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HAUSDORFF DIMENSION OF RANDOM FRACTALS WITH OVERLAPS

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This paper considers random fractals generated by contractive transformations which satisfy a separation condition weaker than the open set condition. This condition allows overlaps in the iterations. Estimates of the Hausdorff dimension of this kind of random fractals are obtained.

1. INTRODUCTION

The term *fractal* was first introduced by Mandelbrot to denote sets with highly irregular structures. Mandelbrot and others have then used fractals to model various natural phenomena (for example, [**19, 2**]). A mathematical development of fractals via the theory of (non-random) self-similar sets and measures was given by Hutchinson [**12**]. A family of contractive maps $\{\omega_j\}_{j=1}$ on $\mathbb{R}^m$ is called an *iterated function system* ([**2**]). An iterated function system generates an invariant compact subset $K = \bigcup_{j=1}^{N} S_j(K)$, which is usually referred to as its *fractal set.* For the given probability weights $\{w_j\}_{j=1}^{N}$, it generates an invariant measure

$$\mu = \sum_{j=1}^{N} w_j \mu \circ S_j^{-1}.$$

The invariant sets and measures play a central role in the theory of fractals. In practice it is often assumed that the maps are similitudes, and in the iteration, they satisfy a nonoverlapping condition called the *open set condition* ([**12**]). One of the advantages of the open set condition is that the points in K can be represented in a symbolic space (except for μ-zero sets) and the dynamics of the iterated function system can be identified with the shift operator in this space. Without the open set condition, the iteration has overlaps and such representation will break down, and it is more difficult to handle this

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situation ([**15**]). The simplest example of an iterated function system with overlaps is given by the maps

$$S_1(x) = \rho x, \quad S_2(x) = \rho x + (1-\rho), \quad 1/2 < \rho < 1, x \in \mathbb{R}.$$

The invariant measure μ_ρ associated with the weights $\omega_1 = \omega_2 = 1/2$ is called the *Bernoulli convolution* ([**4, 8**]). Recently there has been a great deal of interest in the problem whether μ_ρ is absolutely continuous. Erdös [**4**] first proved that if ρ^{-1} is a Pisot–Vijayaraghavan number, then μ_ρ is singular. Solomyak ([**27, 22**]) proved that for almost all ρ, μ_ρ is absolutely continuous. The density argument used in [**22**] has also been used to consider a variety of iterated function systems with overlaps ([**23, 24**]). There are other examples of the overlapping case, such as those given in [**14**] and [**26**].

In another direction Lau and Ngai introduced a *weak separation condition* on iterated function systems of similitudes ([**16**]). This condition is weaker than the open set condition and includes many of the important overlapping cases. Under the weak separation condition, several significant results have been obtained (see the review paper [**15**] and the references therein).

But natural phenomena are more complex and often are better modelled by random sets. In the early 1980's, there was a large amount of works on self-similar random sets in the study of self-similar processes such as stable processes and Brownian motion. Many results have been obtained for fractal properties such as singularity spectrum, generalised dimensions for self-similar random sets (see, for example, [**1, 5, 10, 20, 21, 13**]). In practical applications, it is necessary to consider sets which are only generated by contractive maps, for example in the study of flux in condensed-matter physics ([**28**]) and fractional diffusion equations for transport phenomena in random media ([**9**]). In this direction Liang and Ren [**18**] investigated a class of general random fractals, namely, *random net fractals*.

But all these works on random fractals require the nonoverlapping condition. Apart from Pesin and Weiss [**25**] which considered a special Cantor-like case, there has been no known result in the overlapping case. In this paper, we investigate a class of random net fractals generated by contractive maps with overlaps. Estimates of their Hausdorff dimension under a random weak separation condition will be given. This condition is related to the weak separation condition introduced by Lau and Ngai [**16**] in the non-random case.

2. Random net fractals generated by contractive maps

2.1. Sequences and trees. Let $\sigma = (i_1, i_2, \ldots, i_n)$ be a sequence of positive integers and let $|\sigma| = n$ denote the length of the sequence. For $\sigma = (i_1, i_2, \ldots, i_n)$, $\sigma|_k = (i_1, i_2, \ldots, i_k)$, $k \leqslant n$ denotes the sequence obtained by restricting σ to its first k terms. If $\sigma' = (i'_1, i'_2, \ldots, i'_p)$, then σ, σ' is the sequence $(i_1, i_2, \ldots, i_n, i'_1, i'_2, \ldots, i'_p)$ obtained by

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juxtaposition of the two sequences σ and σ'. We write $\sigma \prec \tau$ to mean that the sequence τ is an extension of σ, that is $\tau = \sigma, \sigma'$ for some sequence σ'. We adopt a similar convention if τ is an infinite sequence of positive integers and assume that the null sequence $\emptyset \prec \sigma$ for any sequence σ.

A *tree* is a collection $\mathcal{F}$ of finite sequences of positive integers such that $\sigma \in \mathcal{F}$ implies $\sigma' \in \mathcal{F}$ for any $\sigma' \prec \sigma$. The sequences of $\mathcal{F}$ may be identified with the vertices of a directed graph with σ being joined to σ, i in the obvious way. We write

$$\mathcal{F}_k = \{\sigma \in \mathcal{F} : \ |\sigma| = k\}$$

for the set of sequences of length k and

$$N(\sigma) = \#\{i : \ \sigma, i \in \mathcal{F}\}$$

for the number of outgoing edges from the vertex σ in a graph of $\mathcal{F}$. Let $\widetilde{\mathcal{F}}$ be the set of infinite sequences τ such that $\sigma \in \mathcal{F}$ for every finite curtailment $\sigma \prec \tau$. We allow $N(\sigma) = 0$, but always assume that $N(\sigma) < \infty$.

REMARK 1. We always assume that the trees used in this paper are random trees representing a branching process, that is, a typical individual $\sigma \in \mathcal{F}$ has $N(\sigma)$ "offspring", where $N(\sigma), \sigma \in \mathcal{F}$ are independent and identically distributed. Moreover, the offspring of σ may be labelled as $(\sigma, 1), \ldots, (\sigma, N(\sigma))$.

2.2. RANDOM NET FRACTAL GENERATED BY CONTRACTIVE MAPS. Fix a Euclidean space $\mathbb{R}^m$ and a compact subset I of $\mathbb{R}^m$ such that $I = \mathrm{cl}(\mathrm{int}\, I)$. Write $|\cdot|$ for the diameter of a subset of $\mathbb{R}^m$. Let $(\Omega, \mathcal{G}, P)$ be a probability space. Let Con denote the set of all contractive transformations in $\mathbb{R}^m$. For each $\omega \in \Omega$, let

$$\Phi(\omega) = \Big\{\phi_\sigma(\omega) : \ \sigma \in \mathcal{F}, \phi_\sigma(\omega) \in \mathrm{Con} \ \text{ and } \\ d_\sigma(\omega)|x-y| \leqslant \big|\phi_\sigma(\omega, x) - \phi_\sigma(\omega, y)\big| \leqslant c_\sigma(\omega)|x-y|\Big\}$$

be a family of random contractive maps satisfying the following properties:

(1) $I_\emptyset(\omega) = I$ for almost all $\omega \in \Omega$;

(2) for every $\sigma \in \mathcal{F}$, $N(\sigma)$, $d_\sigma(\omega)$ and $c_\sigma(\omega)$ are random variables. Moreover the random vectors

$$\big(d_{\sigma,1}, \ldots, d_{\sigma,N(\sigma)}\big) \text{ and } \big(c_{\sigma,1}, \ldots, c_{\sigma,N(\sigma)}\big), \ \sigma \in \mathcal{F}$$

are independent and identically distributed, and

$$0 < \underline{d}(\omega) = \inf\{d_\sigma(\omega) : \sigma \in \mathcal{F}\} \leqslant \sup\{c_\sigma(\omega) : \sigma \in \mathcal{F}\} = \overline{c}(\omega) < 1.$$

We denote $S_\sigma(\omega) = \phi_{\sigma|_1}(\omega) \circ \cdots \circ \phi_{\sigma|_{|\sigma|}}(\omega)$ for any $\sigma \in \mathcal{F}$. Assume that

$$I_\sigma(\omega) = \phi_{\sigma|_1}(\omega) \circ \cdots \circ \phi_{\sigma|_{|\sigma|}}(\omega)(I) \subset I.$$

We denote $I_k(\omega) = \bigcup_{\sigma \in \mathcal{F}_k} I_\sigma(\omega)$. It is easy to see that $I_k(\omega) \subset I_{k-1}(\omega)$. Define the random set $K(\omega)$ by

$$K(\omega) = \bigcap_{k=0}^{\infty} I_k(\omega).$$

The set $K(\omega)$ is called a *random net fractal* generated by a family $\Phi(\omega)$ of random contractive transformations.

Liang and Ren [**18**] assumed that

(*) For almost all $\omega \in \Omega$ and for every $\sigma \in \mathcal{F}$, $\phi_{\sigma,i}(\omega)(I)$ and $\phi_{\sigma,j}(\omega)(I)$, $i \neq j$, are non-overlapping compact subsets.

They then obtained an estimate of the Hausdorff dimension of the random net fractal $K(\omega)$.

For $k \in \mathbb{N}, \omega \in \Omega, \sigma \in \mathcal{F}$, we define a stopping time $t_k(\sigma, \omega)$ by

$$t_k(\sigma,\omega) = \min\Big\{i : \ \big|S_\sigma(\omega)(I)\big| \leqslant \underline{d}^k(\omega)\Big\}$$

and let

$$\Lambda_k(\omega) = \big\{\sigma|_{t_k(\sigma,\omega)} : \ \sigma \in \mathcal{F}\big\}.$$

We now replace condition (*) by the following weaker condition which allows overlaps.

Definition 2.1: For almost all $\omega \in \Omega$, a family $\Phi(\omega)$ of random contractive transformations is said to have the random weak separation property if every closed $\underline{d}^k(\omega)$-ball intersects with at most l distinct $S_\sigma(\omega)(I)$, $\sigma \in \Lambda_k(\omega)$, where $S_\sigma(\omega)(I)$ can be repeated (that is, it is allowable that $S_\sigma(\omega)(I) = S_{\sigma'}(\omega)(I)$ for $\sigma, \sigma' \in \Lambda_k(\omega), \sigma \neq \sigma'$).

Remark 2. In the case of non-random fractal, the condition (*) corresponds to that for net fractals without overlaps (it obviously satisfies the open set condition). For net fractals with overlaps, iterated function systems can still satisfy the open set condition (see Rao and Wen [**26**]). There are also many net fractals which do not satisfy the open set condition but have the weak separation property (see [**16**] for several examples).

Remark 3. $\Phi(\omega)$ without overlaps obviously has the random weak separation property. In this case, we can take $l = 1$. If all $\phi_\sigma(\omega)$ are similitudes with the same contraction ratio and $\Phi(\omega)$ is an iterated function system, from Corollary 2.6 of [**17**], our definition of the random weak separation property becomes the random version of a condition equivalent to the weak separation property introduced by Lau and Ngai [**16**].

For $\sigma \in \Lambda_k(\omega)$, we denote

$$[\sigma] = \big\{\sigma' \in \Lambda_k(\omega) : \ S_\sigma(I) = S_{\sigma'}(I)\big\},$$

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and

$$M_k(\omega) = \max\{\#[\sigma] : \sigma \in \Lambda_k(\omega)\}.$$

For any set $A \subset \mathbb{R}^m$, if $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover A, we say that $\{U_i\}$ is a δ-*cover* of A. For any $\delta > 0, s \geqslant 0$, we define

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \ \{U_i\} \ \text{ is a } \delta\text{-cover of } A\right\}.$$

Then

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

is called the s-*dimensional Hausdorff measure* of A. Moreover

$$\dim_H(A) = \inf\{s : \ \mathcal{H}^s(A) = 0\} = \sup\{s : \ \mathcal{H}^s(A) = \infty\}$$

is called the *Hausdorff dimension* of A (see [**6**]).

We now state the main result of this paper.

THEOREM 2.2. *Consider a family $\Phi(\omega)$ of random contractive transformations which have the random weak separation property and satisfy (1) and (2). Let $K(\omega)$ be the random net fractal generated by $\Phi(\omega)$. Then $K(\omega) = \emptyset$ with probability q and*

$$t + \liminf_{k \longrightarrow \infty} \frac{\log M_k(\omega)}{k \log \underline{d}(\omega)} \leqslant \dim_H K(\omega) \leqslant \min\{s, m\}$$

with probability $1 - q$, where q is the unique non-negative solution less than 1 of the equation $x = \sum_{k=1}^{\infty} P(N(\emptyset) = k)x^k$ and m is the dimension of the Euclidean space, t and s are the solutions of the expectation equations $E\sum_{i=1}^{N(\emptyset)} d_i^t(\omega) = 1$ and $E\sum_{i=1}^{N(\emptyset)} c_i^s(\omega) \doteq 1$ respectively.

The proof of Theorem 2.2 will be given at the end of this paper.

REMARK 4. If we have the nonoverlapping condition, then

$$\liminf_{k \longrightarrow \infty} \frac{\log M_k(\omega)}{k \log \underline{d}(\omega)} = 0,$$

which is the result obtained by Liang and Ren [**18**].

We next recall [**6**, p. 113] that the δ-*parallel body* A_δ of a non-empty compact set $A \subset I$ is defined by

$$A_\delta = \{x \in I : \ |x - a| \leqslant \delta \text{ for some point } a \in A\}$$

and the *Hausdorff metric* is defined by

$$d_H(A, B) = \inf\{\delta : \ A \subset B_\delta \text{ and } B \subset A_\delta\}.$$

We now have

Proposition 2.3. *For each $\omega \in \Omega$ and any non-empty compact subset A of I, if we define*

$$A_\sigma(\omega) = \phi_{\sigma|_1}(\omega) \circ \cdots \circ \phi_{\sigma|_{|\sigma|}}(\omega)(A) \text{ and } A_k(\omega) = \bigcup_{\sigma \in \mathcal{F}_k} A_\sigma(\omega),$$

then $A_k(\omega) \xrightarrow{d_H} K(\omega)$.

Proof: If δ is such that the δ-parallel body $\big(S_\sigma(\omega)(A)\big)_\delta$ contains $S_\sigma(\omega)(I)$ for each σ, then $\Big(\bigcup_{\sigma \in \mathcal{F}_k} S_\sigma(\omega)(A)\Big)_\delta$ contains $\bigcup_{\sigma \in \mathcal{F}_k} S_\sigma(\omega)(I)$. Hence

$$\begin{aligned} d_H\big(A_k(\omega), I_k(\omega)\big) &= d_H\Big(\bigcup_{\sigma \in \mathcal{F}_k} S_\sigma(\omega)(A), \bigcup_{\sigma \in \mathcal{F}_k} S_\sigma(\omega)(I)\Big) \\ &\leqslant \sup_{\sigma \in \mathcal{F}_k} d_H\big(S_\sigma(\omega)(A), S_\sigma(\omega)(I)\big) \\ &\leqslant \sup_{\sigma \in \mathcal{F}_k} c_{\sigma|_1} \ldots c_{\sigma|_{|\sigma|}} d_H(A, I) \\ &\leqslant \bar{c}^k d_H(A, I) \longrightarrow 0 \quad (\text{as } k \longrightarrow \infty). \end{aligned}$$

From the definition of $K(\omega)$, we then have $A_k(\omega) \xrightarrow{d_H} K(\omega)$. ∎

The above proposition allows us to construct $K(\omega)$ starting from any point $x_0 \in I$. If we take $A_0(\omega) = \{x_0\}$ and

$$A_k(\omega) = \big\{S_\sigma(\omega)(x_0) : \sigma \in \mathcal{F}_k\big\},$$

then $A_k(\omega) \xrightarrow{d_H} K(\omega)$.

Regarding a random net fractal $K(\omega)$ as a branching process, Liang and Ren [**18**] proved the following result:

Lemma 2.4. ([**18**]) *For each $\omega \in \Omega$, let $Z_n(\omega) = \sum_{\sigma \in \mathcal{F}_n} N_\omega(\sigma)$.*

(1) *If $E\big(N_\omega(\emptyset)\big) \leqslant 1$, then either $K(\omega) = \emptyset$ almost surely or $K(\omega)$ is a point almost surely*

(2) *If $E\big(N_\omega(\emptyset)\big) > 1$, then $Z_n(\omega) \to \infty$ almost surely and $K(\omega) \neq \emptyset$ almost surely.*

3. Random self-similar fractals and auxiliary random variables

In the definition of $K(\omega)$, if for all $\omega \in \Omega$, $\sigma \in \mathcal{F}_k$ and $d_\sigma(\omega) = c_\sigma(\omega)$ (the common value being denoted as $\rho_\sigma(\omega)$), then $\phi_\sigma(\omega)$ is a similitude, that is, $\big|\phi_\sigma(\omega, x) - \phi_\sigma(\omega, y)\big| = \rho_\sigma(\omega)|x - y|$. In this case, $\phi_\sigma(\omega, x) = \rho_\sigma(\omega) R_\sigma(\omega) x + b_\sigma(\omega)$, where $R_\sigma(\omega)$ is an orthogonal matrix and $b_\sigma(\omega) \in \mathbb{R}^m$. We also assume

$$0 < \underline{\rho}(\omega) = \inf\big\{\rho_\sigma(\omega) : \sigma \in \mathcal{F}\big\} \leqslant \sup\big\{\rho_\sigma(\omega) : \sigma \in \mathcal{F}\big\} = \overline{\rho}(\omega) < 1$$

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and

$$\bar{b} = \sup\{b_\sigma : \sigma \in \mathcal{F}\} < \infty.$$

Then $K(\omega)$ is called a *random self-similar fractal.*

Let $K(\omega)$ be a random self-similar fractal. Without loss of generality, we can assume $|I| = 1$ and define the random variable

$$B_\sigma(\omega) = |I_\sigma(\omega)| = \prod_{n=1}^{|\sigma|} \rho_{\sigma|_n}(\omega).$$

In this section, we suppose that for almost all $\omega \in \Omega$,

$$B_{\sigma|_n}(\omega) \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ if } \sigma = (i_1, i_2, \dots) \in \widetilde{\mathcal{F}}.$$

Since $\overline{\rho}(\omega) < 1$, from [**5**, Lemma 8.2], $g(s) = E\left(\sum_{i=1}^{N_\sigma(\emptyset)} \rho_i^s(\omega)\right)$ is strictly decreasing in s. Hence there exists $s > 0$ such that

$$E\left(\sum_{i=1}^{N_\sigma(\emptyset)} \rho_i^s\right) = 1.$$

For any $n \in \mathbb{N}$, $\omega \in \Omega$, let $\mathcal{L}_n$ denote the σ-algebra of subsets of Ω:

$$\mathcal{L}_n = \mathcal{B}\big(\mathcal{L}_{n-1}; N_\omega(\sigma);\ |I_{\sigma,i}(\omega)| :\ \sigma \in \mathcal{F}_{n-1},\ 1 \leqslant i \leqslant N_\omega(\sigma)\big),$$

where

$$\mathcal{L}_1 = \mathcal{B}\big(N_\omega(\emptyset);\ |I_i(\omega)| :\ 1 \leqslant i \leqslant N_\omega(\emptyset)\big).$$

Then Liang and Ren [**18**] proved

Lemma 3.1. ([**18**]) *For any $\omega \in \Omega$ and for each $k \in \mathbb{N}$,*

(1) $E\left(\sum_{\sigma \in \mathcal{F}_k} B_\sigma^s(\omega) \mid \mathcal{L}_{k-1}\right) = \left(\sum_{\sigma \in \mathcal{F}_{k-1}} B_\sigma^s(\omega)\right)\left(E\sum_{i=1}^{N(\emptyset)}\right)\rho_i^s(\omega),$

(2) $E\sum_{\sigma \in \mathcal{F}_k} B_\sigma^s(\omega) = \left(E\sum_{i=1}^{N(\emptyset)} \rho_i^s(\omega)\right)^k,$

(3) $\lim_{k\to\infty} \sum_{\sigma \in \mathcal{F}_k} B_\sigma^s(\omega) = X(\omega)$ *almost surely, where $X(\omega)$ is a bounded random variable,*

(4) *for any $\sigma \in \mathcal{F}$, define a random variable $X_\sigma(\omega)$ by*

$$X_\sigma(\omega) = \lim_{n\to\infty} T_{\sigma;n}(\omega), \quad \text{where } T_{\sigma;n}(\omega) = \sum_{\sigma,\tau \in \mathcal{F}_{|\sigma|+n}} \prod_{k=1}^{n} \rho^s_{\sigma,(\tau|_k)}(\omega).$$

Then $X_\sigma(\omega)$ exists almost surely and $EX_\sigma(\omega) \leqslant 1$.

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For $k \in \mathbb{N}, \omega \in \Omega, \sigma \in \mathcal{F}$, the stopping time is then

$$t_k(\sigma,\omega) = \min\{i : \rho_{\sigma|_1}(\omega)\cdots\rho_{\sigma|_i}(\omega) \leqslant \underline{\rho}^k(\omega)\}$$

and

$$\Lambda_k(\omega) = \{\sigma|_{t_k(\sigma,\omega)} : \sigma \in \mathcal{F}\}.$$

REMARK 5. As in the proof of (3) in Lemma 3.1, we can have

$$\lim_{k\to\infty} \sum_{\sigma\in\Lambda_k(\omega)} B_\sigma^s(\omega) = X(\omega) \text{ almost surely.}$$

We define

$$\underline{t}_k(\omega) = \min\{|\sigma| : \sigma \in \Lambda_k(\omega)\}.$$

If we take

$$A_k(\omega) = \{S_\sigma(\omega)(I) : \sigma \in \Lambda_k(\omega)\}$$

then as in the proof of Proposition 2.3, it can be proved that $A_k(\omega) \xrightarrow{d_H} K(\omega)$.

We now follow the method used in [**18**] and [**20**] by defining for almost all ω a bounded Borel measure μ_ω on $\mathbb{R}^m$ such that

(a) μ_ω has total mass $X(\omega)$,

(b) $\mu_\omega(K(\omega)) = X(\omega)$.

Let $C_c(\mathbb{R}^m) = \{f \in C_c(\mathbb{R}^m) : f \text{ has compact support}\}$. For $f \in C_c(\mathbb{R}^m)$, any $x_0 \in I$, consider the limit

$$\lim_{k\to\infty} \sum_{\sigma\in\Lambda_k(\omega)} f(x_\sigma(\omega))B_\sigma^s(\omega), \quad \text{where } x_\sigma(\omega) = S_\sigma(\omega)(x_0).$$

Let

$$\Omega' = \{\omega \in \Omega : \forall\, \sigma \in \mathcal{F},\ X_\sigma(\omega) \text{ exists and } \lim_{n\to\infty}|I_{\sigma|_n}(\omega)| = 0\}.$$

From Lemma 3.1, $X_\sigma(\omega)$ exists almost surely. Hence $P(\Omega') = 1$. For any $f \in C_c(\mathbb{R}^m)$ and $p, q \in \mathbb{N}$, we write

$$\varepsilon_{p,q}(\omega) = \Bigl|\sum_{\sigma\in\Lambda_p(\omega)} f(x_\sigma(\omega))B_\sigma^s(\omega) - \sum_{\sigma\in\Lambda_q(\omega)} f(x_\sigma(\omega))B_\sigma^s(\omega)\Bigr|,\ \omega \in \Omega.$$

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Fix $k \in \mathbb{N}$ and assume $p, q \geqslant k$, then for any $\omega \in \Omega$

$$\begin{aligned}
\varepsilon_{p,q}(\omega) = \Bigg| \sum_{\sigma \in \Lambda_k(\omega)} B_\sigma^s(\omega) \Bigg(\sum_{\sigma,\sigma' \in \Lambda_p(\omega)} f\left(x_{\sigma,\sigma'}(\omega)\right) \prod_{j=1}^{|\sigma'|} \rho^s_{\sigma,(\sigma'|_j)}(\omega) \\
- \sum_{\sigma,\sigma' \in \Lambda_q(\omega)} f\left(x_{\sigma,\sigma'}(\omega)\right) \prod_{j=1}^{|\sigma'|} \rho^s_{\sigma,(\sigma'|_j)}(\omega) \Bigg) \Bigg| \\
\leqslant \sum_{\sigma \in \Lambda_k(\omega)} B_\sigma^s(\omega) \Big[\sup_{\sigma,\sigma' \in \Lambda_p(\omega)} \left| f\left(x_{\sigma,\sigma'}(\omega)\right) - f\left(x_\sigma(\omega)\right) \right| T_{\sigma;\underline{t}_p - k}(\omega) \\
+ \left| f(x_\sigma(\omega)) \right| \cdot \left| T_{\sigma;t_p-k}(\omega) - T_{\sigma;t_q-k}(\omega) \right| \\
+ \sup_{\sigma,\sigma' \in \Lambda_q(\omega)} \left| f\left(x_{\sigma,\sigma'}(\omega)\right) - f\left(x_\sigma(\omega)\right) \right| T_{\sigma;\underline{t}_q - k}(\omega) \Big] \\
\leqslant \sum_{\sigma \in \Lambda_k(\omega)} B_\sigma^s(\omega) \operatorname{diam}\Big(f\left(I_\sigma(\omega)\right)\Big) \left(T_{\sigma;t_p-k}(\omega) + T_{\sigma;t_q-k}(\omega)\right) \\
+ \|f\|_\infty \left| T_{\sigma;t_p-k}(\omega) - T_{\sigma;t_q-k}(\omega) \right|.
\end{aligned}$$

If $\omega \in \Omega'$, then

$$\begin{aligned}
\limsup_{p,q \to \infty} \varepsilon_{p,q}(\omega) &\leqslant 2 \sum_{\sigma \in \Lambda_k(\omega)} B_\sigma^s(\omega) \operatorname{diam}\Big(f\left(I_\sigma(\omega)\right)\Big) X_\sigma(\omega) \\
&= 2 \sup_{\sigma \in \Lambda_k(\omega)} \operatorname{diam}\Big(f\left(I_\sigma(\omega)\right)\Big) X(\omega).
\end{aligned}$$

Since f is continuous, $\lim_{k \to \infty} \operatorname{diam}\Big(f\left(I_\sigma(\omega)\right)\Big) = 0$. Thus $\lim_{p,q \to \infty} \varepsilon_{p,q}(\omega) = 0$ if $\omega \in \Omega'$. That is, for almost all ω, the limit $\lim_{k \to \infty} \sum_{\sigma \in \Lambda_k(\omega)} f\left(x_\sigma(\omega)\right) B_\sigma^s(\omega)$ exists. Write

$$F_\omega(f) = \lim_{k \to \infty} \sum_{\sigma \in \Lambda_k(\omega)} f\left(x_\sigma(\omega)\right) B_\sigma^s(\omega).$$

Clearly, $F_\omega(f)$ is a positive linear functional and for any $f \in C_c(\mathbb{R}^m)$ such that $I \subset f^{-1}(1)$, we have $F_\omega(f) = \lim_{k \to \infty} \sum_{\sigma \in \Lambda_k(\omega)} B_\sigma^s(\omega) = X(\omega)$. By the Riesz representation theorem, for almost all $\omega \in \Omega$, there is the Borel measure μ_ω on $\mathbb{R}^m$ such that

$$F_\omega(f) = \int_{\mathbb{R}^m} f(x) d\mu_\omega(x),$$

and it is obvious that $supp(\mu_\omega) = K(\omega)$. If $E(N_\omega(\emptyset) > 1$ for any compact subset A of $\mathbb{R}^m$, we then have

$$\mu_\omega(A) = \lim_{k \to \infty} \sum_{\substack{\sigma \in \Lambda_k(\omega) \\ x_\sigma(\omega) \in A}} B_\sigma^s(\omega) X_\sigma(\omega) \quad \text{almost surely.} \tag{3.1}$$

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Recalling that we have defined $\widetilde{\mathcal{F}}$ in Subsection 2.1 to be the set of infinite sequences τ such that $\sigma \in \mathcal{F}$ for every finite curtailment $\sigma \prec \tau$. Let π be the projection of $\widetilde{\mathcal{F}}$ to $\mathbb{R}^m$ defined by

$$\pi(\tau) = \bigcap_{k=1}^{\infty} \phi_{\tau|_1}(\omega) \circ \cdots \circ \phi_{\tau|_k}(\omega)(I).$$

For a cylinder set $C_\sigma(\omega)$ with base $\sigma \in \mathcal{F}$, we denote by P_ω the measure on $\widetilde{\mathcal{F}}$ such that $P_\omega\big(C_\sigma(\omega)\big) = B_\sigma^s(\omega) X_\sigma(\omega)$. Then $\mu_\omega = P_\omega \circ \pi^{-1}$. Let $C_{[\sigma]}(\omega) = \bigcup_{\sigma' \in [\sigma]} C_{\sigma'}(\omega)$; then for $\sigma \in \Lambda_k(\omega)$ we have

$$\mu_\omega\Big(\pi\big(C_\sigma(\omega)\big)\Big) = P_\omega\big(C_{[\sigma]}(\omega)\big) = \sum_{\sigma' \in [\sigma]} B_{\sigma'}^s X_{\sigma'}(\omega). \tag{3.2}$$

Moreover, if the random weak separation property holds, it intersects with at most l distinct $S_\sigma(\omega)(I)$, $\sigma \in \Lambda_k(\omega)$ for any $\underline{\rho}^k$-ball $B(x, \underline{\rho}^k)$. This means that there are at most l different $[\sigma]$'s such that $\pi\big(C_\sigma(\omega)\big)$ intersects with $B(x, \underline{\rho}^k)$. If we denote these different $[\sigma]$'s as $[\sigma_1], [\sigma_2], \dots, [\sigma_i]$, where $i \leqslant l$, then

$$\mu_\omega\big(B(x, \underline{\rho}^k)\big) \leqslant \sum_{j=1}^{i} P_\omega\big(C_{[\sigma_j]}(\omega)\big) \leqslant \sum_{j=1}^{i} \sum_{\sigma' \in [\sigma_j]} B_{\sigma'}^s X_{\sigma'}(\omega). \tag{3.3}$$

4. Hausdorff dimension of self-similar random fractals

Consider a branching process with probability distribution $N_\omega(\emptyset)$ and generating function

$$f(x) = \sum_{k=1}^{\infty} P(N(\emptyset) = k) x^k. \tag{4.1}$$

It follows from [**11**] and [**5**, p. 567] that the extinction probability q of the process equals the smallest non-negative root of the equation $f(x) = x$. If $E\big(N_\omega(\emptyset)\big) \leqslant 1$ then $q = 1$ (except in the trivial case when $N_\omega(\emptyset) = 1$ almost surely), but if $E\big(N_\omega(\emptyset)\big) > 1$ then $0 \leqslant q < 1$.

Theorem 4.1. *Suppose a family $\Phi(\omega)$ of random similitudes has the random weak separation property and satisfies (1) and (2) in Subsection 2.2, and $K(\omega)$ is the self-similar random fractal generated by $\Phi(\omega)$. Then $K(\omega) = \emptyset$ with probability q and*

$$s + \liminf_{k \to \infty} \frac{\log M_k(\omega)}{k \log \underline{\rho}(\omega)} \leqslant \dim_H K(\omega) \leqslant \min\{s, m\}$$

with probability $1 - q$, where m is the dimension of the Euclidean space, $\dim_H K(\omega)$ is the Hausdorff dimension of $K(\omega)$, s is the solution of the expectation equation $E \sum_{i=1}^{N(\emptyset)} \rho_i^s(\omega) = 1$.

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PROOF: For any $\omega \in \Omega$, if the branching process becomes extinct, then $K(\omega) = \emptyset$ and the extinction probability is equal to q. Otherwise, with probability $1-q$, either $K(\omega)$ is a point or $K(\omega) \neq \emptyset$ for an infinite number of points by Lemma 2.4. For the first case $s = 0$, hence our conclusion obviously holds. Now we need to consider the second case $E\big(N_\omega(\emptyset)\big) > 1$ in which the process does not become extinct.

To obtain an upper bound for $\dim K(\omega)$, we note that, for any $k \in \mathbb{N}$, $\{I_\sigma(\omega) : \sigma \in \Lambda_k(\omega)\}$ is a natural cover of $K(\omega)$, hence

$$\mathcal{H}^s\big(K(\omega)\big) \leqslant \lim_{k\to\infty} \sum_{\sigma\in\Lambda_k(\omega)} \big|I_\sigma(\omega)\big|^s = X(\omega) < \infty \quad \text{almost surely}$$

by Lemma 3.1. Thus $\dim K(\omega) \leqslant s$. Because $K(\omega)$ is a subset of $\mathbb{R}^m$, it is obvious that $\dim K(\omega) \leqslant m$. Hence $\dim K(\omega) \leqslant \min\{s, m\}$.

We next obtain a lower bound for $\dim K(\omega)$. For a fixed $c > 0$ and $\beta < s$, from the proof of [**18**, Theorem 3.1] and the definition of $\Lambda_k(\omega)$, we know that for k sufficiently large and $\sigma \in \Lambda_k(\omega)$, we have

$$B_\sigma(\omega)^s X_\sigma(\omega) = \big(\rho_{\sigma|_1} \cdots \rho_{\sigma|_{|\sigma|}}\big)^s X_\sigma(\omega) \leqslant c(\rho_{\sigma|_1} \cdots \rho_{\sigma|_{|\sigma|}})^\beta \leqslant c(\underline{\rho}^k)^\beta \quad \text{almost surely.}$$

It was shown in Section 3 that there exists a Borel measure μ_ω on $K(\omega)$ almost surely. From the random weak separation property, (3.2) and (3.3), for any $\underline{\rho}^k$-ball $B(x, \underline{\rho}^k)$, we have

$$\mu_\omega\big(B(x, \underline{\rho}^k)\big) \leqslant lM_k(\omega)c(\underline{\rho}^k)^\beta \quad \text{almost surely.}$$

Hence

$$\frac{\log \mu_\omega\big(B(x, \underline{\rho}^k)\big)}{k \log \underline{\rho}} \geqslant \frac{\log lc + \log M_k(\omega)}{k \log \underline{\rho}} + \beta \quad \text{almost surely}$$

if k is sufficiently large. But $\log lc/(k \log \underline{\rho}) \to 0$ almost surely, as $k \to \infty$; thus with probability one

$$\liminf_{k\to\infty} \frac{\log \mu_\omega\big(B(x, \underline{\rho}^k)\big)}{k \log \underline{\rho}} \geqslant \liminf_{k\to\infty} \frac{\log M_k(\omega)}{k \log \underline{\rho}} + \beta.$$

Therefore with probability one

$$\liminf_{k\to\infty} \frac{\log \mu_\omega\big(B(x, \underline{\rho}^k)\big)}{k \log \underline{\rho}} \geqslant \liminf_{k\to\infty} \frac{\log M_k(\omega)}{k \log \underline{\rho}} + s.$$

Since $\mu_\omega\big(K(\omega)\big) = X(\omega) > 0$, by the Kinney–Pitcher–Billingsley theorem ([**3**] or [**7**, Proposition 10.1]), we have

$$\dim K(\omega) \geqslant \liminf_{k\to\infty} \frac{\log M_k(\omega)}{k \log \underline{\rho}} + s.$$

∎

REMARK 6. In the non-random case, $S_1(x) = x/2, S_2(x) = x/2 + 1/2, S_3(x) = x/2 + 1$, $x \in \mathbb{R}$, is an iterated function system related to the wavelet two-scale dilation equation.

It does not satisfy the open set condition but has the weak separation property ([**16**]). In an extensive computer simulation, we find that the maximum number of repetitions $M_k = a_k$, where $\{a_k\}$ is the Fibonacci sequence. As a result,

$$\liminf_{k\to\infty} \frac{\log M_k(\omega)}{k \log \underline{\rho}} = -\frac{(\log\sqrt{5}+1)/2}{\log 2}.$$

The Hausdorff dimension of the invariant set is then equal to 1. It obviously satisfies the nonrandom version of our result.

We can now prove Theorem 2.2 for the case of random net fractals generated by contractive maps.

PROOF OF THEOREM 2.2: From the definition of $K(w)$ in Subsection 2.2, we denote

$$\Phi'(\omega) = \Big\{\phi'_\sigma(\omega) : \sigma \in \mathcal{F}, \big|\phi'_\sigma(\omega,x) - \phi'_\sigma(\omega,y)\big| = d_\sigma(\omega)|x-y|\Big\}$$

and

$$\Phi''(\omega) = \Big\{\phi''_\sigma(\omega) : \sigma \in \mathcal{F}, \big|\phi''_\sigma(\omega,x) - \phi''_\sigma(\omega,y)\big| = c_\sigma(\omega)|x-y|\Big\}.$$

Then $\Phi'(\omega)$ and $\Phi''(\omega)$ are two families of random similitudes. It is seen that

$$\phi'_{\sigma|_1}(\omega)\circ\cdots\circ\phi'_{\sigma|_{|\sigma|}}(\omega)(I) \subset \phi_{\sigma|_1}(\omega)\circ\cdots\circ\phi_{\sigma|_{|\sigma|}}(\omega)(I) \subset \phi''_{\sigma|_1}(\omega)\circ\cdots\circ\phi''_{\sigma|_{|\sigma|}}(\omega)(I)$$

for any $\sigma \in \mathcal{F}$. If we denote by $K'(\omega)$ and $K''(\omega)$ the self-similar random fractals generated by $\Phi'(\omega)$ and $\Phi''(\omega)$ respectively, then we have $K'(\omega) \subset K(\omega) \subset K''(\omega)$. Theorem 2.2 now follows from Theorem 4.1. $\square$

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