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ON THE COMPONENT GROUP OF THE AUTOMORPHISM GROUP OF A LIE GROUP

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Abstract

Let G be a Lie group, $G_\circ$ the connected component of G that contains the identity, and Aut G the group of all topological automorphisms of G. In the case when $G/G_\circ$ is finite and G has a faithful representation, we obtain a necessary and sufficient condition for G so that Aut G has finitely many components in terms of the maximal central torus in $G_\circ$.

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1. Introduction

Let G be a Lie group, and let Aut G be the group of all topological automorphisms of G. If X and Y are subsets of G, we denote by $N(X, Y)$ the collection of all those elements α of Aut G such that $\alpha(x)x^{-1} \in Y$ and $\alpha^{-1}(x)x^{-1} \in Y$ for every x in X. Let $\mathscr{F}$ be the set consisting of elements of the form $N(K, V)$, where K ranges over all compact subsets of G and V ranges over all neighborhoods of the identity element of G. Then, Aut G is a topological group with $\mathscr{F}$ as a fundamental system of neighborhoods of the identity element of Aut G. In [6], Hochschild proves that Aut G is a Lie group and has at most countably many components whenever $G/G_\circ$ is finitely generated, where $G_\circ$ denotes the connected component of G that contains the identity element. It is the aim of this paper to study the case when Aut G has only finitely many components. In the case when $G/G_\circ$ is finite and G has a faithful representation, we obtain a necessary and sufficient condition for G so that Aut G has finitely many components.

Let G be as above and T be the maximal central torus in $G_\circ$. By a theorem of Iwasawa ([11, Theorem 1′]), every automorphism in $(\mathrm{Aut}\, G)_\circ$ fixes every element

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of T. So, if $(\operatorname{Aut} G)_\circ|T$ is infinite, then Aut G has infinitely many components, where $(\operatorname{Aut} G)_\circ|T$ denotes the image of the map from $(\operatorname{Aut} G)_\circ$ into Aut T sending every element of $(\operatorname{Aut} G)_\circ$ onto its restriction of T. This suggests that we concentrate our study on the maximal central torus T in $G_\circ$. The example of the n-dimensional torus shows that the dimension of T is important in determining whether or not Aut G has finitely many components. However, the following example demonstrates that the mere consideration of the dimension of T is not enough.

EXAMPLE. Let S be the 1-dimensional torus, $S_1 = S_2 = S$, and $T = S_1 \times S_2$. Let $\sigma \in \operatorname{Aut} T$ be such that $\sigma^2 = \mathrm{id}_T$, and let Δ_σ be the subgroup of Aut T generated by σ. Let $G_\sigma = T \rtimes \Delta_\sigma$ and $\theta \in \operatorname{Aut} G_\sigma$. It is straightforward to check that $\theta(\sigma)\sigma^{-1} \in T$ and $\theta|T$ is a member of the centralizer $C(\sigma)$ of σ in Aut T. It follows that Aut G_σ is topologically isomorphic to $T \rtimes C(\sigma)$; and hence, Aut G_σ has finitely many components if and only if $C(\sigma)$ has finitely many components. In particular, let δ and γ be automorphisms of T defined by $\delta(x, y) = (\bar{x}, \bar{y})$ and $\gamma(x, y) = (\bar{x}, y)$, where we denote the conjugate of a complex number w by $\bar{w}$. Then, it is straightforward to check that $C(\delta)$ has infinitely many components and $C(\gamma)$ has finitely many components; in turn, Aut G_δ has infinitely many components and Aut G_γ has finitely many components. We note that the maximal central tori of $(G_\delta)_\circ$ and $(G_\gamma)_\circ$ are both T. A close examination of these two groups reveals the following:

(i) Both S_1 and S_2 are smallest nontrivial subtori of T in G_δ (respectively G_γ) that commute with Δ_δ (respectively Δ_γ).
(ii) There is an isomorphism of S_1 onto S_2 that commutes with Δ_δ (namely, the map sending every complex number to its conjugate). On the contrary, there is no isomorphism of S_1 onto S_2 that commutes with Δ_γ.

This example motivates us to introduce what is called a Δ-decomposition of the maximal central torus that plays an important role in our study. To see this, we notice that since G has finitely many components, the restrictions of all inner automorphisms of G to T form a finite subgroup, denoted by Δ, of Aut T. In Section 2, we prove that there are nontrivial Δ-invariant closed subtori $T_1, T_2, \ldots, T_m$ such that each T_i has no proper Δ-invariant closed subtorus of T (we call such subtori of T the *Δ-simple subtori* of T), $T = T_1T_2\cdots T_m$, and $T_i \cap (T_1 \cdots T_{i-1}T_{i+1} \cdots T_m)$ is finite for every $i \in \{1, \cdots, m\}$. We call such a decomposition of T into product of Δ-simple subtori a *Δ-decomposition* of T. Two Δ-simple subtori T_a and T_b of T are said to be *almost Δ-isomorphic* if there is a continuous homomorphism α of T_a onto T_b such that α has finite kernel and $\alpha \circ \delta = \delta \circ \alpha$ for every δ in Δ. Two Δ-simple subtori T_a and T_b of T are said to be *Δ-isomorphic* if there is a topological isomorphism α of T_a onto T_b such that $\alpha \circ \delta = \delta \circ \alpha$ for every δ in Δ. If, in addition, $T_a = T_b$, then we call such a map α a Δ-*automorphism* of T_a. We say that T is Δ-*rigid* if T satisfies the following two properties:

(a) no two distinct Δ-simple subtori of T are almost Δ-isomorphic; and
(b) for every Δ-simple subtorus T_c of T, there are only finitely many Δ-automorphisms of T_c.

Let T_c be a Δ-simple subtorus of T and $\mathrm{Aut}_\Delta T_c$ be the group of all Δ-automorphisms of T_c. In view of Condition (b) above, in Section 3, we completely determine when $\mathrm{Aut}_\Delta T_c$ has finitely many elements and when $\mathrm{Aut}_\Delta T_c$ has infinitely many elements from the dimension of the torus T_c by way of characterizing $\mathrm{Aut}_\Delta T_c$ as the group of units in the ring of algebraic integers in a certain algebraic number field, in the case when $\Delta \mid T_c$ is abelian. In order to see this, let us first fix some notations. If $\delta \in \Delta$, we denote by δ° the differential of δ. Let $\Delta^\circ = \{\delta^\circ : \delta \in \Delta\}$, W be the $\mathbb{Q}$-span of the kernel of the exponential map from $\mathscr{L}(T)$ onto T, $W_c = W \cap \mathscr{L}(T_c)$, and D be the centralizer of $\Delta^\circ \mid W_c$ in $\mathrm{Hom}_{\mathbb{Q}}(W_c, W_c)$, the collection of all $\mathbb{Q}$-linear endomorphisms of W_c. Then, we have the following results.

THEOREM 12. *Using the same notation as above, if $\Delta \mid T_c$ is abelian, then the map sending every element α of $\mathrm{Aut}_\Delta T_c$ onto $\alpha^\circ \mid W_c$ is a group isomorphism of $\mathrm{Aut}_\Delta T_c$ onto the group of units in the ring of algebraic integers in D.*

COROLLARY 13. *We use the same notation as above and suppose that $\Delta \mid T_c$ is abelian. If the order of $\Delta \mid T_c$ is 1, 2, 3, 4, or 6, then $\mathrm{Aut}_\Delta T_c$ has 2, 2, 6, 4, or 6 elements, respectively. Moreover, these are the only cases in each of which $\mathrm{Aut}_\Delta T_c$ has finitely many elements.*

Suppose that G is a Lie group such that G has a faithful representation and G has finitely many components. Then, G is a semidirect product $E \cdot M$, where E is a simply connected solvable normal analytic subgroup of G and M is a maximal reductive subgroup of G; moreover, there is a finite subgroup D of M such that $M = M_\circ D$ ([12, p. 42]; [7, Chapter XVIII]). Let $S = [M_\circ, M_\circ]$ and $K = Z(M_\circ)_\circ$, where $Z(M_\circ)$ is the center of $M_\circ$. Then, K is a maximal torus in the radical of $G_\circ$ ([7, Chapter XVIII]). Recall that Δ is the finite subgroup of $\mathrm{Aut}\, T$ consisting of the restrictions of all inner automorphisms of G to T. If we denote by $I_G(D)$ the group of all those inner automorphisms of G determined by elements in D, then Δ is precisely the group of restrictions of automorphisms in $I_G(D)$ to T. Since D normalizes K, the restrictions of automorphisms in $I_G(D)$ to K form a finite subgroup of $\mathrm{Aut}\, K$. By an abuse of notation, we still denote this group by Δ. Then, both T and K have Δ-decompositions. Now, we are in a position to state our main theorem, which is proved in Section 4.

THEOREM 17. *Using the same notation as above, in order that* Aut G *has finitely many components, it is necessary and sufficient that* G *satisfies the following two conditions:*

(a) T *is* Δ*-rigid; and*
(b) *there is no* Δ*-simple subtorus of* T *that is almost* Δ*-isomorphic to any* Δ*-simple subtorus of* K *that is not in* T.

Since any two maximal tori in the radical of $G_\circ$ are conjugate, we see that Theorem 17 is independent of the choice of K. If, in addition, G is connected, we have the following version of Theorem 17.

COROLLARY 18. *Suppose that* G *is an analytic group that has a faithful representation, and* T *is the maximal central torus in* G. *Then, in order that* Aut G *has finitely many components, it is necessary and sufficient that* T *is trivial, or that the dimension of* T *is* 1 *and* T *is exactly the maximal torus in the radical of* G.

We also give an example to show that Theorem 17 may fail if G does not have a faithful representation.

As an application of our main result, we give a necessary and sufficient condition for an analytic group G so that Aut G is almost algebraic; provided that G has a faithful representation. (Cf. [4, 16, 17].) More precisely, we have the following result.

THEOREM 19. *Suppose that* G *is an analytic group that has a faithful representation, and* T *is the maximal central torus in* G. *Then,* Aut G *is almost algebraic if and only if* T *is trivial, or the dimension of* T *is* 1 *and* T *is exactly the maximal torus in the radical of* G.

NOTATION. Let G be a locally compact group. We denote the connected component of G that contains the identity element by $G_\circ$. Aut G denotes the group of topological automorphisms of G equipped with the topology described above. If f is a function of G into G, we denote the restriction of f to C by $f \mid C$, for every subset C of G. If A and B are subsets of G and $x \in G$, we denote the inner automorphism of G that is determined by x by $I_G(x)$, the set $\{I_G(x) : x \in A\}$ by $I_G(A)$, the set $\{I_G(a) \mid B : a \in A\}$ by $I_G(A) \mid B$, and the subgroup of G that is generated by $\{aba^{-1}b^{-1} : a \in A, b \in B\}$ by $[A, B]$. If F is another topological group, we denote the collection of all continuous homomorphisms of G into F by Hom (G, F). If, in addition, G and F are analytic groups, $g \in$ Hom (G, F), and Γ is a subset of Hom (G, F), then we denote the Lie algebra of G by $\mathscr{L}(G)$, the differential of g by g°, and the set $\{\gamma^\circ : \gamma \in \Gamma\}$ by Γ°. We denote the identity map on any set J by id_J. As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural numbers, integers, rational numbers, real numbers, and complex numbers, respectively.

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2. A decomposition of a torus

Throughout this section, T denotes a fixed torus, and Δ denotes a fixed finite subgroup of $\operatorname{Aut} T$. First, we introduce what is called a Δ-decomposition of T that plays an important role in our study.

Let $\exp\colon \mathscr{L}(T)\to T$ be the exponential map. It is well-known that $\mathscr{L}(T)$ possesses an $\mathbb{R}$-basis whose $\mathbb{Z}$-span is precisely ker(exp), and ker(exp) is Δ°-invariant. Clearly, the $\mathbb{Q}$-span W of ker(exp) is Δ°-invariant. (We will fix the notation exp and W throughout this section.) By the representation theory of finite groups, W is a semisimple Δ°-module; that is, W can be written as a direct sum $W_1 \oplus \cdots \oplus W_m$ of simple Δ°-submodules. It is easy to see that W_j possesses a $\mathbb{Q}$-basis B_j contained in ker(exp), for each $j \in \{1, \cdots, m\}$.

Fix an $i \in \{1, \cdots, m\}$. Let B be the $\mathbb{Z}$-span of $\bigcup_{j=1}^m B_j$ and $A_i = (W_i \otimes_{\mathbb{Q}} \mathbb{R}) + B$. Since $\bigcup_{j=1}^m B_j$ is clearly an $\mathbb{R}$-basis for $\mathscr{L}(T)$, A_i/B is compact. Together with the fact that B is contained in ker(exp), one concludes readily that $A_i + \ker(\exp)$ is closed in $\mathscr{L}(T)$; and hence, $T_i = \exp(A_i + \ker(\exp))$ is a closed subtorus of T. It is straightforward to check that $\mathscr{L}(T_i) = W_i \otimes_{\mathbb{Q}} \mathbb{R}$, $W_i = \mathscr{L}(T_i) \cap W$, and T_i is Δ-invariant. Moreover, we notice that T_i has no proper Δ-invariant closed subtorus. In order to see this, we let E be a Δ-invariant closed subtorus of T_i, $\exp_E$ be the exponential map of $\mathscr{L}(E)$ onto E, and X be the $\mathbb{Q}$-span of $\ker(\exp_E)$. Since $\exp_E$ is precisely the restriction map of exp to $\mathscr{L}(E)$, X is a Δ°-invariant subspace of the simple Δ°-module W_i; a fortiori, X is a proper subspace of W_i. It follows that E is a proper subgroup of T_i. This proves that T_i has no proper Δ-invariant closed subtorus.

DEFINITION 1. A non-trivial Δ-invariant closed subtorus of T is called a Δ-*simple subtorus* if it has no proper Δ-invariant closed subtorus.

DEFINITION 2. A decomposition $T_1T_2\cdots T_m$ of T into Δ-simple subtori is called a Δ-*decomposition* of T if $T_i \cap (T_1 \cdots T_{i-1}T_{i+1}\cdots T_m)$ is finite for every $i \in \{1.\cdots, m\}$.

Using this terminology, the discussion above can be summarized as follows.

PROPOSITION 3. *A Δ-decomposition of T always exists.*

If W_a is a simple Δ°-submodule of W and $T_a = \exp(W_a \otimes_{\mathbb{Q}} \mathbb{R})$, in view of [1, p. 174, Proposition 1], we may use the same argument as above to obtain that T_a is a Δ-simple subtous of T, $\mathscr{L}(T_a) = W_a \otimes_{\mathbb{Q}} \mathbb{R}$, and $W_a = \mathscr{L}(T_a) \cap W$. Conversely, if T_b is a Δ-simple subtorus of T and $W_b = \mathscr{L}(T_b) \cap W$, then it is straightforward to check that W_b is a simple Δ°-submodule of W and $\mathscr{L}(T_b) = W_b \otimes_{\mathbb{Q}} \mathbb{R}$. Thus, we have the following result.

PROPOSITION 4. *There is a one-to-one correspondence between the collection of all simple Δ°-submodules of W and the collection of all Δ-simple subtori of T (given by $W_a \mapsto \exp(W_a \otimes_{\mathbb{Q}} \mathbb{R})$).*

It seems natural to ask at this point what connection T_a and T_b have if W_a and W_b are Δ°-isomorphic, where T_a, T_b, W_a, and W_b have the same meaning as in the last paragraph.

PROPOSITION 5. *Suppose that W_a and W_b are two simple Δ°-submodules of W, $T_a = \exp(W_a \otimes_{\mathbb{Q}} \mathbb{R})$, and $T_b = \exp(W_b \otimes_{\mathbb{Q}} \mathbb{R})$. Then, the following two statements are equivalent.*

(i) *W_a and W_b are isomorphic as Δ°-modules.*
(ii) *There is a continuous homomorphism α of T_a onto T_b such that α has finite kernel and $\alpha \circ \delta = \delta \circ \alpha$ for every δ in Δ.*

PROOF. Denote by $\exp_a$ (respectively $\exp_b$) the exponential map of $\mathscr{L}(T_a)$ (respectively $\mathscr{L}(T_b)$) onto T_a (respectively T_b). As we saw before, W_a (respectively W_b) possesses an $\mathbb{R}$-basis $B_a = \{b_1, \cdots, b_l\}$ (respectively B_b) for $\mathscr{L}(T_a)$ (respectively $\mathscr{L}(T_b)$) such that the $\mathbb{Z}$-span of B_a (respectively B_b) is precisely $\ker(\exp_a)$ (respectively $\ker(\exp_b)$). Let f be a Δ°-module isomorphism of W_a onto W_b. Relative to the bases B_a and B_b, the matrix M associated with f is clearly a matrix over $\mathbb{Q}$. Choose a large enough positive integer c such that cM is a matrix over $\mathbb{Z}$. Define $g\colon \mathscr{L}(T_a) \to \mathscr{L}(T_b)$ by $g(\sum_{i=1}^{l} r_i b_i) = \sum_{i=1}^{l} c r_i f(b_i)$ for every $r_1, \cdots, r_l$ in $\mathbb{R}$. It is straightforward to check that g is a linear isomorphism of $\mathscr{L}(T_a)$ onto $\mathscr{L}(T_b)$, $g(\ker(\exp_a))$ is contained in $\ker(\exp_b)$, and $g \circ \delta^\circ = \delta^\circ \circ g$ for every δ in Δ. As a result, g induces a continuous homomorphism β of T_a onto T_b with discrete kernel such that $\beta \circ \delta = \delta \circ \beta$ for every δ in Δ. Since T_a is compact, β has finite kernel. This proves (i) implies (ii). Conversely, let α be a continuous homomorphism of T_a onto T_b that satisfies (ii). From the fact that the $\mathbb{Q}$-span of $\ker(\exp_a)$ (respectively $\ker(\exp_b)$) is exactly W_a (respectively W_b), it is straightforward to check that the restriction of the differential of α to W_a is a Δ°-module isomorphism of W_a onto W_b. This proves (ii) implies (i), and the proof of the proposition is complete.

DEFINITION 6. Two Δ-simple subtori T_a and T_b of T are said to be *almost Δ-isomorphic* if the second statement (ii) in Proposition 5 holds.

DEFINITION 7. Two Δ-simple subtori T_a and T_b of T are said to be *Δ-isomorphic* if there is a topological isomorphism α of T_a onto T_b such that $\alpha \circ \delta = \delta \circ \alpha$ for every δ in Δ. If, in addition, $T_a = T_b$, then we call such a map α a *Δ-automorphism* of T_a.

The following definition singles out a special class of Δ-tori that plays a central role in our study.

DEFINITION 8. We say that T is Δ-*rigid* if T satisfies the following two properties:

(a) no two distinct Δ-simple subtori of T are almost Δ-isomorphic; and
(b) for every Δ-simple subtorus T_c of T, there are only finitely many Δ-automorphisms of T_c.

In view of Proposition 4 and 5 above, and [1, Proposition 1, 2, pp. 174–175], T satisfies Condition (a) above if and only if T has a unique Δ-decomposition $T_1T_2\cdots T_m$ (up to order) such that if T_i and T_j are almost Δ-isomorphic for some $i, j \in \{1, \cdots, m\}$, then $i = j$.

3. The Δ-automorphism group of a Δ-simple subtorus

In this section, T denotes a fixed torus and Δ denotes a fixed finite subgroup of $\operatorname{Aut} T$. We fix a Δ-simple subtorus T_c of T, and let $\operatorname{Aut}_\Delta T_c$ be the group of all Δ-automorphisms of T_c. Although the results in this section are not needed in the proof of our main theorem, in view of Condition (b) in Definition 8, it is interesting to see when $\operatorname{Aut}_\Delta T_c$ has finitely many elements and when $\operatorname{Aut}_\Delta T_c$ has infinitely many elements. Firstly, we make the following observation.

PROPOSITION 9. *Let* $\alpha \in \operatorname{Aut}_\Delta T_c$. *Then,*

(A) α° *is semisimple; and*
(B) α *is ergodic* (*that is,* α° *has no eigenvalue which is a root of unity*) *if and only if* α *has infinite order.*

PROOF. Let $W_c = W \cap \mathscr{L}(T_c)$. Since W_c is the $\mathbb{Q}$-span of the kernel of the exponential map $\exp_c: \mathscr{L}(T_c) \to T_c$ and $\ker(\exp_c)$ is α°-invariant, we see that $\alpha^\circ(W_c) = W_c$. Let $\beta = \alpha^\circ \mid W_c$, and let u be the unipotent part and s be the semisimple part in the multiplicative Jordan decomposition of β. Clearly, there is a non-zero element v_1 of W_c such that $u(v_1) = v_1$. Let V_1 be the $\mathbb{Q}$-span of $\delta^\circ(v_1)$ as δ ranges over all elements of Δ. If $\delta \in \Delta$, since β commutes with δ°, so does u; and hence, $u(\delta^\circ(v_1)) = \delta^\circ(v_1)$. This proves that u is the identity on V_1. On the other hand, since W_c is a simple Δ°-module (by Proposition 4), the non-zero Δ°-submodule V_1 of W_c must coincide with W_c. Consequently, u is the identity on W_c; and hence, $\beta = s$. Clearly, this implies that α° is semisimple, and (A) is proved. Since β is semisimple, the matrix M associated with the map of $W_c \otimes_{\mathbb{Q}} \mathbb{C}$ onto $W_c \otimes_{\mathbb{Q}} \mathbb{C}$ that is induced from β is a diagonal matrix relative to a suitable basis for $W_c \otimes_{\mathbb{Q}} \mathbb{C}$. From the fact that the diagonal of M consists of all eigenvalues of α°, one concludes readily that if α is ergodic then α has infinite order. Conversely, suppose that λ is an eigenvalue of α° such that $\lambda^n = 1$ for some positive integer n. Then, clearly, the linear map $\beta^n - \mathrm{id}_{W_c}$ is not one-to-one. It follows that there is a non-zero element v_2 of W_c such that $\beta^n(v_2) = v_2$. Let V_2 be the $\mathbb{Q}$-span of $\delta^\circ(v_2)$ as δ ranges over all elements of Δ. Then, $\beta^n(\delta^\circ(v_2)) = \delta^\circ(v_2)$ for every $\delta \in \Delta$. This proves that β^n is identity on V_2. Being a non-zero Δ°-submodule of W_c, V_2 must coincide with W_c. Consequently, β has finite order; a fortiori, α has finite order. This proves that if α has infinite order then α is ergodic. The proof of the proposition is thereby complete.

In the case when $\Delta \mid T_c$ is abelian, we may completely determine when $\mathrm{Aut}_\Delta T_c$ has finitely many elements and when $\mathrm{Aut}_\Delta T_c$ has infinitely many elements from the dimension of the torus T_c, by way of characterizing $\mathrm{Aut}_\Delta T_c$ as the group of units in the ring of algebraic integers in a certain algebraic number field. In preparation for this, we need some results from the representation theory of finite groups. If V is a vector space over a division ring E, we use $\dim_E V$ to denote the dimension of V over E, $\mathrm{Hom}_E(V, V)$ to denote the collection of all E-linear endomorphisms of V, and $GL_E(V)$ to denote the collection of all those elements in $\mathrm{Hom}_E(V, V)$ that are invertible. Let $W_c = W \cap \mathscr{L}(T_c)$. Let K be any extension field of $\mathbb{Q}$. If $g \in \mathrm{Hom}_{\mathbb{Q}}(W_c, W_c)$, then we denote by g^K the K-linear endomorphism of $W_c \otimes_{\mathbb{Q}} K$ that is defined by $g^K(\sum k_i w_i) = \sum k_i g(w_i)$, where the k_i's are elements of K, the w_i's are elements of W_c, and we write $w_i \otimes k_i$ as $k_i w_i$ for simplicity. The group Δ° acts on $W_c \otimes_{\mathbb{Q}} K$ as the group $\{(\delta^\circ \mid W_c)^K : \delta^\circ \in \Delta^\circ\}$ of K-linear automorphisms of $W_c \otimes_{\mathbb{Q}} K$. This gives a Δ°-module structure on $W_c \otimes_{\mathbb{Q}} K$. In particular, $W_c \otimes_{\mathbb{Q}} \mathbb{C}$ is a semisimple Δ°-module; that is, $W_c \otimes_{\mathbb{Q}} \mathbb{C}$ can be written as a direct sum $V_1 \oplus \cdots \oplus V_s$ of simple Δ°-submodules. For each $i \in \{1, \cdots, s\}$, denote by $\lambda_i : \Delta^\circ \to GL_{\mathbb{C}}(V_i)$ the $\mathbb{C}$-representation of Δ° that corresponds to the Δ°-module structure of V_i, χ_i the character of Δ° that is afforded by λ_i, and $\mathbb{Q}(\chi_i)$ the subfield of $\mathbb{C}$ that is generated by $\mathbb{Q}$ and $\{\chi_i(\delta^\circ) : \delta \in \Delta\}$. Let $L = \mathbb{Q}(\chi_1)$. Then, $L = \mathbb{Q}(\chi_i)$ for every $i \in \{1, \cdots, s\}$, L is a finite degree Galois extension of $\mathbb{Q}$, and the Galois group $\mathscr{G}(L/\mathbb{Q})$ is abelian ([10, p. 152 and (9.21)]). Let $r = [L : \mathbb{Q}]$, and $\mathscr{G}(L/\mathbb{Q}) = \{\sigma_1, \cdots, \sigma_r\}$, where $\sigma_1 = \mathrm{id}_L$. For each $i \in \{1, \cdots, r\}$, define the map $S_{\sigma_i} : W_c \otimes_{\mathbb{Q}} L \to W_c \otimes_{\mathbb{Q}} L$ by $S_{\sigma_i}(\sum l_j w_j) = \sum \sigma_i(l_j) w_j$, where the l_j's are elements of L, and the w_j's are elements of W_c. It is straightforward to check that

(1) $f^L \circ S_{\sigma_i} = S_{\sigma_i} \circ f^L$ for every $f \in \mathrm{Hom}_{\mathbb{Q}}(W_c, W_c)$ and $i \in \{1, \cdots, r\}$.

Let U_1 be a fixed simple Δ°-submodule of $W_c \otimes_{\mathbb{Q}} L$. For every $i \in \{1, \cdots, r\}$, let $U_i = S_{\sigma_i}(U_1)$. In view of (1), $U_1, \cdots, U_r$ are Δ°-submodules of $W_c \otimes_{\mathbb{Q}} L$. In fact, we have the following results.

(2) $U_1, \cdots, U_r$ are simple Δ°-submodules of $W_c \otimes_{\mathbb{Q}} L$, $W_c \otimes_{\mathbb{Q}} L = U_1 \oplus \cdots \oplus U_r$, and U_i and U_j are not Δ°-isomorphic for any two distinct $i, j \in \{1, \cdots, r\}$; and

(3) there is an integer k (called the *Schur index* of χ_1 over $\mathbb{Q}$) such that $s = rk$, and $V_1, \cdots, V_s$ can be so chosen that, for each $i \in \{1, \cdots, r\}$, the Δ°-module $U_i \otimes_L \mathbb{C}$ can be written as the direct sum $V_{ik-k+1} \oplus V_{ik-k+2} \oplus \cdots \oplus V_{ik}$ of simple Δ°-submodules,

and any two of $V_{ik-k+1}, V_{ik-k+2}, \cdots, V_{ik}$ are Δ°-isomorphic ([3, (70.15)]; [10, (9.21)]). It follows that

(4) $\dim_{\mathbb{Q}} W_c = \dim_L(W_c \otimes_{\mathbb{Q}} L) = r(\dim_L U_1) = rk(\dim_{\mathbb{C}} V_1)$.

Let D be the centralizer of $\Delta^\circ \mid W_c$ in $\mathrm{Hom}_{\mathbb{Q}}(W_c, W_c)$. Since W_c is a simple Δ°-module, by Schur's lemma, D is a division ring. Let

$$F = \left\{ \sum_{\delta \in \Delta} a_\delta (\delta^\circ \mid W_c) : a_\delta \in \mathbb{Q} \right\}.$$

Since $\Delta^\circ \mid W_c$ is contained in F, W_c is a simple F-module. Viewing W_c as a vector space over D, the double centralizer theorem promises that $F = \mathrm{Hom}_D(W_c, W_c)$. From this, we see that

(5) $\dim_{\mathbb{Q}} F = (\dim_{\mathbb{Q}} D)(\dim_D(\mathrm{Hom}_D(W_c, W_c))) = (\dim_{\mathbb{Q}} W_c)^2/(\dim_{\mathbb{Q}} D)$.

On the other hand, if we denote by $\rho\colon \Delta^\circ \to GL_{\mathbb{Q}}(W_c)$ the $\mathbb{Q}$-representation of Δ° that corresponds to the Δ°-module structure of W_c, then ρ extends to a $\mathbb{C}$-representation $\rho^{\mathbb{C}}\colon \mathbb{C}[\Delta^\circ] \to \mathrm{Hom}_{\mathbb{C}}(W_c \otimes_{\mathbb{Q}} \mathbb{C}, W_c \otimes_{\mathbb{Q}} \mathbb{C})$ of the group algebra $\mathbb{C}[\Delta^\circ]$ of Δ° over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, by (2) and (3), Wedderburn's theorem insures that

$$\begin{aligned}\dim_{\mathbb{Q}} F = \dim_{\mathbb{C}}(\rho^{\mathbb{C}}(\mathbb{C}[\Delta^\circ])) &= \sum_{i=1}^{r} \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(V_{ik-k+1}, V_{ik-k+1})) \\ &= r(\dim_{\mathbb{C}} V_1)^2.\end{aligned}$$

Together with (4), we have

(6) $k(\dim_{\mathbb{Q}} F) = (\dim_{\mathbb{Q}} W_c)(\dim_{\mathbb{C}} V_1)$.

In the case when $\Delta \mid T_c$ is abelian, we have the following result.

LEMMA 10. *Using the same notation as above, if $\Delta \mid T_c$ is abelian, then $D = F$ is a field, $\dim_L U_1 = 1$, and $[L : \mathbb{Q}] = \dim_{\mathbb{Q}} D = \dim_{\mathbb{Q}} W_c$.*

PROOF. Since $\Delta \mid T_c$ is abelian, F is commutative. Together with the fact that W_c is a simple F-module, we see that F is a field ([9, Proposition 1.7, p. 418]). Being a finite dimensional extension field over $\mathbb{Q}$, F may be viewed as an algebraic number field. Let $\{A_1, \cdots, A_p\}$ be an integral basis for F, x a non-zero element of W_c, and $x_i = A_i(x)$ for every $i \in \{1, \cdots, p\}$.

We claim that $\{x_1, \cdots, x_p\}$ is a $\mathbb{Q}$-basis for W_c. To this end, let $a_1, \cdots, a_p \in \mathbb{Q}$ such that $\sum_{i=1}^p a_i x_i = 0$; that is, $(\sum_{i=1}^p a_i A_i)(x) = 0$. Suppose that $\sum_{i=1}^p a_i A_i$ is not zero. Being an element of the field F, $\sum_{i=1}^p a_i A_i$ must be a linear automorphism of W_c. It follows that $x = 0$, which is a contradiction. Therefore, $\sum_{i=1}^p a_i A_i = 0$. The fact that $\{A_1, \cdots, A_p\}$ is a basis for F therefore forces each of $a_1, \cdots, a_p$ to be zero. This proves that $\{x_1, \cdots, x_p\}$ is $\mathbb{Q}$-linearly independent. Next, let S be the $\mathbb{Q}$-span of $\{x_1, \cdots, x_p\}$. If $A \in F$ and $a_1, \cdots, a_p \in \mathbb{Q}$, then $A(\sum_{i=1}^p a_i x_i) = (\sum_{i=1}^p a_i A A_i)(x)$.

Since each AA_i can be written as a linear combination of $A_1, \cdots, A_p$ over $\mathbb{Q}$, one concludes readily that $A(\sum_{i=1}^p a_i x_i)$ is again an element of S. This proves that S is F-invariant; a fortiori, $S = W_c$. This proves our claim that $\{x_1, \cdots, x_p\}$ is a $\mathbb{Q}$-basis for W_c; and hence, $\dim_{\mathbb{Q}} F = \dim_{\mathbb{Q}} W_c = p$. Together with (5), we see that $\dim_{\mathbb{Q}} D = \dim_{\mathbb{Q}} F$. Being an abelian group, $\Delta^\circ \mid W_c$ is clearly contained in D. It follows that F is a $\mathbb{Q}$-vector subspace of D of the same dimension; and hence, $F = D$. On the other hand, the fact that $\Delta \mid T_c$ is abelian forces $\dim_{\mathbb{C}} V_1$ to be 1 ([10, (2.6)]). Together with (4) and (6), we see that $r = p$ and $\dim_L U_1 = 1$. This completes the proof.

LEMMA 11. *Using the same notation as above, if $\Delta \mid T_c$ is abelian, then $\Delta \mid T_c$ is cyclic, L is the cyclotomic extension of $\mathbb{Q}$ of order n, and $[L : \mathbb{Q}] = \varphi(n)$, where n is the order of $\Delta \mid T_c$ and φ is the Euler function.*

PROOF. Let u be a fixed non-zero element of U_1. Let f be an element of D. Clearly, f^L is a Δ°-module endomorphism of $W_c \otimes_{\mathbb{Q}} L$. In view of (2) and Lemma 10, $f^L \mid U_i$ is a Δ°-module endomorphism of U_i, for every $i \in \{1, \cdots, p\}$, where $p = \dim_{\mathbb{Q}} W_c$. Since $\dim_L U_1 = 1$ (by Lemma 10), there is a unique element $\Omega(f)$ of L such that $f^L(u) = \Omega(f)u$. Clearly, Ω is a field homomorphism of D into L. We are going to show that Ω is, in fact, a field isomorphism of D onto L. To this end, we first notice that

(7) $\{S_{\sigma_1}(u), \cdots, S_{\sigma_p}(u)\}$ is an L-basis for $W_c \otimes_{\mathbb{Q}} L$.

Also, by (1), we see that

(8) $f^L(S_{\sigma_i}(u)) = \sigma_i(\Omega(f))S_{\sigma_i}(u)$, for every element f of D and $i \in \{1, \cdots, p\}$.

Now, if g and h are elements of D such that $\Omega(g) = \Omega(h)$, then (7) and (8) imply that $g^L = h^L$; a fortiori, $g = h$. This proves that Ω is one-to-one. To see that Ω is onto, let l be an element of L. Define γ to be the L-linear endomorphism of $W_c \otimes_{\mathbb{Q}} L$ that is determined by $\gamma(S_{\sigma_i}(u)) = \sigma_i(l)S_{\sigma_i}(u)$ for every $i \in \{1, \cdots, p\}$. Let μ be any element of $\Delta^\circ \mid W_c$. Since $\Delta \mid T_c$ is abelian, μ is an element of D. If $l_1, \cdots, l_p$ are elements of L, by (8), we have that

$$
\begin{aligned}
(\gamma \circ \mu^L)\left(\sum_{i=1}^p l_i S_{\sigma_i}(u))\right) &= \gamma\left(\sum_{i=1}^p l_i \sigma_i(\Omega(\mu))S_{\sigma_i}(u)\right) \\
&= \sum_{i=1}^p l_i \sigma_i(l)\sigma_i(\Omega(\mu))S_{\sigma_i}(u) \\
&= \sum_{i=1}^p l_i \sigma_i(l)\mu^L(S_{\sigma_i}(u)) \\
&= \mu^L\left(\sum_{i=1}^p l_i \sigma_i(l)S_{\sigma_i}(u)\right)
\end{aligned}
$$

$$= (\mu^L \circ \gamma)\left(\sum_{i=1}^{p} l_i S_{\sigma_i}(u)\right).$$

This proves that γ is a Δ°-module endomorphism of $W_c \otimes_{\mathbf{Q}} L$. Next, we let $l_{\sigma_1}, \cdots, l_{\sigma_p} \in L$ be such that $\sum_{i=1}^{p} l_{\sigma_i} S_{\sigma_i}(u)$ is an element of W_c, and let j be a fixed element of $\{1, \cdots, p\}$. Then

$$\sum_{i=1}^{p} l_{\sigma_i} S_{\sigma_i}(u) = S_{\sigma_j}\left(\sum_{i=1}^{p} l_{\sigma_i} S_{\sigma_i}(u)\right) = \sum_{i=1}^{p} \sigma_j(l_{\sigma_i}) S_{\sigma_j \circ \sigma_i}(u);$$

and hence, $\sigma_j(l_{\sigma_i}) = l_{\sigma_j \circ \sigma_i}$ for every $i \in \{1, \cdots, p\}$. It follows that

$$\begin{aligned} S_{\sigma_j}\left(\gamma\left(\sum_{i=1}^{p} l_{\sigma_i} S_{\sigma_i}(u)\right)\right) &= S_{\sigma_j}\left(\sum_{i=1}^{p} l_{\sigma_i} \sigma_i(l) S_{\sigma_i}(u)\right) \\ &= \sum_{i=1}^{p} \sigma_j(l_{\sigma_i})(\sigma_j \circ \sigma_i)(l) S_{\sigma_j \circ \sigma_i}(u) \\ &= \sum_{i=1}^{p} l_{\sigma_j \circ \sigma_i}(\sigma_j \circ \sigma_i)(l) S_{\sigma_j \circ \sigma_i}(u) \\ &= \sum_{i=1}^{p} l_{\sigma_i} \sigma_i(l) S_{\sigma_i}(u) = \gamma\left(\sum_{i=1}^{p} l_{\sigma_i} S_{\sigma_i}(u)\right). \end{aligned}$$

Since j is an arbitrary element of $\{1, \cdots, p\}$, the calculation above shows that $\gamma(W_c)$ is contained in W_c. From this, one concludes readily that $\gamma \mid W_c$ is an element of D such that $\Omega(\gamma \mid W_c) = l$. This proves that Ω is onto; and hence, Ω is a field isomorphism of D onto L. Moreover, since $\Omega(q(\mathrm{id}_{W_c})) = q$ for every rational number q, we see that

(9) the restriction of the map Ω to the ring of algebraic integers in D is a ring isomorphism of the ring of algebraic integers in D onto the ring of algebraic integers in L.

Being a finite abelian group, $\Delta^\circ \mid W_c$ can be written as a direct product of cyclic subgroups $G_1, \cdots, G_d$ each of whose orders is a power of a prime. For a fixed prime q, suppose that there are two distinct groups among $G_1, \cdots, G_d$, say, G_1 and G_2, such that $|G_1| = q^{n_1}$, $|G_2| = q^{n_2}$, and $n_1 \geq n_2$. Let μ_1 be a generator of G_1 and μ_2 be a generator of G_2. Since Ω is one-to-one, $\Omega(\mu_1)$ is clearly a primitive q^{n_1}th root of unity in L. On the other hand, $(\mu_2)^{q^{n_1}} = \mathrm{id}_{W_c}$ implies that $\Omega(\mu_2)$ is a q^{n_1}th root of unity in L; and hence, $\Omega(\mu_2) = \Omega(\mu_1)^e$ for some positive integer e. Since Ω is one-to-one, μ_2 and μ_1^e must be the same—a contradiction. This proves that for any fixed prime q, there is at most one group among $G_1, \cdots, G_d$ that has its order a power of q. Clearly, this implies that $\Delta^\circ \mid W_c$ is cyclic; and hence, $\Delta \mid T_c$ is cyclic.

Let n be the order of $\Delta \mid T_c$, π be a generator of $\Delta^\circ \mid W_c$, and $\theta = \Omega(\pi)$. Clearly, θ is a primitive nth root of unity; and hence, the cyclotomic extension $\mathbb{Q}(\theta)$ of $\mathbb{Q}$

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of order n is contained in L. On the other hand, since $\chi_1(\mu)^n = 1$ for every μ in $\Delta^\circ \mid W_c$ ([10, (2.15)]), L is contained in $\mathbb{Q}(\theta)$. (For the definition of χ_1 and L, see the paragraph following the proof of Proposition 9.) Consequently, $L = \mathbb{Q}(\theta)$; and hence, $[L : \mathbb{Q}] = \varphi(n)$, where φ is the Euler function. This completes the proof.

Now, we are in position to characterize $\mathrm{Aut}_\Delta T_c$ as the group of units in the ring of algebraic integres in D, provided that $\Delta \mid T_c$ is abelian.

THEOREM 12. *Using the same notation as above, if $\Delta \mid T_c$ is abelian, then the map Ψ sending every element α of $\mathrm{Aut}_\Delta T_c$ onto $\alpha^\circ \mid W_c$ is a group isomorphism of $\mathrm{Aut}_\Delta T_c$ onto the group $\mathcal{U}$ of units in the ring of algebraic integers in D.*

PROOF. It is clear that the map Ψ in question is a group monomorphism of $\mathrm{Aut}_\Delta T_c$ into D. Let α be a Δ-automorphism of T_c, $\beta = \Psi(\alpha)$, $\exp_c : \mathscr{L}(T_c) \to T_c$ be the exponential map, and $c(x)$ be the characteristic polynomial of β. Since $\ker(\exp_c)$ contains an $\mathbb{R}$-basis B_c for $\mathscr{L}(T_c)$ such that the $\mathbb{Z}$-span of B_c is $\ker(\exp_c)$, and $\ker(\exp_c)$ is β-invariant, we see that $c(x)$ is in $\mathbb{Z}[x]$. Being a root of the monic polynomial $c(x)$, β is an algebraic integer in D. Applying the same argument to α^{-1}, we see that $\beta^{-1} = \Psi(\alpha^{-1})$ is also an algebraic integer in D. This proves that $\beta \in \mathcal{U}$. So, in order to prove the theorem, it remains to show that $\Psi(\mathrm{Aut}_\Delta T_c) = \mathcal{U}$. Let π be a generator of $\Delta^\circ \mid W_c$ and $\theta = \Omega(\pi)$, where $\Omega : D \to L$ is the field isomorphism in the proof of Lemma 11. Let n be the order of $\Delta \mid T_c$. Since $L = \mathbb{Q}(\theta)$ and θ is a primitive nth root of unity by the proof of Lemma 11, the ring of algebraic integers in L is precisely $\mathbb{Z}[\theta]$ ([3, (21.13)]). On the other hand, since $\Omega(q(\mathrm{id}_{W_c})) = q$ for every rational number q, one concludes readily from (9) that the ring of algebraic integers in D is precisely $\mathbb{Z}[\pi]$. Since $\ker(\exp_c)$ is π-invariant, relative to B_c, the matrix associated with π is an integral matrix. It follows that, relative to B_c, the matrix associated with every algebraic integer in D is an integral matrix. Now, let λ be an element in $\mathcal{U}$. Since λ and λ^{-1} are algebraic integers in D, relative to B_c, the matrices associated with λ and λ^{-1} are integral matrices. It follows that $\lambda(\ker(\exp_c)) = \ker(\exp_c)$. From this, one concludes readily that λ induces a Δ°-automorphism α of T_c such that $\Psi(\alpha) = \lambda$. This shows that Ψ is onto, and the proof of the theorem is complete.

From the last theorem, we may completely determine when $\mathrm{Aut}_\Delta T_c$ has finitely many elements and when $\mathrm{Aut}_\Delta T_c$ has infinitely many elements, provided that $\Delta \mid T_c$ is abelian.

COROLLARY 13. *We use the same notation as above and suppose that $\Delta \mid T_c$ is abelian. If the order of $\Delta \mid T_c$ is 1, 2, 3, 4, or 6, then $\mathrm{Aut}_\Delta T_c$ has 2, 2, 6, 4, or 6 elements, respectively. Moreover, these are the only cases in each of which $\mathrm{Aut}_\Delta T_c$ has finitely many elements.*

PROOF. Let n be the order of $\Delta \mid T_c$, and let ϵ_i be a fixed primitive ith root of unity for every positive integer i. By Dirichlet's units theorem ([15, p. 148]), the group of units of an algebraic number field K is finite if and only if $K = \mathbb{Q}$ or $K = \mathbb{Q}(\sqrt{-q})$ for some positive square free rational number q. In particular, if L has finitely many units, then $[L : \mathbb{Q}]$ is either 1 or 2; and hence, $n = 1, 2, 3, 4$, or 6, and $L = \mathbb{Q}$, $\mathbb{Q}(\epsilon_3)$, $\mathbb{Q}(\epsilon_4)$, or $\mathbb{Q}(\epsilon_6)$ (by Lemma 11). Since $\mathbb{Q}$ has two units, $\mathbb{Q}(\epsilon_3) = \mathbb{Q}(\epsilon_6)$ has 6 units, and $\mathbb{Q}(\epsilon_4)$ has 4 units ([15, p. 130]), the corollary follows immediately from (9) and Theorem 12. This completes the proof.

The following lemma will be used several times in reducing the proof of the main theorem to the proof of a simpler situation. Although it is rather simple, we include it here for completeness.

LEMMA 14. *Let p be an open continuous homomorphism of a locally compact group G onto a (Hausdorff) topological group H. If both* ker p *and H have finitely many components, then so does G.*

PROOF. Put $K = \ker p$, and choose $k_1, \cdots, k_t$ in K such that $K = \bigcup_{i=1}^{t}(K_\circ k_i)$. Then, $G_\circ K = \bigcup_{i=1}^{t}(G_\circ k_i)$. It follows that $G_\circ K$ is closed in G and $G_\circ K$ has finitely many components. So, in order to prove that G has finitely many components, it suffices to show that $G_\circ K$ is a subgroup of G of finite index. But this follows from the facts that $p(G_\circ K) = H_\circ$ and $H/H_\circ$ is finite. This completes the proof.

4. The main theorem

Suppose that G is a Lie group such that G has a faithful representation and G has finitely many components. Then, G is a semidirect product $E_1 \cdot M$, where E_1 is a simply connected solvable normal analytic subgroup of G and M is a maximal reductive subgroup of G; moreover, there is a finite subgroup D of M such that $M = M_\circ D$ ([12, p. 42]), $G_\circ = E_1 M_\circ$, and $M_\circ$ is a maximal reductive analytic subgroup of $G_\circ$ ([7, Chapter XVIII]).

For our purpose, it will be convenient to choose an appropriate E_1. Such a choice is made in the following lemma.

LEMMA 15. *Using the same notation as above, there is a subgroup E of G so that E enjoys all properties that are satisfied by E_1, and $E\,T$ is a characteristic subgroup of G, where T is the maximal central torus in $G_\circ$.*

PROOF. Let u be the canonical map of $G_\circ$ onto $G_\circ/T$. Clearly, $u(G_\circ)$ is an analytic group whose nilradical (that is, the maximal nilpotent normal analytic subgroup) is

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simply connected. By a result of Hochschild, [8], the holomorph $u(G_\circ)\rtimes\operatorname{Aut} u(G_\circ)$ of $u(G_\circ)$ has a faithful representation. And hence, $\operatorname{Aut} u(G_\circ) = P\,(\operatorname{Aut} u(G_\circ))_\circ$ for some finite subgroup P of $\operatorname{Aut} u(G_\circ)$ ([13, Theorem 1]). Denote by R the radical (that is, the maximal solvable normal analytic subgroup) of $G_\circ$. Consider the canonical map v of $u(R)$ onto $u(R)/u([G_\circ, R])$. Being an abelian analytic group, $v(u(R))$ is a direct product of the maximal torus T' in $v(u(R))$ and a vector group V_1. On the other hand, since P induces a finite group P' of linear automorphisms of $\mathscr{L}(v(u(R)))$, P' is completely reducible. It follows that we may choose V_1 such that V_1 is invariant under those automorphisms of $v(u(R))$ that are induced from P. Clearly, $E_2 = v^{-1}(V_1)$ is therefore P-invariant. Next, consider the analytic group $u(R)\rtimes(\operatorname{Aut} u(G_\circ))_\circ$, where every element of $(\operatorname{Aut} u(G_\circ))_\circ$ acts on $u(R)$ by restriction. For simplicity, let us denote both the identity element of $u(R)$ and the identity element of $(\operatorname{Aut} u(G_\circ))_\circ$ by 1. Let $x \in E_2$ and $\lambda \in (\operatorname{Aut} u(G_\circ))_\circ$. Then, $(x, 1)(1, \lambda)(x, 1)^{-1}(1, \lambda)^{-1} = (x\,\lambda(x)^{-1}, 1)$ is contained in the nilradical of $u(R)\rtimes(\operatorname{Aut} u(G_\circ))_\circ$. It follows that $x\,\lambda(x)^{-1}$ is contained in the nilradical of $u(R)$. Since the nilradical of $u(R)$ is simply connected, $x\,\lambda(x)^{-1}$ must be in E_2. This proves that E_2 is $(\operatorname{Aut} u(G_\circ))_\circ$-invariant. As a result, E_2 is a characteristic subgroup of $u(G_\circ)$. Clearly, $E_3 = u^{-1}(E_2)$ is therefore $I_G(D)$-invariant. Let w be the canonical map of E_3 onto $E_3/[G_\circ, R]$. Being an abelian analytic group, $w(E_3)$ is a direct product of the maximal torus T'' in $w(E_3)$ and a vector group V_2. As we saw before, we may choose V_2 so that V_2 is invariant under those automorphisms of $w(E_3)$ that are induced from $I_G(D)$. Let $E = w^{-1}(V_2)$. We see that $E_3 = E\,T$, and it is straightforward to check that E satisfies all the requirements. This completes the proof.

Now, let $S = [M_\circ, M_\circ]$ and $K = Z(M_\circ)_\circ$, where $Z(M_\circ)$ is the center of $M_\circ$. Then, S is semisimple, K is a maximal torus in the radical R of $G_\circ$, and $M_\circ = K\,S$ ([7, Chapter XVIII]). The lemma below is a special case of our main theorem.

LEMMA 16. *Using the same notation as above,* $\operatorname{Aut}(SD)$ *has finitely many components, and* $(\operatorname{Aut}(SD))_\circ = I_{SD}(S)$.

PROOF. Suppose that ϕ is the map of $\operatorname{Aut} S$ into $\operatorname{Aut}\mathscr{L}(S)$ that sends every element of $\operatorname{Aut} S$ onto its differential. By a result of H. Matsumoto ([14, Theorem 2.2]), $\operatorname{Aut}\mathscr{L}(S) = J\,(\operatorname{Aut}\mathscr{L}(S))_\circ$ for some finite subgroup J of $\operatorname{Aut}\mathscr{L}(S)$. Since $\phi(I_S(S)) = (\operatorname{Aut}\mathscr{L}(S))_\circ$, we see that $\operatorname{Aut} S = I_S(S)\,X$, where X is the finite group $\phi^{-1}(J)$. Let $\rho\colon \operatorname{Aut}(SD) \to \operatorname{Aut} S$ be the map defined by $\rho(\theta) = \theta \mid S$. If $\theta \in \ker\rho$, $d \in D$, and $s \in S$, then $dsd^{-1}\theta(d) = \theta(dsd^{-1})\theta(d) = \theta(d)s$; that is, $d^{-1}\theta(d)$ is in the centralizer Y of S in SD. Since Y is a characteristic subgroup of SD, the map $\zeta\colon\ker\rho \to \operatorname{Aut}(DY)$ defined by $\zeta(\theta) = \theta \mid (DY)$ is well-defined. Next, let $\Omega\colon SD \to (SD)/Z(S)$ be the canonical map, where $Z(S)$ is the center of S. Let e be

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the identity element of SD. For every $d \in D$, we define $\sigma(d)$ as follows:

$$\sigma(d) = \begin{cases} \Omega(sd), & \text{if there is } s \in S \text{ such that } sd \in Y; \\ \Omega(e), & \text{otherwise.} \end{cases}$$

Clearly, σ is a well-defined map of D onto $\Omega(Y)$; and hence, $\Omega(Y)$ is finite. On the other hand, since S has a faithful representation, $Z(S)$ is finite. As a result, Y is finite; and hence, $\operatorname{Aut}(DY)$ is finite. Because ζ is clearly one-to-one, ker ρ is finite; and hence, $\rho^{-1}(X)$ is finite. Next, we claim that $I_{SD}(S)$ is closed in $\operatorname{Aut}(SD)$. To this end, suppose that $s_n \in S$ and $\nu \in \operatorname{Aut}(SD)$ are such that $I_{SD}(s_n) \to \nu$ in $\operatorname{Aut}(SD)$. Since ρ is clearly continuous, $I_S(s_n) \to \rho(\nu)$ in $\operatorname{Aut} S$. From the fact that $I_S(S)$ is closed in $\operatorname{Aut} S$, $\rho(\nu) = I_S(s)$ for some $s \in S$. It follows that $s_n z_n \to s$ for some $z_n \in Z(S)$. Since $Z(S)$ is finite, without loss of generality, we may assume that $s_n z \to s$ for some $z \in Z(S)$. Consequently, $\nu = I_{SD}(sz^{-1})$. This proves that $I_{SD}(S)$ is closed in $\operatorname{Aut}(SD)$ as we claimed. Finally, since $\operatorname{Aut}(SD) = I_{SD}(S)\,\rho^{-1}(X)$, the lemma follows immediately from Baire's theorem. This completes the proof.

Recall that T is the maximal central torus in $G_\circ$. Let $\Delta = I_G(G) \mid T$. Clearly, Δ is a finite subgroup of $\operatorname{Aut} T$. In fact, $\Delta = I_G(D) \mid T$. Since D normalizes K, $I_G(D) \mid K$ is also a finite subgroup of $\operatorname{Aut} K$. By an abuse of notation, we still denote $I_G(D) \mid K$ by Δ. By Proposition 3, both T and K have Δ-decompositions. Now, we are in position to prove our main theorem.

THEOREM 17. *Using the same notation as above, in order that* Aut G *has finitely many components, it is necessary and sufficient that* G *satisfies the following two conditions:*

(a) *T is Δ-rigid; and*
(b) *there is no Δ-simple subtorus of T that is almost Δ-isomorphic to any Δ-simple subtorus of K that is not in T.*

PROOF. First we prove the sufficiency part of the theorem. Suppose that G satisfies Conditions (a) and (b). It follows from the conjugacy of maximal reductive subgroups of G that $\operatorname{Aut} G = I_G(G_\circ)A_1$, where $A_1 = \{f \in \operatorname{Aut} G : f(M) = M\}$. So, in order to prove that $\operatorname{Aut} G$ has finitely many components, it suffices to show that A_1 has finitely many components.

Let f be an element of A_1. Since ET is a characteristic subgroup of G (by Lemma 15), for every $x \in E$, we may write $f(x) = f_E(x) f_T(x)$ with $f_E(x) \in E$ and $f_T(x) \in T$. One sees readily that $f_E \in \operatorname{Aut} E$, $f_T \in \operatorname{Hom}(E, T)$, and

(10) $$f_E \circ (I_G(y) \mid E) = (I_G(f(y)) \mid E) \circ f_E, \quad \text{and}$$

(11) $$f_T \circ (I_G(y) \mid E) = (I_G(f(y)) \mid T) \circ f_T,$$

for every $y \in M$. Let $A_2 = \{f \in \operatorname{Aut} G : f(E) = E,\ f(M) = M\}$. Define $\Psi: A_1 \to A_2$ by $\Psi(f)(xy) = f_E(x)f(y)$ for every $x \in E$ and $y \in M$. Clearly, Ψ is a continuous homomorphism of A_1 onto A_2. Next, let $B_1 = \{g \in \operatorname{Hom}(E, T) : g \circ (I_G(y) \mid E) = (I_G(y) \mid T) \circ g$ for every $y \in M\}$, $V = E/[E, E]$, and $\exp: \mathscr{L}(T) \to T$ be the exponential map. For every $g \in B_1$, we denote by g_V the element in $\operatorname{Hom}(V, T)$ that is induced by g, and $\tilde{g}$ the unique element in $\operatorname{Hom}(V, \mathscr{L}(T))$ such that $\exp \circ \tilde{g} = g_V$. For every $y \in M$, we denote by $p(y)$ the automorphism of V and $q(y)$ the automorphism of $\mathscr{L}(T)$ induced by $I_G(y)$. Let $g \in B_1$ and $y \in M$. Clearly, $q(y) \circ \tilde{g} \circ p(y)^{-1} \in \operatorname{Hom}(V, \mathscr{L}(T))$ and $\exp \circ q(y) \circ \tilde{g} \circ p(y)^{-1} = g_V$. The uniqueness of $\tilde{g}$ therefore promises that $q(y) \circ \tilde{g} \circ p(y)^{-1} = \tilde{g}$. Since $\operatorname{Hom}(E, T)$ and $\operatorname{Hom}(V, T)$ are topologically isomorphic, if we let $B_2 = \{h \in \operatorname{Hom}(V, \mathscr{L}(T)) : q(y) \circ h \circ p(y)^{-1} = h$ for every $y \in M\}$, then the foregoing shows that the correspondence $g \mapsto \tilde{g}$ gives a homeomorphism of B_1 onto B_2. Since B_2 is clearly connected, so is B_1. On the other hand, in view of (11), it is straightforward to check that the correspondence $f \mapsto f_T$ gives a homeomorphism of the kernel of Ψ onto B_1. Consequently, the kernel of Ψ is connected. By Lemma 14, in order to prove that A_1 has finitely many components, it suffices to show that A_2 has finitely many components.

Since every automorphism of G induces an automorphism of $G/G_\circ$, and $\operatorname{Aut}(G/G_\circ)$ is finite, in order to prove that A_2 has finitely many components, it suffices to show that $A_3 = \{f \in A_2 : x^{-1}f(x) \in G_\circ$ for every $x \in G\}$ has finitely many components.

Let $f \in A_3$ and $d \in D$. Since $f(M) = M$, we see that $f(d) = dy$ for some $y \in M_\circ$. If $k \in K$, then $dyf(k) = f(dk) = f(dkd^{-1})dy$. Since $K = Z(M_\circ)_\circ$, this implies that $df(k)d^{-1} = f(dkd^{-1})$ for every $d \in D$ and $k \in K$; that is, $f \mid K$ ($\in \operatorname{Aut} K$) commutes with δ for every $\delta \in \Delta$. Next, let W be the $\mathbb{Q}$-span of the kernel of the exponential map $\exp_K: \mathscr{L}(K) \to K$. As we saw in Section 1, K has a Δ-decompositon $T_1T_2 \cdots T_mC_1C_2 \cdots C_n$ of Δ-simple subtori of K such that $T = T_1T_2 \cdots T_m$, and W is the direct sum $(\mathscr{L}(T_1) \cap W) \oplus \cdots \oplus (\mathscr{L}(T_m) \cap W) \oplus (\mathscr{L}(C_1) \cap W) \oplus \cdots \oplus (\mathscr{L}(C_n) \cap W)$ of simple Δ°-submodules, where W is the $\mathbb{Q}$-span of $\ker(\exp_K)$. Let $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, n\}$, and $\pi_i: W \to \mathscr{L}(T_i) \cap W$ be the projection. Since $f^\circ \mid W$ is a Δ°-module automorphism of W, and since both $\mathscr{L}(T_i) \cap W$ and $\mathscr{L}(C_j) \cap W$ are simple Δ°-modules, $f_i = \pi_i \circ (f^\circ \mid (\mathscr{L}(C_j) \cap W))$ must be either trivial or a Δ°-module isomorphism of $\mathscr{L}(C_j) \cap W$ onto $\mathscr{L}(T_i) \cap W$. It follows from Proposition 5 and Condition (b) that f_i must be trivial. From this, one concludes readily that $C = C_1C_2 \cdots C_n$ is A_3-invariant. On the other hand, since E is a simply connected analytic group, $\operatorname{Aut} E$ is an algebraic group. Being a compact subgroup of $\operatorname{Aut} E$, $I_G(C) \mid E$ is also an algebraic group ([2, Proposition 2, p. 230]). It follows that the normalizer $\mathscr{N}$ of $I_G(C) \mid E$ in $\operatorname{Aut} E$ is an algebraic group; and hence, $\mathscr{N}$ has finitely many components. Together with the fact that $I_G(C) \mid E$ is a torus, $\mathscr{N}_\circ$ must be trivial ([11, Theorem 1′]). As a result, $\mathscr{N}$ is finite. Since C is A_3-invariant, it is easy to see that the map

$\Lambda: A_3 \to \operatorname{Aut}(I_G(C) \mid E)$ defined by $\Lambda(f)(I_G(c) \mid E) = (f \mid E) \circ (I_G(c) \mid E) \circ (f \mid E)^{-1}$ for every $f \in A_3$ and $c \in C$ is a continuous homomorphism, and $\Lambda(A_3)$ is contained in the finite group $\mathcal{N}$. So, in order to show that A_3 has finitely many components, it suffices to prove that the kernel of Λ has finitely many components; that is, to prove that $A_4 = \{f \in A_3 : (f \mid E) \circ (I_G(c) \mid E) = (I_G(c) \mid E) \circ (f \mid E)$ for every $c \in C\}$ has finitely many components.

Let $f \in A_4$. If $c \in C$ and $x \in E$, then (10) implies that $f(c)f(x)f(c)^{-1} = cf(x)c^{-1}$; that is, $c^{-1}f(c)$ is contained in the centralizer C_E of E in C. Clearly, $(C_E)_\circ$ is contained in T. The fact that $C \cap T$ is finite therefore forces C_E to be discrete. Together with the fact that the map of C into C_E that sends every element c of C onto $c^{-1}f(c)$ is continuous, one concludes readily that f is the identity on C. This proves that every element of A_4 is the identity on C. Next, we consider the action of A_4 on T. To this end, let $f \in A_4$. Since $f \mid K$ commutes with δ for every δ in Δ, $f^\circ \mid (\mathcal{L}(T) \cap W)$ is a Δ°-module automorphism of $\mathcal{L}(T) \cap W$. Let $i \in \{1, \cdots, m\}$. Since $\mathcal{L}(T_i) \cap W$ is a simple Δ°-submodule of $\mathcal{L}(T) \cap W$, by the uniqueness of the Δ-decomposition of T (by the remark following Definition 8), we see that $f^\circ \mid (\mathcal{L}(T) \cap W)$ must map $\mathcal{L}(T_i) \cap W$ isomorphically onto $\mathcal{L}(T_j) \cap W$ for some $j \in \{1, \cdots, m\}$. Since $f^\circ(\ker(\exp)) = \ker(\exp)$, one concludes readily that $f^\circ(\ker(\exp_i)) = \ker(\exp_j)$, where $\exp: \mathcal{L}(T) \to T$, $\exp_i: \mathcal{L}(T_i) \to T_i$, and $\exp_j: \mathcal{L}(T_j) \to T_j$ are exponential maps. It follows that $f^\circ \mid (\mathcal{L}(T_i) \cap W)$ induces a Δ- isomorphism of T_i onto T_j. By the rigidity of T, j must be equal to i and there are only finitely many choices of $f^\circ \mid (\mathcal{L}(T_i) \cap W)$. From this, we may conclude that the image of the map of A_4 into $\operatorname{Aut} T$ that sends every element f of A_4 onto $f \mid T$ is finite. Thus, in order to prove that A_4 has finitely many components, it suffices to show that $A_5 = \{f \in A_4 : f$ is the identity on $K\}$ has finitely many components.

Let $\pi: M \to M/S$ be the canonical map, and $L = \{\xi \in \operatorname{Aut} \pi(M) : \pi(y)^{-1}\xi(\pi(y)) \in \pi(M_\circ)$ for every $y \in M\}$. Denote by $\eta: A_5 \to L$ the map that sends every element f of A_5 onto the automorphism of $\pi(M)$ that is induced by $f \mid M$. Since $\pi(M)$ is compact and $\pi(M_\circ)$ is a torus, $(\operatorname{Aut} \pi(M))_\circ = I_{\pi(M)}(\pi(M_\circ))$ ([11, Theorem 1′]). Because $I_G(M_\circ)$ is clearly contained in A_5, we see that $(\operatorname{Aut} \pi(M))_\circ$ is contained in $\eta(A_5)$; and hence, $\eta(A_5)$ is closed in L. On the other hand, if we let $D = \{d_1, \cdots, d_l\}$ and let U be the Lie group $\pi(M_\circ)^l \rtimes \operatorname{Aut} \pi(M_\circ)$ with multiplication

$$\begin{aligned}(\pi(y_1), \ldots, \pi(y_l), \xi)&(\pi(y_1'), \ldots, \pi(y_l'), \xi') \\ &= (\pi(y_1)\xi(\pi(y_1')), \ldots, \pi(y_l)\xi(\pi(y_l')), \xi\xi')\end{aligned}$$

for every $y_i, y_i' \in M_\circ$ and $\xi, \xi' \in \operatorname{Aut} \pi(M_\circ)$, then the map Φ of L into U defined by $\Phi(\xi) = (\pi(d_1^{-1})\xi(\pi(d_1)), \cdots, \pi(d_l^{-1})\xi(\pi(d_l)), \xi \mid \pi(M_\circ))$ is a topological isomorphism of L onto the closed subgroup $\Phi(L)$ of U ([6, Theorem 2]). Consequently, $\Phi(\eta(A_5))$ is closed in the compact group $\pi(M_\circ)^l \times \{1\}$, where 1 denotes the identity element of $\operatorname{Aut} \pi(M_\circ)$. It follows that $\eta(A_5)$ has finitely many components.

By Lemma 14, in order to prove that A_5 has finitely many components, it suffices to show that the kernel of η has finitely many components; that is, to show that $A_6 = \{f \in A_5 : f(SD) = SD\}$ has finitely many components.

Since $I_G(S)$ is clearly contained in A_6, we see that $I_{SD}(S)$ is contained in the image of the map of A_6 into $\operatorname{Aut}(SD)$ that sends every element of A_6 onto its restriction to SD. Thus, by Lemma 14 and Lemma 16, in order to prove that A_6 has finitely many components, it suffices to show that $A_7 = \{f \in A_6 : f \text{ is the identity on } M\}$ has finitely many components.

Since A_7 is topologically isomorphic with $A_8 = \{f \in \operatorname{Aut} E : f \circ (I_G(y) \mid E) = (I_G(y) \mid E) \circ f \text{ for every } y \in M\}$ and A_8 is an algebraic group, we see that A_7 has finitely many components. This proves that if G satisfies Conditions (a) and (b), then $\operatorname{Aut} G$ has finitely many components.

Now, we prove the necessity part of the theorem. We first assume that G is not rigid; that is, T has a Δ-decomposition $T_1 T_2 \cdots T_m$ such that either

(i) T_1 and T_2 are almost Δ-isomorphic; or

(ii) there are infinitely many Δ-automorphisms of T_1.

Clearly, in either case, K has a Δ-decomposition of the form $T_1 T_2 \cdots T_m C_1 C_2 \cdots C_n$. As above, we let $C = C_1 C_2 \cdots C_n$. Moreover, we let $H = T_2 T_3 \cdots T_m CSD$ and $F = H \cap T_1$. Clearly, F is finite. Say, F has a elements. Suppose that (i) holds, and f is a continuous homomorphism of T_1 onto T_2 such that f has finite kernel and $f \circ \delta = \delta \circ f$ for every $\delta \in \Delta$. Define $g(t) = f(t)^a$ for every $t \in T_1$. Since f is surjective, g is a non-trivial continuous homomorphism of T_1 into T_2 and F is contained in the kernel of g. Now, define $r(xth) = xtg(t)h$ for every $x \in E, t \in T_1$, and $h \in H$. It is straightforward to check that r is a topological automorphism of G. Again since f is surjective, the map sending every positive integer i onto $r^i \mid T$ is a one-to-one map of the set of positive integers into $\operatorname{Aut} T$. Since $(\operatorname{Aut} G)_\circ \mid T$ is trivial ([11, Theorem 1′]), we may conclude that $\operatorname{Aut} G$ has infinitely many components. Next suppose that (ii) holds, and let Γ_1 be the collection of all Δ-automorphisms of T_1. If $\gamma \in \Gamma_1$, then $\gamma(F)$ is a finite subgroup of T_1 of order less than or equal to a. Because there are only finitely many such subgroups of T_1, we see that $\Gamma_1(F) = \cup\{\gamma(F) : \gamma \in \Gamma_1\}$ is finite. Since there are only finitely many functions of F into $\Gamma_1(F)$ and Γ_1 is infinite, there must be in Γ_1 an infinite subset Γ_2 such that the restrictions of all elements in Γ_2 to F are all the same. Fix an element $\gamma_\circ$ in Γ_2. Let $\Gamma = \{\gamma_\circ^{-1} \circ \gamma : \gamma \in \Gamma_2\}$. Then, Γ consists of infinitely many Δ-automorphisms of T_1 which are the identity on F. we denote by $\tilde{\gamma}$ the topological automorphism of G defined by $\tilde{\gamma}(xth) = x\gamma(t)h$ for every $x \in E, t \in T_1$, and $h \in H$. Since $\{\tilde{\gamma} \mid T : \gamma \in \Gamma\}$ consists of infinitely many elements, we may conclude that $\operatorname{Aut} G$ has infinitely many components.

Finally, we assume that K has a Δ-decomposition $T_1 T_2 \cdots T_m C_1 C_2 \cdots C_n$ such that $T = T_1 T_2 \cdots T_m$ and there is a continuous homomorphism f of C_1 onto T_1 with finite kernel, and $f \circ \delta = \delta \circ f$ for every $\delta \in \Delta$. Let $H = T_1 T_2 \cdots T_m C_2 C_3 \cdots C_n SD$ and let $F = H \cap C_1$. Clearly, F is finite. Say, F has a elements. Define $g(c) = f(c)^a$ for every $c \in C_1$. Since f is surjective, g is a non-trivial continuous homomorphism of C_1 into T_1 and F is contained in the kernel of g. Now, define $r(xch) = xg(c)ch$ for every $x \in E$, $c \in C_1$, and $h \in H$. It is straightforward to check that r is a topological automorphism of G. Again since f is surjective, the map sending every positive integer i onto r^i is a one-to-one map of the set of positive integers into $\operatorname{Aut} G$. Let $A(G, T) = \{\alpha \in \operatorname{Aut} G : \alpha \text{ is the identity on } T\}$ and let $B = EC \cap T$. Clearly, $(\operatorname{Aut} G)_\circ$ and $\{r^i : i \in \mathbb{N}\}$ are contained in $A(G, T)$, and B is finite. Let $\mu: R \to R/B$ be the canonical map, where R is the radical of $G_\circ$. Clearly, $\mu(R) = \mu(EC) \times \mu(T)$. Being an abelian analytic group, $\mu(EC)/[\mu(EC), \mu(EC)]$ is a direct product of its maximal torus τ and a vector group U. Now, let $\alpha \in A(G, T)$, and denote by $\tilde{\alpha}$ the automorphism of $\mu(R)$ that is induced by α. If $z \in EC$, then we may write $\tilde{\alpha}(\mu(z)) = \tilde{\alpha}_1(\mu(z))\tilde{\alpha}_4(\mu(z))$, where $\tilde{\alpha}_1(\mu(z)) \in \mu(EC)$ and $\tilde{\alpha}_4(\mu(z)) \in \mu(T)$. Clearly, $\tilde{\alpha}_1 \in \operatorname{Aut} \mu(EC)$ and $\tilde{\alpha}_4 \in \operatorname{Hom}(\mu(EC), T)$. Identifying $\operatorname{Hom}(\mu(EC), T)$ with $\operatorname{Hom}(\tau, T) \times \operatorname{Hom}(U, T)$, we may further decompose $\tilde{\alpha}_4$ into $(\tilde{\alpha}_2, \tilde{\alpha}_3)$ where $\tilde{\alpha}_2 \in \operatorname{Hom}(\tau, T)$ and $\tilde{\alpha}_3 \in \operatorname{Hom}(U, T)$. This shows that every element α of $A(G, T)$ can be decomposed continuously into three maps $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$, where $\tilde{\alpha}_1 \in \operatorname{Aut}(\mu(EC))$, $\tilde{\alpha}_2 \in \operatorname{Hom}(\tau, T)$, and $\tilde{\alpha}_3 \in \operatorname{Hom}(U, T)$. Clearly, the $\tilde{\alpha}_2$-component of every element in $(\operatorname{Aut} G)_\circ$ is trivial. On the other hand, since $[EC, EC]$ is clearly contained in E, we see that τ is not trivial and the $\tilde{\alpha}_2$-components of $\{r^i : i \in \mathbb{N}\}$ are all distinct. From this, we conclude readily that $\operatorname{Aut} G$ has infinitely many components. The proof of the theorem is thereby completed.

Since any two maximal tori in the radical of $G_\circ$ are conjugate, we see that Theorem 17 is independent of the choice of K. If, in addition, G is connected, we have the following version of Theorem 17.

COROLLARY 18. *Suppose that G is an analytic group that has a faithful representation, and that T is the maximal central torus in G. Then, in order that* Aut G *has finitely many components, it is necessary and sufficient that T is trivial, or that the dimension of T is* 1 *and T is exactly the maximal torus in the radical of G.*

If G does not have a faithful representation, even when G is connected, Theorem 17 fails to hold as shown in the following example due to Dani [4].

EXAMPLE. Let H be the three-dimensional Heisenberg group and Z an infinite cyclic subgroup contained in the center of H. Let S be a subgroup of $\operatorname{Aut}(H/Z)$ that is isomorphic to $SL(2, \mathbb{R})$ and $G = (H/Z) \rtimes S$. Then, the center of G^n is an n-dimensional torus and $\operatorname{Aut} G^n$ has finitely many components for all n ([4], p. 451).

As an application of our main result, we give a necessary and sufficient condition for an analytic group G to have $\operatorname{Aut} G$ being almost algebraic (provided that G has a faithful representation). (cf. [4, 16, 17].) By an algebraic group, we mean an algebraic subgroup of $GL(V)$ for some vector space V over $\mathbb{R}$. A subgroup G of $GL(V)$ is said to be *almost algebraic* if there is an algebraic subgroup of $GL(V)$ that contains G as a subgroup of finite index. If G is a subgroup of $GL(V)$, we denote by G^* the *algebraic group hull* of G; that is, the smallest algebraic subgroup of $GL(V)$ that contains G. We will need, in the sequel, the following results on almost algebraic groups.

(12) If G is an almost algebraic subgroup of $GL(V)$, then $G_\circ = (G^*)_\circ$ and G has finitely many components. ([5, p. 266].)

(13) A subgroup H of $GL(V)$ is almost algebraic if and only if H is closed, H has finitely many components, and the Lie algebra of H is algebraic. ([5, (2.2)].)

(14) If G is an analytic group, then $(\operatorname{Aut} G)_\circ$, identified as a subgroup of $GL(\mathscr{L}(G))$, is almost algebraic. ([16, 17].)

Now, we are ready to state our result.

THEOREM 19. *Suppose that G is an analytic group that has a faithful representation, and T is the maximal central torus in G. Then,* Aut G *is almost algebraic if and only if T is trivial, or the dimension of T is* 1 *and T is exactly the maximal torus in the radical of G.*

PROOF. In view of Corollary 18 and (12) above, it remains to show that $\operatorname{Aut} G$ is almost algebraic if $\operatorname{Aut} G$ has finitely many components. To this end, suppose $H = \operatorname{Aut} G$ has finitely many components. Since H normalizes $H_\circ$, H normalizes $(H_\circ)^*$; and hence, $H(H_\circ)^*$ is a subgroup of $GL(\mathscr{L}(G))$ after identification. By (14), $H_\circ$ is almost algebraic; and hence, by (12), $((H_\circ)^*)_\circ = H_\circ$. Together with the fact that H has finitely many components, one sees that $(H(H_\circ)^*)_\circ = H_\circ$ and $H(H_\circ)^*$ has finitely many components, $H(H_\circ)^*$ is closed, $H(H_\circ)^*/H$ is finite, and $H(H_\circ)^*/(H_\circ)^*$ is finite. By [5, Lemma 1.3], the Lie algebra of $H(H_\circ)^*$ is algebraic. Then, by (13), $H(H_\circ)^*$ is almost algebraic; and hence, $(H(H_\circ)^*)^*/H(H_\circ)^*$ is finite. Together with the fact $H^* = (H(H_\circ)^*)^*$, we have H^*/H is finite. This proves that H is almost algebraic, and the proof of the theorem is complete.

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