-1, c/F)$, $(-1, d/F)$, and $(-1, cd/F)$ are split. Hence, as $(c, c/F) \cong (-1, c/F)$ and $(d, d/F) \cong (-1, d/F)$ we also have $(c, c/F)$ and $(d, d/F)$ are split. The quadratic form isometry $\langle c, d\rangle \cong \langle c+d, cd(c+d)\rangle$ shows

$$(c, d/F) \cong \big(c+d, -cd/F\big) = \big(-1, -cd/F\big) \cong (-1, -1/F).$$

Thus, by taking $a_1 = b_1 = -1$, $a_2 = b_3 = c$, $a_3 = b_2 = d$, we see that F is not tractable. ■

Examples 3.3 (a) $\mathbb{Q}(\sqrt{-7})$ is not tractable, as we already saw in Ex. 2.11(c).

(b) Let x be transcendental over $\mathbb{Q}$. Then, $\mathbb{Q}(x, \sqrt{-3-x^2})$ is not tractable.

Corollary 3.4 *Let F be a field such that $(-1, -1/F)$ is not split, but there exist $a, b \in F^*$, each a sum of two squares, such that $(-a, -b/F)$ is split. Then F has level 4, so F is not tractable.*

Proof Since $(-a, -b/F)$ is split, the quadratic form $\langle 1, a, b\rangle$ is isotropic (see Lemma 1.2(i)). Hence, $-1 = ar^2 + bs^2$ for some $r, s \in F$. Since a and b are sums of two squares, this shows -1 is a sum of four squares. Hence, $s(F) = 4$, by Remark 3.1, so F is not tractable, by Prop. 3.2. ■

Example 3.5 Any field F ($\mathrm{char}(F) \neq 2$, $\mathrm{char}(F) \neq 5$) for which $(-1, -1/F)$ is nonsplit but $(-2, -5/F)$ is split is not tractable. In particular, any subfield F of $\mathbb{Q}_2$ such that $(-2, -5/F)$ is split is not tractable.

Some of our further results will use specialization arguments. Let R be a local integral domain with maximal ideal $\mathfrak{m}$ and quotient field K. Let $\overline{R} = R/\mathfrak{m}$. Let $\mathrm{Br}(R)$ denote the Brauer group of equivalence classes of Azumaya algebras over R, *cf.* [OS] or [DI]. There is a "specialization map" $\rho_R\colon \mathrm{Br}(R) \to \mathrm{Br}(\overline{R})$, which is just scalar extension, $[A] \mapsto [A \otimes_R \overline{R}]$. We also have the scalar extension homomorphism $\mathrm{Br}(R) \to \mathrm{Br}(K)$, $[A] \mapsto [A \otimes_R K]$. The latter map is known to be injective if R is a valuation ring (see [JW, Prop. 2.5] or [S_1, Lemma 1.2]) or if R is a regular local ring (see [OS, Th. 6.19, p. 67]). When this occurs, we view the specialization map as defined on a subgroup of $\mathrm{Br}(K)$. Namely, if $[B] \in \mathrm{Br}(K)$, we say that B is specializable with respect to R if $[B] \in \mathrm{im}\big(\mathrm{Br}(R) \to \mathrm{Br}(K)\big)$, and in that case, define $\rho_R([B]) = \rho_R([A])$ for the unique $[A] \in \mathrm{Br}(R)$ with $[B] = [A \otimes_R K]$. For example, suppose $\mathrm{char}(\overline{R}) \neq 2$, and let $Q = (c, d/K)$ be a quaternion algebra over K. If $c, d \in R - \mathfrak{m}$, then Q is specializable re R, since $Q \cong (c, d/R) \otimes_R K$, where $(c, d/R)$ denotes the R-algebra which is free as an R-module, with base $\{1, i, j, k\}$ and multiplication defined by $i^2 = c$, $j^2 = d$, and $ij = -ji = k$. Since $\mathrm{char}(\overline{R}) \neq 2$, $(c, d/R)$ is an Azumaya algebra over R. We have $\rho_R(Q) = \big(\overline{c}, \overline{d}/\overline{R}\big)$, where $\overline{c}$, $\overline{d}$ are the images of c, d in $\overline{R}$. Note also that if F is a field which is a subring of R, then for $[D] \in \mathrm{Br}(F)$, we have $D \otimes_F K$ is specializable re R, and $\rho_R([D \otimes_F K]) = [D \otimes_F \overline{R}]$.

Proposition 3.6 *Suppose F is not a tractable field. Let K be a field containing F such that K is the quotient field of a ring R which is a valuation ring or a regular local ring with residue field F. Then K is not tractable.*

Proof Take $a_1, \ldots, b_3 \in F^*$ satisfying $(*)$ over F, with $(a_i, b_i/F)$ nonsplit. Of course, $a_1, \ldots, b_3$ also satisfy $(*)$ over K. Let $\overline{R} = R/\mathfrak{m}$, where $\mathfrak{m}$ is the unique maximal ideal of R.

Because the composition $F \hookrightarrow R \to \overline{R}$ is assumed to be an isomorphism, $(a_i, b_i/R)$ is nontrivial, since $\rho_R\big((a_i, b_i/R)\big) \cong (a_i, b_i/F)$, which is nonsplit. Since the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is injective, $(a_i, b_i/K)$ must be nonsplit. Thus, K is not tractable. ■

Example 3.7 If F is not tractable and K is the function field $F(X)$ of an integral variety X over F such that X has a smooth rational point, then K is not tractable. For, the ring $\mathcal{O}_{X,x}$ of rational functions on X defined near x is a regular local ring with quotient field $F(X)$ and residue field F. For example, if K is a purely transcendental extension of the nontractable field F, then K is not tractable.

Proposition 3.8 *Suppose F is a nontractable global field, and let $Q_1, \dots, Q_n$ be the nonsplit quaternion algebras over F which are split by every $F_\mathfrak{p}$ for $\mathfrak{p}$ nondyadic. If $K \supseteq F$ is a field such that some Q_j is not split by K, then K is not tractable.*

Proof The proof of Th. 2.10 shows that there are $a_1, \dots, b_3 \in F^*$ satisfying $(*)$ with $(a_i, b_i/F) \cong Q_j$. Then $a_1, \dots, b_3$ satisfy $(*)$ over K, and since $(a_i, b_i/K) \cong Q_j \otimes_F K$ is nonsplit, K is not tractable. (Note that these Q_j are the only quaternion algebras that appear as $(a_i, b_i/F)$ in a counterexample to tractability of F. For, if any other nonsplit quaternion algebra Q over F were to appear, there would be a nondyadic place $\mathfrak{p}$ such that $Q_\mathfrak{p}$ is nonsplit, showing $F_\mathfrak{p}$ is not tractable, contradicting Cor. 2.3.) ■

Example 3.9 (a) If F is a nontractable global field, with $m \geq 3$ dyadic places and K is an algebraic function field in one variable over F of genus 0, then K is not tractable. For, it is known (see [A, Th. 6, p. 302]) that K has genus 0 over F iff K has the form $K = F(x, \sqrt{c + dx^2})$, where $c, d \in F^*$ and x is transcendental over F; further, for this K, the map $\mathrm{Br}(F) \to \mathrm{Br}(K)$ has kernel $\{[F], [(c, d/F)]\}$, see [Wi_1, Satz, p. 465] or [Sch_1, Kor. 2.3, Bem. p. 5]. Hilbert reciprocity shows that there are $2^{m-1} - 1$ nonsplit quaternion algebras Q_i over F which are split for every $F_\mathfrak{p}$ with $\mathfrak{p}$ nondyadic. So, at least one Q_j is not split by K, and Prop. 3.8 shows that K is not tractable.

(b) If a global field F has exactly two dyadic places $\mathfrak{q}_1, \mathfrak{q}_2$ and $(a, b/F)$ is nonsplit at the $F_{\mathfrak{q}_j}$ and at no other $F_\mathfrak{p}$, then I. Han at UCSD shows in [H] that $F(x, \sqrt{a + bx^2})$ is tractable even though F is not. However, every other function field K of genus 0 over this F is not tractable, by Prop. 3.8, because $(a, b/K)$ is not split.

For F a global field, and Q a quaternion algebra over F, define

$$\mathrm{supp}(Q) = \{\mathfrak{p} \in \Omega_F \mid Q_\mathfrak{p} \text{ is nonsplit}\}.$$

If, further, $\mathrm{char}(F) = 0$ let $\mathcal{R}$ denote the set of real infinite prime spots of F, and let $\mathcal{D}$ denote the set of dyadic prime spots of F.

Proposition 3.10 *Suppose F is a tractable global field. Let $Q = (c, d/F)$ be a nonsplit quaternion algebra over F, and let $K = F(x, \sqrt{c + dx^2})$, where x is transcendental over F. In the following cases, K is not tractable:*

(i) $\mathrm{char}(F) \neq 0$ *and* $|\mathrm{supp}(Q)| \geq 4$.

(ii) $\mathrm{char}(F) = 0$ *and there is a nonsplit quaternion algebra P over F with $P \not\cong Q$ and either*

(a) $\mathrm{supp}(P) \subseteq \mathrm{supp}(Q) \cup \mathcal{D} - \mathcal{R}$*; or*

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(b) $\operatorname{supp}(Q) \cap \mathcal{R} \subseteq \operatorname{supp}(P) \subseteq \operatorname{supp}(Q) \cup \mathcal{D}$.

Proof This is another variation on the proof of Th. 2.10 above. We will construct $a_1, \dots, b_3$ over F which are close to being a counterexample to tractability over F, which yield an actual counterexample over K. In case (i), let P be a quaternion algebra over F with $\operatorname{supp}(P) \subsetneqq \operatorname{supp}(Q)$ and $|\operatorname{supp}(P)| = 2$. Assume now we are in case (i) or (ii)(a). We want to find $a_1, \dots, b_3 \in F^*$ such that $(a_i, b_j/F)$ is split for $i \neq j$ and $(a_1, b_1/F) \cong (a_3, b_3/F) \cong P$, while $(a_2, b_2/F) \sim P \otimes_F Q$ in $\operatorname{Br}(F)$. (Recall that, as F is a global field, there is a quaternion algebra R over F with $R \sim P \otimes_F Q$ in $\operatorname{Br}(F)$. Indeed, R is the quaternion algebra with $\operatorname{supp}(R) = \big(\operatorname{supp}(P) \cup \operatorname{supp}(Q)\big) - \big(\operatorname{supp}(P) \cap \operatorname{supp}(Q)\big)$.)

We first choose $a_1, a_2, a_3 \in F^*$ using the Weak Approximation Theorem so that for $\mathfrak{p} \in \Omega_F$,

(I) if $\mathfrak{p} \in \operatorname{supp}(Q) - \operatorname{supp}(P)$, then $a_1, a_3 \in F_{\mathfrak{p}}^{*2}$, while $a_2 \notin F_{\mathfrak{p}}^{*2}$;

(II) if $\mathfrak{p} \in \operatorname{supp}(P) \cap \operatorname{supp}(Q)$ (so $\mathfrak{p} \notin \mathcal{R}$), then a_1, a_3 are $\mathbb{Z}/2\mathbb{Z}$-independent in $F_{\mathfrak{p}}^*/F_{\mathfrak{p}}^{*2}$ and $a_2 \in F_{\mathfrak{p}}^{*2}$;

(III) if $\mathfrak{p} \in \operatorname{supp}(P) - \operatorname{supp}(Q)$ (so $\mathfrak{p}$ must be dyadic), then a_1, a_2, a_3 are $\mathbb{Z}/2\mathbb{Z}$-independent in $F_{\mathfrak{p}}^*/F_{\mathfrak{p}}^{*2}$.

Now, choose $b_{1,\mathfrak{p}} \in F_{\mathfrak{p}}^*$ so that for $\mathfrak{p}$ in case (I) $b_{1,\mathfrak{p}} = 1$; for $\mathfrak{p}$ in case (II) or in case (III), $(a_1, b_{1,\mathfrak{p}}/F_{\mathfrak{p}})$ is nonsplit, while $(a_2, b_{1,\mathfrak{p}}/F_{\mathfrak{p}})$ and $(a_3, b_{1,\mathfrak{p}}/F_{\mathfrak{p}})$ are split; and for all other $\mathfrak{p}$, $b_{1,\mathfrak{p}} = 1$. By Tate's Lemma 2.9, there is $b_1 \in F^*$ with $(a_1, b_1/F_{\mathfrak{p}}) \cong (a_1, b_{1,\mathfrak{p}}/F_{\mathfrak{p}}) \cong P_{\mathfrak{p}}$ and $(a_2, b_1/F_{\mathfrak{p}})$ and $(a_3, b_1/F_{\mathfrak{p}})$ split for all $\mathfrak{p} \in \Omega_F$. Hence, $(a_1, b_1/F) \cong P$, while $(a_2, b_1/F)$ and $(a_3, b_1/F)$ are split. Likewise, we can find $b_2 \in F^*$ with $(a_2, b_2/F) \cong P \otimes_F Q$ while $(a_1, b_2/F)$ and $(a_3, b_2/F)$ are split; likewise, we obtain $b_3 \in F^*$ with $(a_3, b_3/F) \cong P$ while $(a_1, b_3/F)$ and $(a_2, b_3/F)$ are split. Since K splits Q, $a_1, \dots, b_3$ satisfy $(*)$ over K (even though not over F), with $(a_i, b_i/K) \cong P \otimes_F K$. But, $P \otimes_F K$ is nonsplit, since $\ker\big(\operatorname{Br}(F) \to \operatorname{Br}(K)\big) = \{[F], [Q]\}$. Hence, K is not tractable.

There remains case (ii)(b). In this case, we want to find $a_1, \dots, b_3 \in F^*$ such that $(a_i, b_j/F)$ is split for $i \neq j$ and $(a_1, b_1/F) \cong P$, while $(a_2, b_2/F) \sim (a_3, b_3/F) \sim P \otimes_F Q$ in $\operatorname{Br}(F)$. We find the a_1, a_2, a_3 as before, with the only changes that for $\mathfrak{p}$ in case (I) (so now $\mathfrak{p} \notin \mathcal{R}$), $a_1 \in F_{\mathfrak{p}}^{*2}$, while a_2, a_3 are $\mathbb{Z}/2\mathbb{Z}$-linearly independent in $F_{\mathfrak{p}}^*/F_{\mathfrak{p}}^{*2}$; and, for $\mathfrak{p}$ in case (II), (so this time possibly $\mathfrak{p} \in \mathcal{R}$), $a_1 \notin F_{\mathfrak{p}}^{*2}$ while $a_2, a_3 \in F_{\mathfrak{p}}^{*2}$. The argument then proceeds as before, with the obvious modifications in the choices of the $b_{i,\mathfrak{p}}$. ■

Example 3.11 If F is a tractable algebraic number field, and $Q = (c, d/F)$ is nonsplit at two or more finite nondyadic prime spots, then (ii)(a) shows the corresponding field $K = F(x, \sqrt{c + dx^2})$ is not tractable.

Note 3.12 For fields $K = \mathbb{Q}(x, \sqrt{c + dx^2})$ with $c, d \in \mathbb{Q}^*$, the cases not covered by Prop. 3.10 are (for $Q = (c, d/\mathbb{Q})$),

(i) $\operatorname{supp}(Q) = \varnothing$;

(ii) $\operatorname{supp}(Q) = \{p, \infty\}$, p an odd prime and ∞ the real infinite prime spot;

(iii) $\operatorname{supp}(Q) = \{2, p\}$, p an odd prime;

(iv) $\operatorname{supp}(Q) = \{2, \infty\}$, (*i.e.*, $Q = (-1, -1/\mathbb{Q})$).

In case (i), $K \cong \mathbb{Q}(x)$, and we will show in Th. 3.13 below that K is tractable. In case (ii), K has level 4, so is not tractable, by Prop. 3.2. Since this paper was written, I. Han has shown that in cases (iii) (where $s(K) = \infty$) and (iv) (where $s(K) = 2$) K is tractable. He has shown that analogous results hold for function fields of genus 0 over any tractable global field with at most one ordering. His proof will appear in [H].

Fein, Saltman, and Schacher introduced in [FSS] the notion of a Brauer-Hilbertian field to provide a division algebra analogue to the idea of a Hilbertian field, which is one over which Hilbert's Irreducibility Theorem holds. For background on Hilbertian fields, see [FJ]. (Hilbertian fields must not be confused with the Hilbert fields of Section 2 above!) For any field F and any $c \in F$, there is an associated discrete valuation ring R_c of the rational function field $F(x)$ defined by $R_c = F[x]_{(x-c)} = \{f/g \mid f, g \in F[x], g(c) \neq 0\}$ with residue field $\overline{R_c} = F$. The associated specialization map $\mathrm{Br}(R_c) \to \mathrm{Br}(F)$ is denoted ρ_c. So, for $f_1, g_1, f_2, g_2 \in F[x]$ with $f_1(c)g_1(c)f_2(c)g_2(c) \neq 0$, $\big(f_1/g_1,\, f_2/g_2 \big/ F(x)\big)$ is specializable at c (*i.e.*, with respect to R_c), and $\rho_c\big(\big(f_1/g_1,\, f_2/g_2 \big/ F(x)\big)\big) = \big(f_1(c)/g_1(c),\, f_2(c)/g_2(c) \big/ F\big)$. For a central simple algebra A over F, let $\exp(A)$ denote the exponent of A, which is the order of $[A]$ as an element of $\mathrm{Br}(F)$. Let $\mathrm{Br}'(F)$ denote the subgroup of $\mathrm{Br}(F)$ consisting to those $[A]$ with $\exp(A)$ relatively prime to $\mathrm{char}(F)$. (So, if $\mathrm{char}(F) = 0$, then $\mathrm{Br}'(F) = \mathrm{Br}(F)$.) As defined in [FSS], a field F is *Brauer-Hilbertian* if for every $[A] \in \mathrm{Br}'\big(F(x)\big)$ there are infinitely many $c \in F$ at which A is specializable and $\exp\big(\rho_c(A)\big) = \exp(A)$. It is shown in [FSS, Th. 2.5, Th. 2.6] that every global field is Brauer-Hilbertian, and if F is any Hilbertian field, then every finitely and separably generated extension of F of transcendence degree ≥ 1 is also Brauer-Hilbertian.

Theorem 3.13 *If F is a tractable Brauer-Hilbertian field, then the rational function field $F(x)$ is tractable. If, in addition, $F(x)$ is Brauer-Hilbertian, then every purely transcendental extension of F is tractable.*

Proof If the rational function field $F(x)$ is not tractable, there exist $a_1, \ldots, b_3 \in F(x)$ satisfying $(*)$ over $F(x)$, but with $(a_i, b_i \big/ F(x))$ nonsplit. For any $e \in F(x)^*$, there are only finitely many $d \in F$ such that e is not a unit of the discrete valuation ring R_d. Consequently, as F is Brauer-Hilbertian, we can find $c \in F$ such that all $a_1, \ldots, b_3$ are units of R_c and $\rho_c\big((a_i, b_i \big/ F(x))\big)$ is nonsplit. Let $\overline{a_i}, \overline{b_j}$ denote the images of a_i, b_j in the map $R_c \to \overline{R_c} = F$ (the map is evaluation at c). Then, $\overline{a_1}, \ldots, \overline{b_3}$ satisfy $(*)$ over F, as ρ_c is a group homomorphism, while $\big(\overline{a_i}, \overline{b_i} \big/ F\big) = \rho_c\big((a_i, b_i \big/ F(x))\big)$ is nonsplit. This contradicts the tractability of F.

When F and $F(x)$ are both Brauer-Hilbertian, then by [FSS, Th. 2.6], $F(x_1, \ldots, x_n)$ is Brauer-Hilbertian for every $x_1, \ldots, x_n$ algebraically independent over F. It then follows by induction on n that when F is also tractable, so is $F(x_1, \ldots, x_n)$, for every n. The argument of the previous paragraph provides the induction step. Then, since a direct limit of tractable fields is clearly tractable, every purely transcendental extension of F is also tractable. ■

Example 3.14 If F is a tractable global field, then every purely transcendental extension of F is also tractable. (For, by [FSS, Th. 2.5, Th. 2.6], F and $F(x)$ are Brauer-Hilbertian.)

Note 3.15 Since this paper was written, I. Han has proved that every purely transcendental extension of a tractable field F is tractable (without any Brauer-Hilbertian restrictions on F). His proof will appear in [H].

The next easy lemma gives a way of obtaining further examples of tractable fields. For a field F, let $\mathrm{Quat}(F)$ denote the set of isomorphism classes of quaternion algebras over F, let $\mathrm{Br}_2(F)$ be the 2-torsion in the Brauer group of F, and let WF be the Witt ring of anisotropic quadratic forms over F.

Lemma 3.16 *Let $\{K_i\}_{i\in I}$ be a family of fields each containing a field F. If each K_i is tractable and the canonical map $\alpha\colon \mathrm{Quat}(F) \to \prod_{i\in I} \mathrm{Quat}(K_i)$ is injective, then F is tractable. If the map $\mathrm{Br}_2(F) \to \prod_{i\in I} \mathrm{Br}_2(K_i)$ or the map $WF \to \prod_{i\in I} WK_i$ is injective, then α is injective.*

Proof Suppose there were $a_1, \dots, b_3$ satisfying $(*)$ with $(a_1, b_1/F)$ not split. Then the injectivity of α shows that $(a_1, b_1/K_i)$ is not split for some i, contradicting the tractability of K_i. Hence, F must be tractable. The last sentence of the lemma is clear, since $\mathrm{Quat}(F)$ maps injectively into $\mathrm{Br}_2(F)$ and into WF (by mapping a quaternion algebra to its norm form). ∎

Theorem 3.17 *If R is a real closed field, then any field F of transcendence degree 1 over R is tractable.*

Proof We may assume that F is a finitely generated field extension of R. For, our original F is clearly tractable if every subfield of F finitely generated over R is tractable. Also, we may assume F is formally real (*i.e.*, has at least one ordering). For, since $F(\sqrt{-1})$ is of transcendence degree 1 over the algebraically closed field $R(\sqrt{-1})$, Tsen's theorem (see [G, Th. 3.6, p. 22]) says that every quadratic form over $F(\sqrt{-1})$ of dimension 3 is isotropic. That is, the u-invariant $u\big(F(\sqrt{-1})\big) \le 2$. Hence, by [$\mathrm{EL}_1$, Th. 4.11(2)], $u(F) \le 2$ for the Elman-Lam generalized u-invariant of F. When F is not formally real, this means that every three-dimensional quadratic form over F is isotropic. Hence, every quaternion algebra over F is split, and tractability of F follows vacuously.

Now, for F formally real and finitely generated of transcendence degree 1 over R, let X_F denote the set of orderings of F. For $P \in X_F$ (with the corresponding order relation on F denoted $<_P$), let F_P denote the real closure of F with respect to P. So (*cf.* [J, Section 5.1] or [P, Section 3]), F_P has a unique ordering, in which every positive element is a square; also, the ordering on F_P extends P on F; also, $F_P(\sqrt{-1})$ is algebraically closed. Consequently, $\big(-1,-1/F_P\big)$ is the unique quaternion division algebra over F_P. (In fact, $\mathrm{Br}(F_P) = \{[F_P], \big[\big(-1,-1/F_P\big)\big]\}$.) So, for a quaternion algebra $Q = (c, d/F)$, we have $Q \otimes_F F_P$ is nonsplit iff $c <_P 0$ and $d <_P 0$. Moreover, there is a local-global principle for F (see [Wi_2, Satz 22] when $R = \mathbb{R}$ or [ELP, Th. I] for any real closed R), which says that Q is nonsplit iff $Q \otimes_F F_P$ is nonsplit for some $P \in X_F$. Hence, by Lemma 3.16 F is tractable, since by Cor. 2.1 each F_P is tractable. ∎

Th. 3.17 has an analogue for nondyadic local fields. We thank B. Fein for pointing this out to us, and for sketching the proof given below.

Theorem 3.18 *Let k be any nondyadic local field, and let F be a field of transcendence degree 1 over k. Then F is tractable.*

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Proof We may assume that F is finitely generated over k, and that k is algebraically closed in F. Let $\{V_P\}_{P\in \mathrm{P}_{F/k}}$ be the set of discrete valuation rings of F over k (*i.e.*, $k \subseteq V_P \subseteq F$ and F is the quotient field of V_P). Let $\widehat{F_P}$ denote the quotient field of the completion $\widehat{V_P}$ of V_P, and let $\overline{V}_P$ be the residue field of V_P and of $\widehat{V_P}$. Since $\overline{V_P}$ is a finite degree extension of k, it is tractable by Cor. 2.3. Then, Prop. 2.5 shows that $\widehat{F_P}$ is tractable. But further, since k is a local field, it is known that the map

$$\beta\colon \operatorname{Br}'(F) \to \prod_{P\in \mathrm{P}_{F/k}} \operatorname{Br}'(\widehat{F_P}) \tag{3.1}$$

is injective (see below). Hence, Lemma 3.16 shows that F is tractable.

When the local field k has characteristic 0, the injectivity of the map β of (3.1) is a special case of results of Pop, [Po, Th. 4.1, Th. 3.7, Th. E.1]; but since k is local with $\operatorname{char}(k) = 0$, the argument appears already implicitly in Lichtenbaum's paper [Li]. For, let X be the unique regular integral curve projective over k with function field F, and let $\operatorname{Br}(X)$ denote its cohomological Brauer group, $\operatorname{Br}(X) = H^2_{\text{ét}}(X, \mathrm{G}_m)$ (*cf.* [M, Ch. IV] or [Gr]). Then, the local rings $\mathcal{O}_{X,x}$ of the closed points x of X are the same as the V_P for $P \in \mathrm{P}_{F/k}$, and it is known that $\operatorname{Br}(X)$ is the subgroup of $\operatorname{Br}(F)$ consisting of elements unramified with respect to each of the V_P, *i.e.*, $\operatorname{Br}(X) = \bigcap_{P\in\mathrm{P}_{F/k}} \operatorname{Br}(V_P) \subseteq \operatorname{Br}(F)$ (see [Li, Section 1]). Now, the divisor group $\operatorname{Div}(X)$ is the free abelian group on $\mathrm{P}_{F/k}$, and $\operatorname{Pic}(X) = \operatorname{Div}(X)/\operatorname{Prin}(X)$, where $\operatorname{Prin}(X)$ is the group of principal divisors. For each $P \in \mathrm{P}_{F/k}$ there is a homomorphism $\operatorname{Br}(X) \to \operatorname{Br}(k)$ given by composing the canonical map $\gamma_P\colon \operatorname{Br}(X) \to \operatorname{Br}(V_P) \to \operatorname{Br}(\overline{V_P})$ with the corestriction $\operatorname{cor}\colon \operatorname{Br}(\overline{V_P}) \to \operatorname{Br}(k)$. (Here, $\operatorname{Br}(\overline{V_P}) \cong \mathbb{Q}/\mathbb{Z} \cong \operatorname{Br}(k)$, and the corestriction map is an isomorphism [Se, Prop. 6, p. 193; Th. 1, p. 195; Prop. 1(ii), p. 167].) These maps combine to yield a pairing $\operatorname{Br}(X) \times \operatorname{Div}(X) \to \operatorname{Br}(k) \cong \mathbb{Q}/\mathbb{Z}$ which factors through $\operatorname{Pic}(X)$, hence inducing a map $\Phi_F\colon \operatorname{Br}(X) \to \operatorname{Hom}\big(\operatorname{Pic}(X), \mathbb{Q}/\mathbb{Z}\big)$. Lichtenbaum proves in [Li, Th. 4] that Φ_F is an isomorphism. (His description of Φ_F is somewhat different from the one given here (we have used the one in [CTS]), but one can show that this Φ_F is the same as his.) Note that for any element of $\operatorname{Br}(F)$ which ramifies at some V_P, its image in $\operatorname{Br}(\widehat{F_P})$ has the same ramification with respect to $\widehat{V_P}$, so is nonzero. Hence, for the β of (3.1), any element of $\ker(\beta)$ must be unramified with respect to each V_P; so, $\ker(\beta) \subseteq \bigcap_{P\in\mathrm{P}_{F/k}} \operatorname{Br}(V_P) = \operatorname{Br}(X)$. The maps $\gamma_P\colon \operatorname{Br}(X) \to \operatorname{Br}(\overline{V_P})$ defined above combine to provide a homomorphism $\lambda\colon \operatorname{Br}(X) \to \prod_{P\in\mathrm{P}_{F/k}} \operatorname{Br}(\overline{V_P})$, and $\ker(\beta) \subseteq \ker(\lambda)$ since the map $\operatorname{Br}(\overline{V_P}) \to \operatorname{Br}(\widehat{F_P})$ is injective for each P. However, clearly $\ker(\lambda) \subseteq \ker(\Phi_F) = (0)$, so β is injective. (Lichtenbaum has to argue a bit in [Li, proof of Th. 5] to see that λ is injective, because of his different definition of Φ_F, and Pop gives the argument in more detail in [Po, proof of Th. 4.1].)

The injectivity of β in (3.1) when $\operatorname{char}(k) = p \neq 0$ is deducible from [CTS], as follows: There is still a unique regular integral curve X projective over k with function field F. Moreover, there is a regular integral scheme $\mathcal{X}$ proper and flat over the valuation ring V of k, such that $\mathcal{X} \times_V k \cong X$. This can be seen by first lifting X to a scheme $\mathcal{X}'$ projective and flat over V with $\mathcal{X}' \times_V k \cong X$, and then resolving the singularities of $\mathcal{X}'$, obtaining the desired smooth variety $\mathcal{X}$ over V with a proper birational morphism $f\ \mathcal{X} \to \mathcal{X}'$. This is possible by Lipman's argument for resolution of singularities of 2-dimensional excellent schemes, see [Ar]. Let $\mathcal{X}_s = \mathcal{X} \times_V \overline{V}$, the special fibre of $\mathcal{X}$. As V is Henselian and

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Japanese and $\mathcal{X}$ has dimension 2, $\mathrm{Br}(\mathcal{X}) \cong \mathrm{Br}(\mathcal{X}_s)$ by [Gr, Th. 3.1, p. 98]. Furthermore, since $\mathcal{X}$ is projective over the finite field $\overline{V}$, we have $\mathrm{Br}(\mathcal{X}_s) = 0$ by [Gr, Rem. 2.5(b), p. 96]. (This holds even if $\mathcal{X}_s$ is not regular, as pointed out in [Gr, comments preceding 2.5(c), p. 97]. For an alternate approach of reducing to the nonsingular case, see [S_2, Lemma 3.2].) Hence, $\mathrm{Br}(\mathcal{X}) = 0$. It is shown in [CTS, Cor. 2.4; comments on p. 153] that the map $\big(\mathrm{Br}'(X)/\mathrm{Br}'(\mathcal{X})\big) \times \mathrm{Pic}(X) \to \mathrm{Br}(k) = \mathbb{Q}/\mathbb{Z}$ is nondegenerate in the left factor. Because $\mathrm{Br}(\mathcal{X}) = 0$, this shows that we again have an injective map $\mathrm{Br}'(X) \to \mathrm{Hom}\big(\mathrm{Pic}(X), \mathbb{Q}/\mathbb{Z}\big)$. (This is also part of [Sa, Th. 9.2], at least when $\mathcal{X}$ is smooth.) So, the argument of the preceding paragraph applies here, showing that the map β of (3.1) is injective. ■

Remark 3.19 In the terminology of [Po], a field k of characteristic 0 is *p-adically closed* if k is the quotient field of a Henselian valuation ring V with finite residue field of cardinality a power of p and with value group Γ containing a least positive element γ such that $\Gamma/n\Gamma$ is cyclic and generated by the image of γ for each natural number n. The proof of Th. 3.18 adapts to show that each field F of transcendence degree 1 over a p-adically closed field k ($p \neq 2$) is tractable. For, any finite degree extension of k is tractable by Prop. 2.5, and the map β of (3.1) is injective, by [Po, Th. 4.1, Th. 3.7].

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