On the expansion of $\left(1+\frac{z}{\lfloor 2}+\frac{z^2}{\lfloor 3}+\dots\right)^{-n}$ in positive integral powers of z, when n is a positive integer.

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§1. *The radius of convergence of the power series is* 2π.

The function $\left(1+\frac{z}{\lfloor 2}+\frac{z^2}{\lfloor 3}+\dots\right)^{-n}$ or $\frac{z^n}{(e^z-1)^n}$ is regular within a circle whose centre is the origin of the z plane and radius 2π, and can be expanded in a Taylor's series converging at all points within the circle.

§2. *The coefficient of* z^{n-1} *in* $\left(1+\frac{z}{\lfloor 2}+\frac{z^2}{\lfloor 3}+\dots\right)^{-n}$ *is* $(-1)^{n-1}$.

The coefficient of z^{n-1} is $\frac{1}{2\pi i}\int_C \frac{dz}{(e^z-1)^n}$, C being a closed contour surrounding the origin and lying within the circle of convergence.

Now $$-\frac{1}{n-1}\frac{d}{dz}\frac{1}{(e^z-1)^{n-1}}=\frac{e^z}{(e^z-1)^n}=\frac{1}{(e^z-1)^n}+\frac{1}{(e^z-1)^{n-1}}.$$

$$\therefore \int_C \frac{dz}{(e^z-1)^n}=-\int_C \frac{dz}{(e^z-1)^{n-1}}=(-1)^2\int_C \frac{dz}{(e^z-1)^{n-2}}=\dots$$
$$=(-1)^{n-1}\int_C \frac{dz}{e^z-1}.$$

But the residue of $\frac{1}{e^z-1}$ is 1.

$$\therefore \int_C \frac{dz}{(e^z-1)^n}=(-1)^{n-1}\,2\pi i.$$

Hence the coefficient of z^{n-1} is $(-1)^{n-1}$.

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§3. *If the coefficient of* z^r *in* $\left(1+\frac{z}{\underline{|2}}+\frac{z^2}{\underline{|3}}+\ldots\right)^{-n}$ *is denoted by* ${}_na_r$, *then*

$$ {}_na_r = -{}_{n-1}a_{r-1} + \frac{n-r-1}{n-1}\,{}_{n-1}a_r, $$

n having any of the values 2, 3, 4..., *and r any of the values* 1, 2, 3.... .

For

$$ {}_{n-1}a_{r-1} = \frac{1}{2\pi i}\int \frac{z^{n-r-1}}{(e^z-1)^{n-1}}\,dz = \frac{1}{2\pi i}\int \frac{z^{n-r-1}e^z - z^{n-r-1}}{(e^z-1)^n}\,dz $$

$$ = -\frac{1}{2\pi i}\int \frac{z^{n-r-1}}{(e^z-1)^n}\,dz - \frac{1}{2\pi i}\frac{1}{n-1}\int z^{n-r-1}\frac{d}{dz}\frac{1}{(e^z-1)^{n-1}}\,dz $$

$$ = -{}_na_r + \frac{1}{2\pi i}\frac{n-r-1}{n-1}\int_c \frac{z^{n-r-2}}{(e^z-1)^{n-1}}\,dz $$

$$ = -{}_na_r + \frac{n-r-1}{n-1}\,{}_{n-1}a_r. $$

Hence ${}_na_r = -{}_{n-1}a_{r-1} + \frac{n-r-1}{n-1}\,{}_{n-1}a_r$, provided $n>1$, $r>0$.

It may be added that since the coefficient of z^0 is unity for $n=1, 2, 3, \ldots$ we have

$$ {}_na_0 = {}_{n-1}a_0 = \ldots = {}_2a_0 = {}_1a_0. $$

§4. The values of the suffixes at which we ultimately arrive are seen very clearly if we associate with each coefficient ${}_na_r$ a point whose ordinate is n and abscissa r: this point can be denoted by the same symbol ${}_na_r$ without any danger of confusion. Provided r is not equal to $n-1$, the reduction equation leads from the point ${}_na_r$ to the two points ${}_{n-1}a_r$ and ${}_{n-1}a_{r-1}$, that is, to those reached by a step one unit in length parallel to the axis of n and towards the axis of r, and then another parallel to the axis of r and towards the axis of n. If $r>$ or $=n$, the repeated application of the reduction equation will therefore determine ${}_na_r$ as the sum of multiples of ${}_1a_r$, ${}_1a_{r-1}$, ... ${}_1a_{r-n+1}$, that is, as the sum of multiples of

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certain of Bernoulli's numbers, since when $n=1$ the expansion becomes

$$1-\frac{z}{2}+\frac{B_1}{\underline{|2}}z^2-\frac{B_2}{\underline{|4}}z^4+\dots .$$

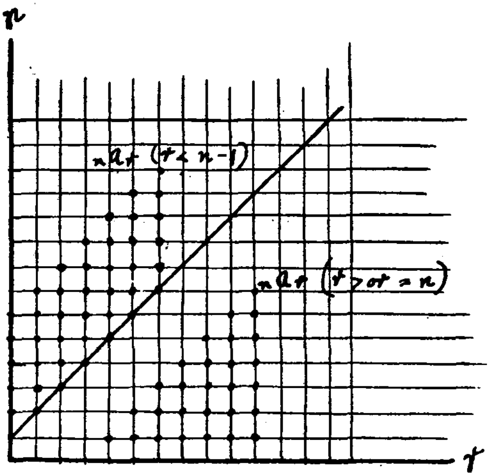

Fig. 1.

For powers of z lower than z^{n-1} the case is different. When the suffix r is less than the suffix n by unity, the reduction equation assumes the form

$${}_ka_{k-1}=-{}_{k-1}a_{k-2},$$

and again when the suffix r is zero,

$${}_ka_0={}_{k-1}a_0.$$

These discontinuities in the form of the reduction equation preclude the crossing of the lines $r=n-1$ and $r=0$, and lead to the determination of ${}_na_r$ when $r<n-1$, simply as a multiple of ${}_1a_0$.

§ 5. *The coefficients of powers of z higher than z^{n-1} in the expansion of* $\left(1+\frac{z}{\underline{|2}}+\frac{z^2}{\underline{|3}}+\dots\right)^{-n}$ *are given, for values of* $r>n-1$, *by*

$$ {}_nc_r = \frac{(-1)^{n-1}}{\underline{|n-1}} (\delta+1)(2\delta+1)\dots(\overline{n-1}\,\delta+1)\,{}_1c_r, $$

where ${}_nc_r \equiv \underline{|r-n}\,{}_na_r$, ${}_kc_k = {}_ka_k$, *and* δ *is a symbolic operator such that* $\delta({}_mc_k) = {}_mc_{k-1}$.

When $r > n-1$, the reduction equation

$$ {}_na_r = -\,{}_{n-1}a_{r-1} - \frac{r-n+1}{n-1}\,{}_{n-1}a_r $$

may be written

$$ \underline{|r-n}\,{}_na_r = -\underline{|r-n}\,{}_{n-1}a_{r-1} - \frac{1}{n-1}\,\underline{|r-\overline{n-1}}\,{}_{n-1}a_r, $$

or
$$ {}_nc_r = -\,{}_{n-1}c_{r-1} - \frac{1}{n-1}\,{}_{n-1}c_r, $$

where
$$ {}_nc_r \equiv \underline{|r-n}\,{}_na_r \text{ when } r > n, \text{ and } {}_kc_k \equiv {}_ka_k. $$

This modified reduction equation holds good for all values of $n > 1$.

Now let Δ and δ be symbolic operators such that

$$ \Delta({}_mc_k) = {}_{m-1}c_k \text{ and } \delta({}_mc_k) = {}_mc_{k-1}. $$

Then
$$ {}_nc_r = -\left(\delta + \frac{1}{n-1}\right)\Delta\,{}_nc_r, $$

so that the operator $-\left(\delta + \frac{1}{n-1}\right)\Delta$ acting on a coefficient whose primary suffix in n reproduces that coefficient: and $\delta\Delta\,{}_nc_r$ and $\Delta\,{}_nc_r$ being coefficients whose primary suffix is $n-1$, and the operators being commutative, it follows that

$$ {}_nc_r = (-1)^2\left(\delta + \frac{1}{n-1}\right)\left(\delta + \frac{1}{n-2}\right)\Delta^2{}_nc_r. $$

Thus
$$ {}_nc_r = (-1)^{n-1}\left(\delta + \frac{1}{n-1}\right)\left(\delta + \frac{1}{n-2}\right)\dots(\delta+1)\,\Delta^{n-1}{}_nc_r. $$

$$ = \frac{(-1)^{n-1}}{\underline{|n-1}}(\delta+1)(2\delta+1)\dots(\overline{n-1}\,\delta+1)\,{}_1c_r. $$

This formula gives the coefficient ${}_na_r$ for values of $r > n-1$ as the sum of multiples of certain numbers of Bernoulli.

§ 6. *For values of* r *from* $n-2$ to 1

$$ {}_na_r = (-1)^r \frac{\underline{|n-r-1}}{\underline{|n-1}} S_r(1, 2, 3 \ldots n-1), $$

where $S_r(1, 2, 3 \ldots n-1)$ *means the sum of the products of the numbers* $1, 2, 3 \ldots n-1$ *taken* r *at a time.*

When $0 < r < n-1$ the reduction equation may be written

$$ \frac{{}_na_r}{\underline{|n-r-1}} = -\frac{{}_{n-1}a_{r-1}}{\underline{|n-r-1}} + \frac{1}{n-1} \frac{{}_{n-1}a_r}{\underline{|n-r-2}}, $$

or
$$ {}_nb_r = -{}_{n-1}b_{r-1} + \frac{1}{n-1} {}_{n-1}b_r, $$

where
$$ {}_nb_r \equiv \frac{{}_na_r}{\underline{|n-r-1}}. $$

When, however, $r=0$, and again when $r=n-1$, the reduction equation is discontinuous in form. In the first case

$$ {}_ka_0 = {}_{k-1}a_0 \quad (k>1), $$

and in the second case

$$ {}_ka_{k-1} = -{}_{k-1}a_{k-2} \quad (k>1); $$

and the corresponding modified equations are

$$ {}_kb_0 = \frac{1}{k-1} {}_{k-1}b_0 \quad (k>1), $$

and
$$ {}_kb_{k-1} = -{}_{k-1}b_{k-2} \quad (k>1). $$

Hence the modified equation may be regarded as continuous in form if we introduce the symbols

$$ {}_kb_{-1}, {}_kb_{-2}, \ldots \text{ etc.}, {}_{k-1}b_{k-1}, {}_{k-1}b_{k-2}, \ldots \text{ etc.}, $$

all of them having the value zero. By the introduction of these

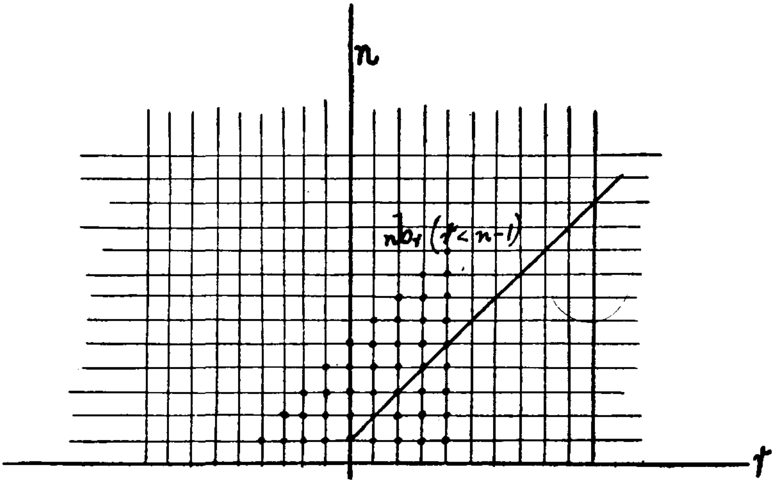

Fig. 2.

symbols, the array of points associated with the coefficients obtained from ${}_nb_r$ by means of the reduction equation is no longer bounded by the lines $r=0$, $r=n-1$, but extends to the line $n=1$. Of the points ${}_1b_r$, ${}_1b_{r-1}$, … ${}_1b_1$, ${}_1b_0$, ${}_1b_{-1}$, …, etc., ultimately reached, all except ${}_1b_0$ are associated with coefficients which vanish by virtue of their definition.

Now let Δ and δ be symbolic operators such that $\Delta({}_mb_k)={}_{m-1}b_k$ and $\delta({}_mb_k)={}_mb_{k-1}$.

Then $${}_nb_r=\left(-\delta+\frac{1}{n-1}\right)\Delta\,{}_nb_r=\left(-\delta+\frac{1}{n-1}\right)\left(-\delta+\frac{1}{n-2}\right)\Delta^2{}_nb_r=\dots$$

$$=\left(-\delta+\frac{1}{n-1}\right)\left(-\delta+\frac{1}{n-2}\right)\dots(-\delta+1)\,\Delta^{n-1}{}_nb_r$$

$$=\left(-\delta+\frac{1}{n-1}\right)\left(-\delta+\frac{1}{n-2}\right)\dots(-\delta+1)\,{}_1b_r$$

$$=\frac{1}{\underline{|n-1}}\,(1-\delta)(1-2\delta)\dots(1-\overline{n-1}\delta)\,{}_1b_r.$$

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But $\delta^{\lambda}({}_1b_r)=0$, for all values of λ except r, and $\delta^r({}_1b_r)=1$.

$$\therefore \quad {}_nb_r=\frac{(-1)^n}{\underline{|n-1}}\,S_r(1,\ 2,\ 3\ldots n-1),$$

where $S_r(1,\ 2,\ 3\ldots n-1)$ means the sum of the products of the numbers $1,\ 2,\ 3\ldots n-1$ taken r at a time.

$$\text{Hence} \quad {}_na_r=(-1)^r\,\frac{\underline{|n-r-1}}{\underline{|n-1}}\,S_r(1,\ 2,\ 3,\ldots\ n-1)$$

for all values of $r<n-1$.

It may be observed that the formula holds for $r=n-1$, but the coefficients for which this is the case have been evaluated already.

§7. A number of identities involving Bernoulli's numbers can be obtained by expanding $\left(1-\frac{z}{2}+\frac{B_1}{\underline{|2}}z^2-\frac{B_2}{\underline{|4}}z^4+\ldots\right)^n$ and equating the coefficients of various powers of z to those of the expansion of $\left(1+\frac{z}{\underline{|2}}+\frac{z^2}{\underline{|3}}+\ldots\right)^{-n}$. For example, when $n=2$ the coefficients of z^{2r} give

$$(2r+1)\,\frac{B_r}{\underline{|2r}}=\frac{B_{r-1}}{\underline{|2r-2}}\,\frac{B_1}{\underline{|2}}+\frac{B_{r-2}}{\underline{|2r-4}}\,\frac{B_2}{\underline{|4}}+\ldots+\frac{B_1}{\underline{|2}}\,\frac{B_{r-1}}{\underline{|2r-2}}.$$

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