Some triangle theorems by complex numbers

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1. *Introduction*

The following theorems appear in [1, pp. 62-63]:

Theorem 1

If similar triangles $A_1B_0C_0$, $A_0B_1C_0$, $A_0B_0C_1$ are erected externally on the sides of $\Delta A_0B_0C_0$, then the circumcircles of these three triangles have a common point, F . (See Figure 1.)

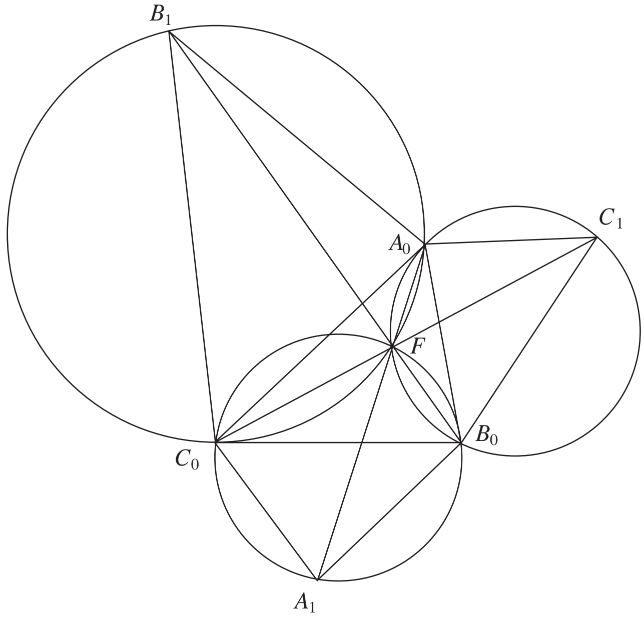

FIGURE 1

Theorem 2

In the situation of Theorem 1, the circumcentres of the three triangles form a triangle similar to the three triangles. (See Figure 2.)

If the three triangles are equilateral, then F is the *first isogonic centre* of $\Delta A_0B_0C_0$. If the angles of $\Delta A_0B_0C_0$ are all less than 120°, then F is inside $\Delta A_0B_0C_0$ and is then also the *Fermat point* of $\Delta A_0B_0C_0$, the point such that $|FA_0| + |FB_0| + |FC_0|$ is least possible. Also, the fact that the (circum)centres of the three triangles then form another equilateral triangle is Napoleon's theorem. For all of this, see [2, chapter 11]; see also [3, chapter XII, §§ 352–356].

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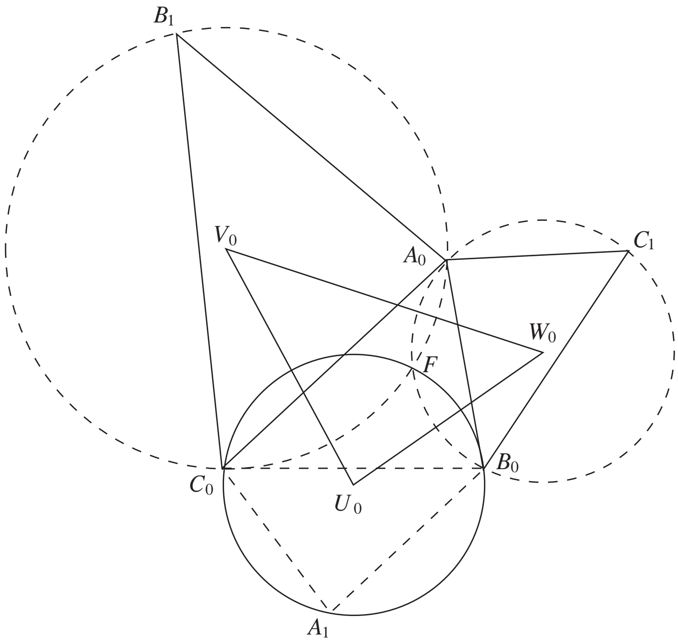

FIGURE 2

We shall prove Theorems 1 and 2, and more, in Section 2. For example, in the equilateral case it is known that A_0A_1, B_0B_1 and C_0C_1 all pass through F, but we shall show that this is in fact true in the general case. We shall also cover the case when the three similar triangles are erected internally on the sides of the original triangle. This can all be done by angle-chasing, but then the proofs are diagram dependent, so instead we shall mostly use algebra (complex numbers), when one proof covers all cases.

In Section 3, we shall prove a result about the areas of the various triangles, generalising another theorem from [1, p. 64].

In Section 4, we shall erect three more similar triangles $A_2B_1C_1$, $A_1B_2C_1$, $A_1B_1C_2$ on the sides of $\Delta A_1B_1C_1$, then three more similar triangles $A_3B_2C_2$, $A_2B_3C_2$, $A_2B_2C_3$ on the sides of $\Delta A_2B_2C_2$, and so on. We shall find that the circumcircles of the similar triangles all share the same point F, and that they fall into three coaxial systems, with one circle in common.

In the final section we shall look at the equilateral case, and in particular at some properties of Napoleon triangles.

For synthetic proofs of much of the material in sections 2 and 3, see [4]. For an alternative generalisation of the Napoleon configuration, involving similar triangles arranged differently around a given triangle, see [5]. For the Kiepert configuration, involving similar isosceles triangles arranged around a given triangle, see [2, Theorem 11.4].

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2. *Similar triangles*

We work in the complex plane, and adopt the convention that complex numbers a, b, … correspond to points labelled A, B, … .

Now if in ΔABC the labels go around the triangle in anticlockwise order, we say the triangle is *positively oriented*; so then ΔBCA, for example, is also positively oriented, but ΔBAC, for example, is *negatively* oriented. Then ΔABC is similar to $\Delta A'B'C'$ if the angles at A and A' are equal, likewise the angles at B and B', and at C and C'. They are *directly* similar if also both are oriented the same way, and *oppositely* similar otherwise.

So if, in ΔABC we have $\angle BAC = \alpha$ (the sign of α depending on the orientation), then

$$\frac{c - a}{b - a} = \frac{|AC|}{|AB|}e^{i\alpha},$$

from which it follows that ΔABC and $\Delta A'B'C'$ are directly similar if, and only if,

$$\frac{c - a}{b - a} = \frac{c' - a'}{b' - a'},$$

or equivalently

$$\frac{a - b}{c - b} = \frac{a' - b'}{c' - b'}, \qquad \text{or} \qquad \frac{b - c}{a - c} = \frac{b' - c'}{a' - c'},$$

by the case side-angle-side. The reader might like to show that these conditions are equivalent to

$$\begin{vmatrix} a & a' & 1 \\ b & b' & 1 \\ c & c' & 1 \end{vmatrix} = 0.$$

Then triangles ABC and $A'B'C'$ are oppositely similar if, and only if,

$$\frac{\bar{c} - \bar{a}}{\bar{b} - \bar{a}} = \frac{c' - a'}{b' - a'},$$

with equivalent conditions as above.

Now, in the situation of Theorem 1, let us choose axes and scaling so that $c_0 = 0$ and $b_0 = 1$. Then we can express everything in terms of a_0 and a_1. For, since triangles $A_1B_0C_0$ and $A_0B_1C_0$ are directly similar, we have

$$\frac{b_0 - c_0}{a_1 - c_0} = \frac{b_1 - c_0}{a_0 - c_0}, \qquad \text{that is,} \qquad \frac{1}{a_1} = \frac{b_1}{a_0}, \qquad \text{or} \qquad b_1 = \frac{a_0}{a_1}, \tag{1}$$

and, since triangles $A_1B_0C_0$ and $A_0B_0C_1$ are directly similar, we have

$$\frac{c_0 - b_0}{a_1 - b_0} = \frac{c_1 - b_0}{a_0 - b_0}, \text{ that is, } \frac{-1}{a_1 - 1} = \frac{c_1 - 1}{a_0 - 1}, \text{ or } c_1 = \frac{a_1 - a_0}{a_1 - 1}. \tag{2}$$

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We now prove the following, which incorporates Theorem 1:

Theorem 3

If directly similar triangles $A_1B_0C_0$, $A_0B_1C_0$ and $A_0B_0C_1$ are erected on the sides of $\Delta A_0B_0C_0$, then the three lines A_0A_1, B_0B_1 and C_0C_1 meet at a point F, which also lies on the three circumcircles $\odot A_1B_0C_0$, $\odot A_0B_1C_0$ and $\odot A_0B_0C_1$. Further, $|A_0A_1| : |B_0B_1| : |C_0C_1| = |B_0C_0|^{-1} : |C_0A_1|^{-1} : |A_1B_0|^{-1}$.

An alternative way of saying the last part is that

$$|A_0A_1|\,|B_0C_0| = |B_0B_1|\,|C_0A_1| = |C_0C_1|\,|A_1B_0|.$$

Note that, because the triangles are directly similar, they must either all be erected externally, or all be erected internally on the sides of $\Delta A_0B_0C_0$. Note too that in the case where the similar triangles are equilateral, we have that F is the first or the second isogonic centre/Fermat point of $\Delta A_0B_0C_0$, and also we have the well-known result that, in this case, $|A_0A_1| = |B_0B_1| = |C_0C_1|$.

Proof of Theorem 3

Let $\angle B_0C_0A_1 = \alpha$. Then

$$\frac{|A_1C_0|}{|B_0C_0|}e^{i\alpha} = \frac{a_1 - c_0}{b_0 - c_0} = \frac{a_1 - 0}{1 - 0} = a_1.$$

But also, using (1),

$$\frac{a_0 - a_1}{b_1 - b_0} = \frac{a_0 - a_1}{a_0/a_1 - 1} = a_1.$$

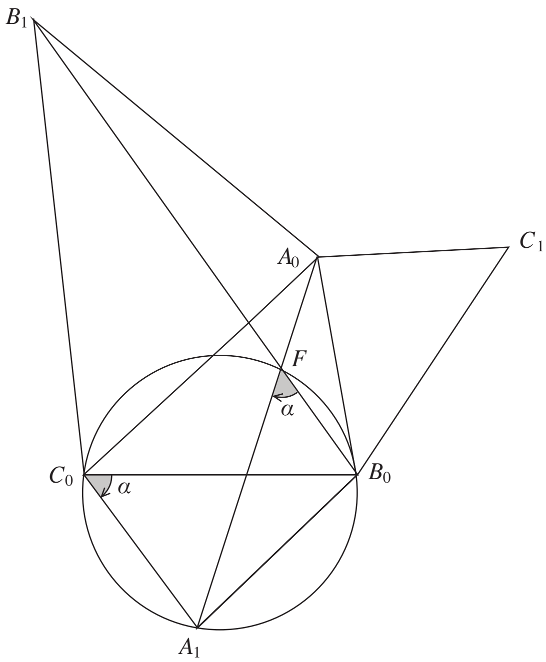

FIGURE 3

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So $|A_1C_0| : |B_0C_0| = |A_0A_1| : |B_1B_0|$, or $|B_0C_0|^{-1} : |C_0A_1|^{-1} = |A_0A_1| : |B_0B_1|$; and similarly $|C_0A_1|^{-1} : |A_1B_0|^{-1} = |B_0B_1| : |C_0C_1|$. Also, if A_0A_1 and B_1B_0 meet at F, then the line B_1B_0, rotated about F through the angle α, becomes the line A_0A_1. So A_1, B_0, C_0 and F are concyclic, by the converse of the theorem about angles in the same segment if C_0 and F are on the same side of A_1B_0, as in Figure 3, for example, or by the converse of the theorem about the internal and opposite external angles of a cyclic quadrilateral if C_0 and F are on opposite sides of A_1B_0, as in Figure 4, for example. (An alternative, purely algebraic, proof of this result is suggested at the end of Section 3.) Said another way, if A_0A_1 meets $\odot A_1B_0C_0$ again at F, then F also lies on B_0B_1; and similarly it also lies on C_0C_1. Putting *this* another way, if A_0A_1, B_0B_1 and C_0C_1 meet at F, then F lies on $\odot A_1B_0C_0$; and similarly it also lies on $\odot A_0B_1C_0$ and on $\odot A_0B_0C_1$.

Next we compute the circumcentres U_0, V_0 and W_0 of triangles $A_1B_0C_0$, $A_0B_1C_0$ and $A_0B_0C_1$. Now $|u_0 - c_0| = |u_0 - b_0| = |u_0 - a_1|$, that is, $|u_0| = |u_0 - 1| = |u_0 - a_1|$. Squaring,

$$u_0\overline{u_0} = (u_0 - 1)(\overline{u_0} - 1) = (u_0 - a_1)(\overline{u_0} - \overline{a_1}),$$

or

$$0 = 1 - u_0 - \overline{u_0} = a_1\overline{a_1} - u_0\overline{a_1} - \overline{u_0}a_1.$$

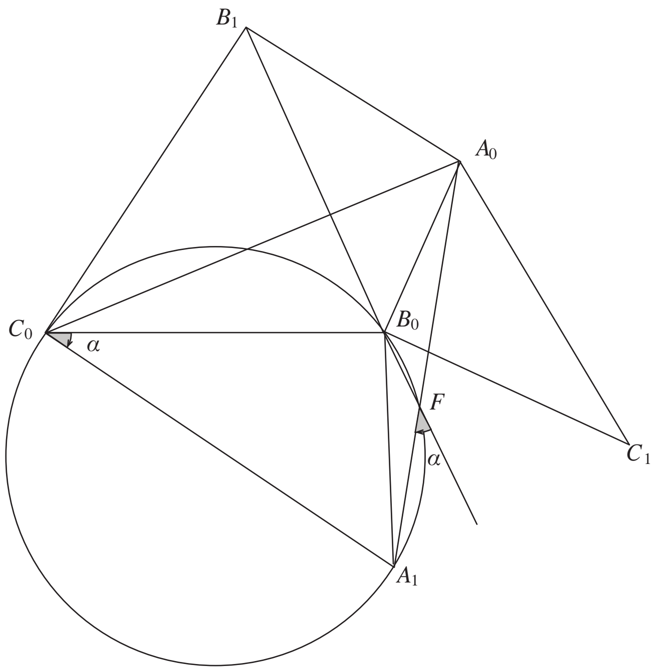

FIGURE 4

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Treating these as simultaneous equations in u_0 and $\overline{u_0}$, we eliminate the latter to find

$$u_0 = \frac{a_1(1 - \overline{a_1})}{a_1 - \overline{a_1}}. \tag{3}$$

Now we use the fact that triangles $U_0B_0C_0$, $V_0B_1C_0$ and $W_0B_0C_1$ are directly similar. So

$$\frac{u_0 - c_0}{b_0 - c_0} = \frac{v_0 - c_0}{b_1 - c_0}, \quad \text{or} \quad u_0 = \frac{v_0}{b_1}, \quad \text{that is,} \quad v_0 = \frac{a_0(1 - \overline{a_1})}{a_1 - \overline{a_1}}. \tag{4}$$

And

$$\frac{u_0 - c_0}{b_0 - c_0} = \frac{w_0 - c_1}{b_0 - c_1}, \text{ or } u_0 = \frac{w_0 - c_1}{1 - c_1}, \text{ that is, } w_0 = c_1 + u_0 - c_1u_0.$$

Using (2) and (3), we obtain

$$w_0 = \frac{a_1 - a_0\overline{a_1}}{a_1 - \overline{a_1}}. \tag{5}$$

We now prove something a little more specific than Theorem 2:

Theorem 4

$\Delta A_1B_0C_0$ and $\Delta U_0V_0W_0$ are oppositely similar. (Again, see Figure 2.)

Proof: Firstly,

$$\frac{\overline{a_1} - \overline{c_0}}{\overline{b_0} - \overline{c_0}} = \overline{a_1}.$$

Secondly, using (3), (4) and (5), and multiplying numerator and denominator through by $a_1 - \overline{a_1}$, we have

$$\frac{u_0 - w_0}{v_0 - w_0} = \frac{a_1(1 - \overline{a_1}) - (a_1 - a_0\overline{a_1})}{a_0(1 - \overline{a_1}) - (a_1 - a_0\overline{a_1})} = \overline{a_1}.$$

3. *Areas*

Proposition: The area of $\Delta Z_1Z_2Z_3$ is given by

$$\frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}.$$

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Proof: If $z_k = x_k + iy_k$, where $x_k, y_k \in \mathbb{R}$, for $k = 1, 2, 3$, then a standard result is that the area $\mathscr{A}$ of $\Delta Z_1Z_2Z_3$ is given by

$$\mathscr{A} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

See, for example, [6, pp. 205-206]. Note that $\mathscr{A}$ is positive or negative according as $\Delta Z_1Z_2Z_3$ is positively or negatively oriented; also, $\mathscr{A} = 0$ if, and only if, Z_1, Z_2 and Z_3 are collinear.

Then, by column operations,

$$\mathscr{A} = \frac{1}{2}\begin{vmatrix} x_1 + iy_1 & y_1 & 1 \\ x_2 + iy_2 & y_2 & 1 \\ x_3 + iy_3 & y_3 & 1 \end{vmatrix} = \frac{i}{4}\begin{vmatrix} x_1 + iy_1 & -2iy_1 & 1 \\ x_2 + iy_2 & -2iy_2 & 1 \\ x_3 + iy_3 & -2iy_3 & 1 \end{vmatrix} = \frac{i}{4}\begin{vmatrix} x_1 + iy_1 & x_1 - iy_1 & 1 \\ x_2 + iy_2 & x_2 - iy_2 & 1 \\ x_3 + iy_3 & x_3 - iy_3 & 1 \end{vmatrix},$$

as required.

Let us use this to compute the area of $\Delta A_0B_0C_0$, This is

$$\frac{i}{4}\begin{vmatrix} a_0 & \overline{a_0} & 1 \\ b_0 & \overline{b_0} & 1 \\ c_0 & \overline{c_0} & 1 \end{vmatrix} = \frac{i}{4}\begin{vmatrix} a_0 & \overline{a_0} & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{i}{4}\left(a_0 - \overline{a_0}\right) = -\frac{1}{2}\mathscr{I}\left(a_0\right),$$

that is, $-\frac{1}{2} \times$ the imaginary part of a_0. This has the magnitude ($\frac{1}{2} \times$ base $\times$ height), as expected, and the minus sign is explained by the fact that, if $\mathscr{I}(a_0) > 0$, that is, if A_0 is above the real axis, then $\Delta A_0B_0C_0$ is negatively oriented.

Next we compute the area of $\Delta U_0V_0W_0$. Using (3), (4) and (5), this is

$$\frac{i}{4}\begin{vmatrix} u_0 & \overline{u_0} & 1 \\ v_0 & \overline{v_0} & 1 \\ w_0 & \overline{w_0} & 1 \end{vmatrix} = \frac{i}{4}\begin{vmatrix} \dfrac{a_1\left(1 - \overline{a_1}\right)}{a_1 - \overline{a_1}} & \dfrac{\overline{a_1}\left(1 - a_1\right)}{\overline{a_1} - a_1} & 1 \\ \dfrac{a_0\left(1 - \overline{a_1}\right)}{a_1 - \overline{a_1}} & \dfrac{\overline{a_0}\left(1 - a_1\right)}{\overline{a_1} - a_1} & 1 \\ \dfrac{a_1 - a_0\overline{a_1}}{a_1 - \overline{a_1}} & \dfrac{\overline{a_1} - \overline{a_0}a_1}{\overline{a_1} - a_1} & 1 \end{vmatrix}$$

$$= \frac{i}{4\left(a_1 - \overline{a_1}\right)^2}\begin{vmatrix} a_1 - a_1\overline{a_1} & a_1\overline{a_1} - \overline{a_1} & 1 \\ a_0 - a_0\overline{a_1} & \overline{a_0}a_1 - \overline{a_0} & 1 \\ a_1 - a_0\overline{a_1} & \overline{a_0}a_1 - \overline{a_1} & 1 \end{vmatrix}$$

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$$= \frac{i}{4(a_1 - \overline{a_1})^2} \begin{vmatrix} a_0\overline{a_1} - a_1\overline{a_1} & a_1\overline{a_1} - \overline{a_0}a_1 & 0 \\ a_0 - a_1 & \overline{a_1} - \overline{a_0} & 0 \\ a_1 - a_0\overline{a_1} & \overline{a_0}a_1 - \overline{a_1} & 1 \end{vmatrix}$$

$$= \frac{i}{4(a_1 - \overline{a_1})^2} \begin{vmatrix} (a_0 - a_1)\overline{a_1} & a_1(\overline{a_1} - \overline{a_0}) & 0 \\ a_0 - a_1 & \overline{a_1} - \overline{a_0} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{(a_0 - a_1)(\overline{a_1} - \overline{a_0})i}{4(a_1 - \overline{a_1})^2} \begin{vmatrix} \overline{a_1} & a_1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{(a_0 - a_1)(\overline{a_0} - \overline{a_1})i}{4(a_1 - \overline{a_1})}. \qquad (6)$$

Now suppose the points A_1', B_1' and C_1' are the reflections of A_1, B_1 and C_1 in the respective sides B_0C_0, C_0A_0 and A_0B_O of $\Delta A_0B_0C_0$. Then triangles $A_1'B_0C_0$, $A_0B_1'C_0$ and $A_0B_0C_1'$ are directly similar (being oppositely similar to triangles $A_1B_0C_0$, etc.) and are erected internally or externally on the sides of $\Delta A_0B_0C_0$ according as triangles $A_1B_0C_0$, etc. are erected externally or internally, respectively. Further, the circumcentres U, V and W of the new triangles are the reflections of U_0, V_0 and W_0 in the respective sides of $\Delta A_0B_0C_0$. Formulæ for these new circumcentres are therefore obtained from formulæ for the old ones by just replacing a_1 by $\overline{a_1}$, throughout. So, by (6),

$$\text{area}(U_0V_0W_0) + \text{area}(UVW) = \frac{(a_0 - a_1)(\overline{a_0} - \overline{a_1})i}{4(a_1 - \overline{a_1})} + \frac{(a_0 - \overline{a_1})(\overline{a_0} - a_1)i}{4(\overline{a_1} - a_1)}$$

$$= \frac{i}{4}(a_0 - \overline{a_0}) = -\frac{1}{2}\mathcal{I}(a_0) = \text{area}(A_0B_0C_0).$$

But each triangle in the list $U_0V_0W_0$, $A_1B_0C_0$, $A_1'B_0C_0$, UVW has the opposite orientation to the adjacent ones; hence ΔUVW has the opposite orientation to $\Delta U_0V_0W_0$. (See Figure 5.) So what we have proved, above, is:

Theorem 5

$$\left\| \text{area}(U_0V_0W_0) \right| - \left| \text{area}(UVW) \right\| = \left| \text{area}(A_0B_0C_0) \right|.$$

This generalises [1, Theorem 3.38 (p. 64)], which just deals with the case of Napoleon triangles.

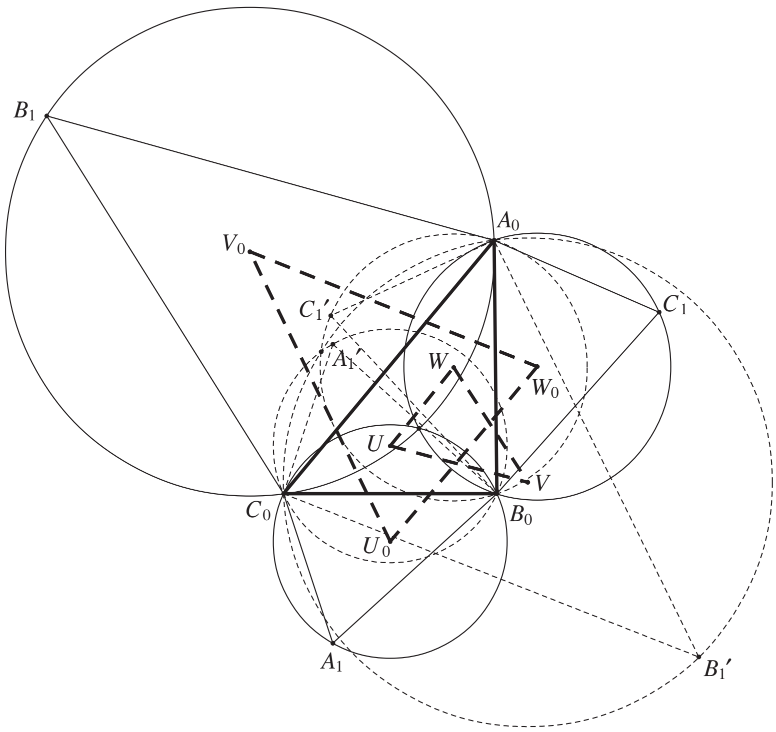

FIGURE 5

We finish this section by using determinants to compute f, the complex number representing F. Now F is collinear with A_0 and A_1, and also with B_0 and B_1, so, by (1),

$$\begin{vmatrix} f & \overline{f} & 1 \\ a_0 & \overline{a_0} & 1 \\ a_1 & \overline{a_1} & 1 \end{vmatrix} = 0 = \begin{vmatrix} f & \overline{f} & 1 \\ 1 & 1 & 1 \\ \frac{a_0}{a_1} & \frac{\overline{a_0}}{\overline{a_1}} & 1 \end{vmatrix}.$$

We need to eliminate $\overline{f}$. So, by row and column operations,

$$\begin{vmatrix} \overline{a_1}f & \overline{a_1 f} & \overline{a_1} \\ a_0 & \overline{a_0} & 1 \\ a_1 & \overline{a_1} & 1 \end{vmatrix} = 0 = \begin{vmatrix} a_1 f & \overline{a_1 f} & 1 \\ a_0 & \overline{a_0} & 1 \\ a_1 & \overline{a_1} & 1 \end{vmatrix}.$$

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Subtracting the one determinant from the other,

$$\begin{vmatrix} (a_1 - \overline{a_1})f & 0 & 1 - \overline{a_1} \\ a_0 & \overline{a_0} & 1 \\ a_1 & \overline{a_1} & 1 \end{vmatrix} = 0,$$

whence

$$f = \frac{(a_0\overline{a_1} - \overline{a_0}a_1)(\overline{a_1} - 1)}{(\overline{a_0} - \overline{a_1})(a_1 - \overline{a_1})}. \tag{7}$$

The reader might like to use this to show that $|f - u_0| = |u_0|$, thus giving an alternative proof that F lies on $\odot A_1B_0C_0$.

4. *Repetition of the construction*

In this section we are going to use a well-known result about equal fractions:

$$\text{If } \omega = \frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_n}{y_n}, \text{ then } \omega = \frac{\sum_k \lambda_k x_k}{\sum_k \lambda_k y_k}$$

for any $\lambda_1, \lambda_2, \dots, \lambda_n$ with $\sum_k \lambda_k y_k \neq 0$.

(I speculated about whether this really is well-known in the introduction of [7].)

In fact, what we need is the following corollary of the above result:

$$\text{If } \omega \neq 0, \sum_k \lambda_k x_k = 0 \text{ if, and only if, } \sum_k \lambda_k y_k = 0.$$

Proofs are immediate, on noting that $x_k = \omega y_k$, for all k, so that $\sum_k \lambda_k x_k = \omega \sum_k \lambda_k y_k$.

So now suppose we repeat our previous construction: start with $\Delta A_0B_0C_0$ and erect directly similar triangles $A_1B_0C_0$, $A_0B_1C_0$, $A_0B_0C_1$, $A_2B_1C_1$, $A_1B_2C_1$, $A_1B_1C_2$, … , $A_{k+1}B_kC_k$, $A_kB_{k+1}C_k$, $A_kB_kC_{k+1}$, … .

Lemma

For all k, A_k, B_k and C_k are the midpoints of $A_{k+1}A_{k+2}$, $B_{k+1}B_{k+2}$ and $C_{k+1}C_{k+2}$, respectively. (See Figure 6, where we have drawn the case $k = 0$.)

Proof:

Since triangles $A_{k+1}B_kC_k$, $A_kB_{k+1}C_k$, $A_kB_kC_{k+1}$ and $A_{k+2}B_{k+1}C_{k+1}$ are directly similar, we have

$$\frac{a_{k+1} - b_k}{c_k - b_k} = \frac{a_k - b_{k+1}}{c_k - b_{k+1}} = \frac{a_k - b_k}{c_{k+1} - b_k} = \frac{a_{k+2} - b_{k+1}}{c_{k+1} - b_{k+1}}.$$

Now $(c_k - b_k) - (c_k - b_{k+1}) - (c_{k+1} - b_k) + (c_{k+1} - b_{k+1}) = 0$, so we must also have

$$(a_{k+1} - b_k) - (a_k - b_{k+1}) - (a_k - b_k) + (a_{k+2} - b_{k+1}) = 0,$$

or $a_k = \frac{1}{2}(a_{k+1} + a_{k+2})$, as required; and similarly for the other cases.

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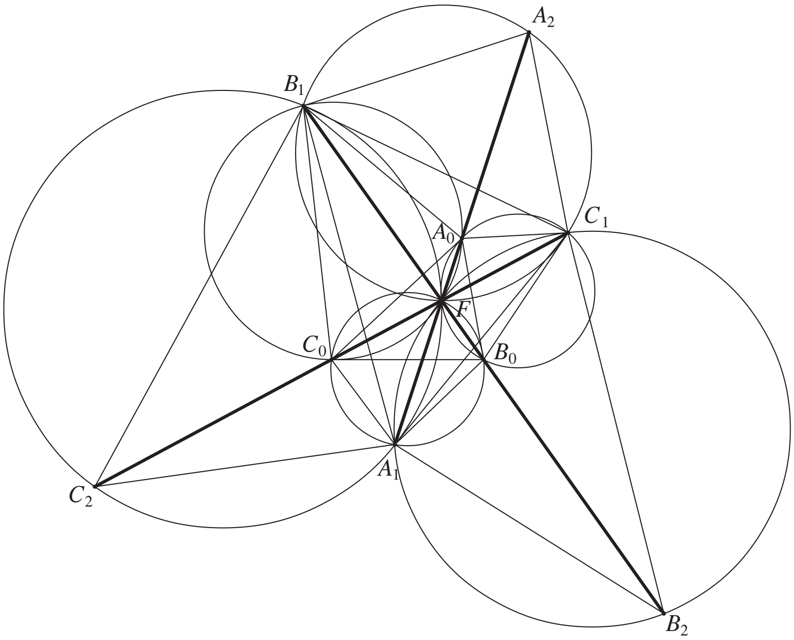

FIGURE 6

This last result provides a solution to the following construction problem: given ΔABC and points A_1, B_1 and C_1, find points A_0, B_0 and C_0 so that triangles $A_1B_0C_0$, $A_0B_1C_0$ and $A_0B_0C_1$ are directly similar to ΔABC. The solution is to construct A_2, B_2 and C_2 so that triangles $A_2B_1C_1$, $A_1B_2C_1$ and $A_1B_1C_2$ are directly similar to ΔABC, and then A_0, B_0 and C_0 are the mid-points of A_1A_2, B_1B_2 and C_1C_2, respectively.

Theorem 6

The points A_k all lie on one line, the points B_k lie on another, and the points C_k lie on a third. These three lines meet in the point F, which also lies on the circumcircles $\odot A_{k+1}B_kC_k$, $\odot A_kB_{k+1}C_k$ and $\odot A_kB_kC_{k+1}$, for all k. (Again, see Figure 6.)

Proof: By the lemma, A_k, A_{k+1} and A_{k+2} are collinear, for all k, whence the points A_k all lie on one line; and we know A_0A_1 passes through F. Similarly for the B_k and the C_k. Then apply Theorem 3 to the triangle $A_kB_kC_k$ with its similar triangles $A_{k+1}B_kC_k$, $A_kB_{k+1}C_k$ and $A_kB_kC_{k+1}$ to see that $\odot A_{k+1}B_kC_k$, $\odot A_kB_{k+1}C_k$ and $\odot A_kB_kC_{k+1}$ all pass through F.

Now let U_k, V_k and W_k be the circumcentres of triangles $A_{k+1}B_kC_k$, $A_kB_{k+1}C_k$ and $A_kB_kC_{k+1}$, respectively.

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Theorem 7

For all k, $\Delta U_k V_k W_k$ is the medial triangle of $\Delta U_{k+1} V_{k+1} W_{k+1}$. (See Figure 7, where we have drawn the case $k = 0$.)

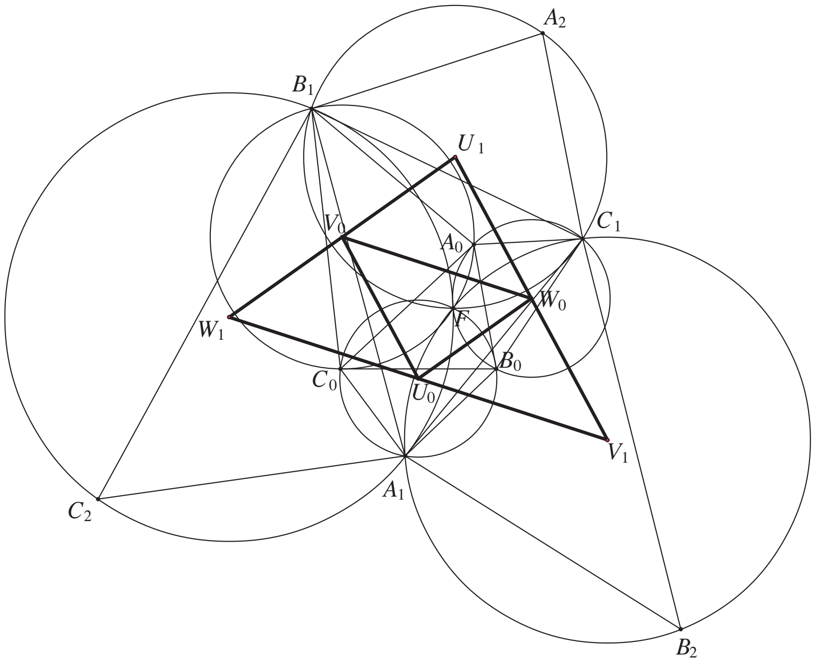

FIGURE 7

Proof:

Since triangles $U_k B_k C_k$, $V_{k+1} B_{k+2} C_{k+1}$ and $W_{k+1} B_{k+1} C_{k+2}$ are directly similar,we have

$$\frac{u_k - b_k}{c_k - b_k} = \frac{v_{k+1} - b_{k+2}}{c_{k+1} - b_{k+2}} = \frac{w_{k+1} - b_{k+1}}{c_{k+2} - b_{k+1}}.$$

Now

$$2(c_k - b_k) - (c_{k+1} - b_{k+2}) - (c_{k+2} - b_{k+1})$$

$$= (2c_k - c_{k+1} - c_{k+2}) - (2b_k - b_{k+1} - b_{k+2}) = 0 - 0 = 0,$$

so we must also have $2(u_k - b_k) - (v_{k+1} - b_{k+2}) - (w_{k+1} - b_{k+1}) = 0$, that is,

$$(2u_k - v_{k+1} - w_{k+1}) - (2b_k - b_{k+1} - b_{k+2}) = (2u_k - v_{k+1} - w_{k+1}) = 0$$

So $u_k = \frac{1}{2}(v_{k+1} + w_{k+1})$, as required; and similarly for v_k and for w_k.

Now a triangle and its medial triangle share the same medians and the same centroid. So, by Theorem 7, the triangles $U_k V_k W_k$ have the same medians and the same centroid, for all k. (The reader might like to apply the

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technique of Theorem 7 to the directly similar triangles $U_k B_k C_k$, $U_{k+1} B_{k+1} C_{k+1}$ and $U_{k+2} B_{k+2} C_{k+2}$ to give a direct proof that U_k is the midpoint of $U_{k+1} U_{k+2}$.)

Let the medians through all the U_k, all the V_k and all the W_k be ℓ, m and n, respectively, and let the centroid, where ℓ, m and n meet, be G. So $g = \frac{1}{3}(u_k + v_k + w_k)$, all k. Let L, M and N be the reflections of F in ℓ, m and n, respectively. Then we get:

Theorem 8

For all k, $\odot A_{k+1} B_k C_k$ belongs to the coaxial system of circles through F and L: $\odot A_k B_{k+1} C_k$ belongs to the coaxial system of circles through F and M; and $\odot A_k B_k C_{k+1}$ belongs to the system of coaxial circles through F and N. Further, there is a circle Σ, centre G, through F, L, M and N, which belongs to all three systems of coaxial circles. (See Figure 8, where we have just drawn the circles for $k = 0$ and 1.)

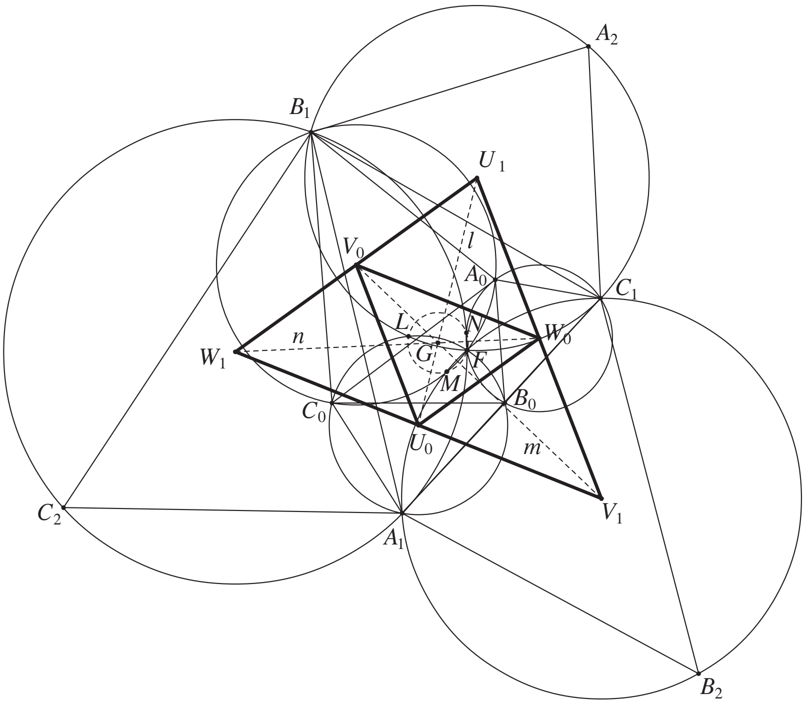

FIGURE 8

Proof: $\odot A_{k+1} B_k C_k$ has its centre U_k on ℓ, which is the perpendicular bisector of FL, so since the circle passes through F, it must pass through L as well. Similarly the circles with centres on m and passing through F are circles which must pass through M, and the circles with centres on n and passing through F are circles which must pass through N. Likewise, since G lies on ℓ, m and n, the circle centre G through F must also pass through L, M and N.

Next, the six lines through F yield three harmonic pencils:

Theorem 9

$$F\{B_0C_0;\ A_0L\} = F\{C_0A_0;\ B_0M\} = F\{A_0B_0;\ C_0N\} = -1.$$

Proof:

It is only necessary to prove $F\{B_0C_0; A_0L\} = -1$; the other cases are similar. Let A_0A_1 meet B_0C_0 (the real axis) in P, and let $\{B_0C_0;\ PQ\} = -1$. We show first that $U_0G \perp FQ$. For $g = \frac{1}{3}(u_0 + v_0 + w_0)$, so by (3), (4) and (5),

$$u_0 - g = \frac{2u_0 - v_0 - w_0}{3} = \frac{2a_1(1 - \overline{a_1}) - a_0(1 - \overline{a_1}) - (a_1 - a_0\overline{a_1})}{3(a_1 - \overline{a_1})}$$

$$= \frac{(a_0 - a_1)(2\overline{a_1} - 1)}{3(a_1 - \overline{a_1})}.$$

Then, remembering that $p = \overline{p}$,

$$\begin{vmatrix} p & p & 1 \\ a_0 & \overline{a_0} & 1 \\ a_1 & \overline{a_1} & 1 \end{vmatrix} = 0, \qquad \text{whence } p = \frac{a_0\overline{a_1} - \overline{a_0}a_1}{a_0 - \overline{a_0} - a_1 + \overline{a_1}}.$$

Then $\dfrac{p}{1 - p} = -\dfrac{q}{1 - q}$, so that

$$q = \frac{p}{2p - 1} = \frac{a_0\overline{a_1} - \overline{a_0}a_1}{2a_0\overline{a_1} - 2\overline{a_0}a_1 - a_0 + \overline{a_0} + a_1 - \overline{a_1}}.$$

Thus, using (7),

$$f - q = \frac{(a_0\overline{a_1} - \overline{a_0}a_1)(\overline{a_1} - 1)}{(\overline{a_0} - \overline{a_1})(a_1 - \overline{a_1})} - \frac{a_0\overline{a_1} - \overline{a_0}a_1}{2a_0\overline{a_1} - 2\overline{a_0}a_1 - a_0 + \overline{a_0} + a_1 - \overline{a_1}}$$

$$= \frac{(a_0\overline{a_1} - \overline{a_0}a_1)(2\overline{a_1} - 1)(a_0\overline{a_1} - \overline{a_0}a_1 - a_0 + \overline{a_0} + a_1 - \overline{a_1})}{(\overline{a_0} - \overline{a_1})(a_1 - \overline{a_1})(2a_0\overline{a_1} - 2\overline{a_0}a_1 - a_0 + \overline{a_0} + a_1 - \overline{a_1})}.$$

So

$$\frac{f - q}{u_0 - g} = \frac{3(a_0\overline{a_1} - \overline{a_0}a_1)(a_0\overline{a_1} - \overline{a_0}a_1 - a_0 + \overline{a_0} + a_1 - \overline{a_1})}{(a_0 - a_1)(\overline{a_0} - \overline{a_1})(2a_0\overline{a_1} - 2\overline{a_0}a_1 - a_0 + \overline{a_0} + a_1 - \overline{a_1})}.$$

The product of the first two brackets in the denominator, being of the form $z\overline{z} = |z^2|$, is real, and the other three brackets, each being of the form $z - \overline{z} = 2i\mathcal{I}(z)$, are pure imaginary, so the whole expression is pure imaginary, whence $U_0G \perp FQ$, as claimed.

Now $U_0G \perp FQ$, so it follows that L, F and Q are collinear. Then

$$F\{B_0C_0;\ A_0L\} = F\{B_0C_0;\ PQ\} = \{B_0C_0;\ PQ\} = -1.$$

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We finish this section with a brief look at a limiting or degenerate case of the above, when we allow $F \to \infty$. So here the lines A_0A_1, B_0B_1 and C_0C_1 are parallel, and the circles $\odot A_{k+1}B_kC_k$, $\odot A_kB_{k+1}C_k$ and $\odot A_kB_kC_{k+1}$, which all pass through F and (respectively) L, M or N, become lines through (respectively) L, M or N. Further, Σ becomes a line through L, M and N. See Figure 9. Another way of stating Theorem 9 is that $\{B_0C_0;\, A_1L\} = -1$ on $\odot A_1B_0C_0$, and when $F \to \infty$ this just becomes $\{B_0C_0;\, A_1L\} = -1$, this time on a line rather than on a circle.

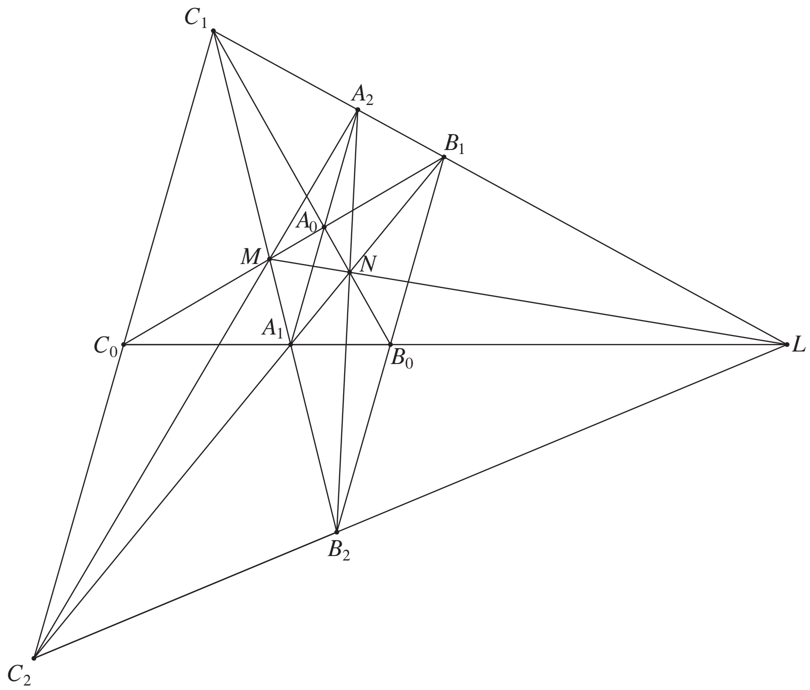

FIGURE 9

5. *Napoleon triangles*

We now restrict to the case where the similar triangles $A_1B_0C_0$, $A_0B_1C_0$, $A_0B_0C_1$ are equilateral. In this case Napoleon's theorem says that the (circum)centre triangle $U_0V_0W_0$ is also equilateral. If the similar triangles are erected outwardly (respectively, inwardly) on the sides of $\Delta A_0B_0C_0$, then $\Delta U_0V_0W_0$ is the *outer* (respectively, inner) *Napoleon triangle* of $\Delta A_0B_0C_0$. As before, let U, V, W be the reflections of U_0, V_0, W_0 in B_0C_0, C_0A_0, A_0B_0, respectively. Then if one of the triangles UVW, $U_0V_0W_0$ is the outer Napoleon triangle, the other is the inner: we shall say that they are *opposite* Napoleon triangles of $\Delta A_0B_0C_0$.

So, taking $c_0 = 0$ and $b_0 = 1$ as before, let us also take $a_0 = a$ and $a_1 = \varepsilon$, a primitive sixth root of 1. So $\varepsilon = \frac{1}{2}(1 \pm i\sqrt{3})$, $\varepsilon^2 - \varepsilon + 1 = 0$,

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$\varepsilon\bar{\varepsilon} = 1$ and $(\varepsilon - \bar{\varepsilon})^2 = -3$. Then by (1) and (2),

$$b_1 = \frac{a}{\varepsilon} = a(1 - \varepsilon), \text{ and } \qquad c_1 = \frac{\varepsilon - a}{\varepsilon - 1} = a\varepsilon - \varepsilon^2 = a\varepsilon - \varepsilon + 1. \qquad (8)$$

Also, by (3),

$$u_0 = \frac{\varepsilon(1 - \bar{\varepsilon})}{\varepsilon - \bar{\varepsilon}} = \frac{1}{3}\varepsilon^2(\bar{\varepsilon} - \varepsilon) = \frac{1}{3}(\varepsilon + 1), \text{ so } u = \overline{u_0} = \frac{1}{3}(2 - \varepsilon). \qquad (9)$$

And by (7),

$$f = \frac{(a\bar{\varepsilon} - \bar{a}\varepsilon)(\bar{\varepsilon} - 1)}{(\bar{a} - \bar{\varepsilon})(\varepsilon - \bar{\varepsilon})} = \frac{(a\bar{\varepsilon} - \bar{a}\varepsilon)(\varepsilon - 2)}{3(\bar{a} - \bar{\varepsilon})}. \qquad (10)$$

Next, triangles $B_0C_0U_0$, $C_0A_0V_0$ and $A_0B_0W_0$ are similar, so

$$\frac{u_0 - c_0}{b_0 - c_0} = \frac{v_0 - a_0}{c_0 - a_0} = \frac{w_0 - b_0}{a_0 - b_0}.$$

Here, the denominators sum to zero, hence so do the numerators, that is, $u_0 + v_0 + w_0 = a_0 + b_0 + c_0$. Dividing by 3, we see that the centroid G of $\Delta U_0V_0W_0$ is equal to the centroid of $\Delta A_0B_0C_0$, that is, $g = \frac{1}{3}(a + 1)$. Similarly, the centroid of ΔUVW is also G.

Theorem 10

U, V and W lie on the circle Σ of Theorem 8.

Proof: It is only necessary to prove that U lies on Σ. Since Σ is the circle centre G through F, we need to show that $|g - u| = |g - f|$. We have, on the one hand, by (10),

$$\begin{aligned} 3(\bar{a} - \bar{\varepsilon})(g - f) &= (\bar{a} - \bar{\varepsilon})(a + 1) - (a\bar{\varepsilon} - \bar{a}\varepsilon)(\varepsilon - 2) \\ &= (a\bar{a} + \bar{a} - a\bar{\varepsilon} + \varepsilon^2) - a(\varepsilon - \bar{\varepsilon}) - \bar{a}(\varepsilon + 1) \\ &= a\bar{a} - a\varepsilon - \bar{a}\varepsilon + \varepsilon^2 \\ &= (a - \varepsilon)(\bar{a} - \varepsilon). \end{aligned}$$

And on the other hand, by (9),

$$3(a - \varepsilon)(\bar{g} - \bar{u}) = (a - \varepsilon)((\bar{a} + 1) - (\varepsilon + 1)) = (a - \varepsilon)(\bar{a} - \varepsilon),$$

whence

$$3(a - \varepsilon)(\bar{g} - \bar{u}) = 3(\bar{a} - \bar{\varepsilon})(g - f).$$

The result follows on comparing the moduli of either side.

The last result says that the first (respectively, second) isogonic centre of $\Delta A_0B_0C_0$ lies on the circumcircle of its inner (respectively, outer) Napoleon triangle, which is perhaps not quite the way round one might have expected.

We now show that, for all k, ΔUVW is one of the Napoleon triangles of $\Delta A_kB_kC_k$. Explicitly:

Theorem 11

For all k, ΔUVW and $\Delta U_kV_kW_k$ are opposite Napoleon triangles of $\Delta A_kB_kC_k$.

Proof: The result is true by definition when $k = 0$. For $k = 1$, we need to show that U is the reflection of U_1 in B_1C_1. This follows if we show that ΔUB_1C_1 is directly congruent to $\Delta U_1C_1B_1$; since the latter is directly similar to $\Delta U_0C_0B_0$, it will be sufficient to show that $\dfrac{u - b_1}{c_1 - b_1} = \dfrac{u_0 - c_0}{b_0 - c_0}$, or, using (8) and (9),

$$\frac{\frac{1}{3}(2 - \varepsilon) - a(1 - \varepsilon)}{a\varepsilon - \varepsilon + 1 - a(1 - \varepsilon)} = \frac{\varepsilon + 1}{3},$$

that is,

$$\frac{2 - \varepsilon - 3a + 3a\varepsilon}{2a\varepsilon - \varepsilon + 1 - a} = \varepsilon + 1.$$

But

$$\begin{aligned}(\varepsilon + 1)(2a\varepsilon - \varepsilon + 1 - a) &= 2a\varepsilon^2 - \varepsilon^2 + \varepsilon - a\varepsilon + 2a\varepsilon - \varepsilon + 1 - a \\ &= 2a(\varepsilon - 1) - (\varepsilon - 1) + \varepsilon + a\varepsilon - \varepsilon + 1 - a \\ &= 2 - \varepsilon - 3a + 3a\varepsilon,\end{aligned}$$

as required. Likewise, V and W are the reflections of V_1 and W_1 in C_1A_1 and A_1B_1, respectively, which completes the case $k = 1$. Finally, the progression from $\Delta U_kV_kW_k$ to $\Delta U_{k+1}V_{k+1}W_{k+1}$ is exactly the same as the progression from $\Delta U_0V_0W_O$ to $\Delta U_1V_1W_1$, and we have finished.

Note that it is by no means certain that ΔUVW is the outer, or the inner, Napoleon triangle of $\Delta A_kB_kC_k$ for all k. Indeed, it is perfectly possible for it to be the outer one for some k and the inner one for $k + 1$. See Figure 10, where we have erected triangles $A_1B_0C_0$, $A_0B_1C_0$ and $A_0B_0C_1$ inwardly on the sides of $\Delta A_0B_0C_0$, but in order to have triangles $A_{k+1}B_kC_k$, $A_kB_{k+1}C_k$ and $A_kB_kC_{k+1}$ directly similar for all k, it is necessary to erect the triangles $A_2B_1C_1$, $A_1B_2C_1$ and $A_1B_1C_2$ outwardly on the sides of $A_1B_1C_1$. The next result will enable us to gain some control over this situation.

Theorem 12

$$\text{area}\left(A_kB_kC_k\right) - \text{area}\left(A_{k-1}B_{k-1}C_{k-1}\right) = \frac{i}{4}(1 - 2\varepsilon)\left|A_kA_{k-1}\right|^2.$$

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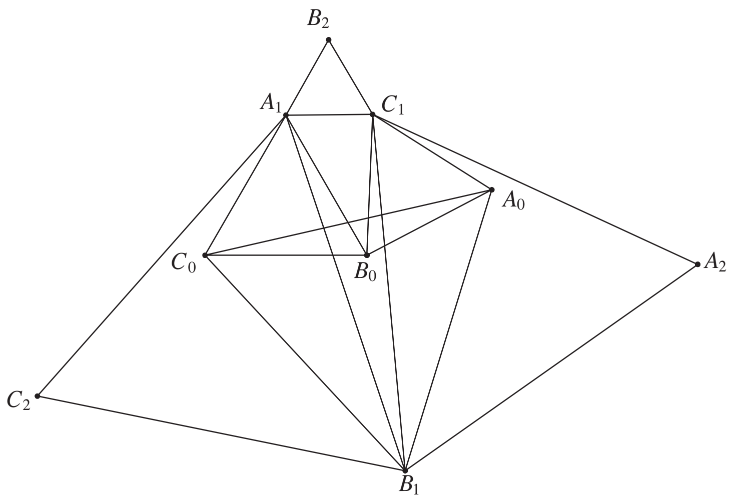

FIGURE 10

Proof: It is enough to prove the result when $k = 1$. Now

$$\frac{4}{i}\left(\text{area}\left(A_1B_1C_1\right) - \text{area}\left(A_0B_0C_0\right)\right)$$

$$= \begin{vmatrix} \varepsilon & \bar{\varepsilon} & 1 \\ a\bar{\varepsilon} & \bar{a}\varepsilon & 1 \\ a\varepsilon + \bar{\varepsilon} & \bar{a}\bar{\varepsilon} + \varepsilon & 1 \end{vmatrix} - \begin{vmatrix} a & \bar{a} & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \bar{a}\varepsilon^2 + \left(a + \bar{\varepsilon}^2\right) + \left(a\bar{a}\bar{\varepsilon}^2 + a\right) - \left(\bar{a} + \varepsilon^2\right) - a\bar{\varepsilon}^2 - \left(a\bar{a}\varepsilon^2 + \bar{a}\right) - \left(a - \bar{a}\right)$$

$$= \left(a\bar{a} - a + 1\right)\bar{\varepsilon}^2 - \left(a\bar{a} - \bar{a} + 1\right)\varepsilon^2 + a - \bar{a}$$

$$= -\left(a\bar{a} - a + 1\right)\varepsilon - \left(a\bar{a} - \bar{a} + 1\right)(\varepsilon - 1) + a - \bar{a}$$

$$= \left(1 - 2\varepsilon\right)a\bar{a} + (\varepsilon + 1)a + (\varepsilon - 2)\bar{a} + 1 - 2\varepsilon$$

$$= \left(1 - 2\varepsilon\right)\left(a\bar{a} - \bar{\varepsilon}a - \varepsilon\bar{a} + \varepsilon\bar{\varepsilon}\right)$$

$$= \left(1 - 2\varepsilon\right)(a - \varepsilon)\left(\bar{a} - \bar{\varepsilon}\right)$$

$$= \left(1 - 2\varepsilon\right)|A_0A_1|^2,$$

and the result follows.

Note that Theorem 12, indirectly provides another proof that $|A_0A_1| = |B_0B_1| = |C_0C_1|$.

Now $\varepsilon = \frac{1}{2}\left(1 \pm i\sqrt{3}\right)$; let us take $\varepsilon = \frac{1}{2}\left(1 + i\sqrt{3}\right)$. Then $\dfrac{a_1 - c_0}{b_0 - c_0} = \varepsilon$, so $\angle B_0C_0A_1 = +60°$, whence $\Delta A_1B_0C_0$ is negatively

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oriented. Further, $i\left(1 - 2\varepsilon\right) = +\sqrt{3}$, so

$$\text{area}\left(A_kB_kC_k\right) - \text{area}\left(A_{k+1}B_{k+1}C_{k+1}\right) = \frac{\sqrt{3}}{4}\left|A_kA_{k-1}\right|^2 .$$

As we saw earlier, A_{k-2} is the mid-point of A_kA_{k-1}, so $\left|A_kA_{k-1}\right| = 2\left|A_{k-1}A_{k-2}\right|$, whence

$$\text{area}\left(A_kB_kC_k\right) - \text{area}\left(A_{k-1}B_{k-1}C_{k-1}\right) = 4^{k-2}\sqrt{3}\left|A_1A_0\right|^2 ,$$

and, summing the geometric progression,

$$\text{area}\left(A_kB_kC_k\right) - \text{area}\left(A_0B_0C_0\right) = \frac{\left(4^k - 1\right)\sqrt{3}}{12}\left|A_1A_0\right|^2 .$$

Thus

$$\text{area}\left(A_0B_0C_0\right) < \text{area}\left(A_1B_1C_1\right) < \ldots < \text{area}\left(A_kB_kC_k\right) < \ldots ,$$

and $\text{area}\left(A_kB_kC_k\right) \to \infty$ as $k \to \infty$, from which one of three things happens:

(i) $\text{area}\left(A_kB_kC_k\right) > 0$ for all k; or

(ii) for some $k' > 0$, $\text{area}\left(A_kB_kC_k\right)$ is negative or positive according as $k < k'$ or $k \geqslant k'$, respectively; or

(iii) same as (ii), except that this time $\text{area}\left(A_k{'}B_k{'}C_k{'}\right) = 0$.

In case (i), $\Delta U_kV_kW_k$ is the outer and ΔUVW is the inner Napoleon triangle of $\Delta A_kB_kC_k$, for all k. Case (ii) is the same as case (i) for $k \geqslant k'$, but the other way around for $k < k'$. Case (iii) is the same as case (ii), except that here $A_{k'}B_{k'}C_{k'}$ is a straight line, not a triangle. See Figure 11, where $\Delta A_0B_0C_0$ is negatively oriented, $A_1B_1C_1$ is a straight line, and $\Delta A_2B_2C_2$ is positively

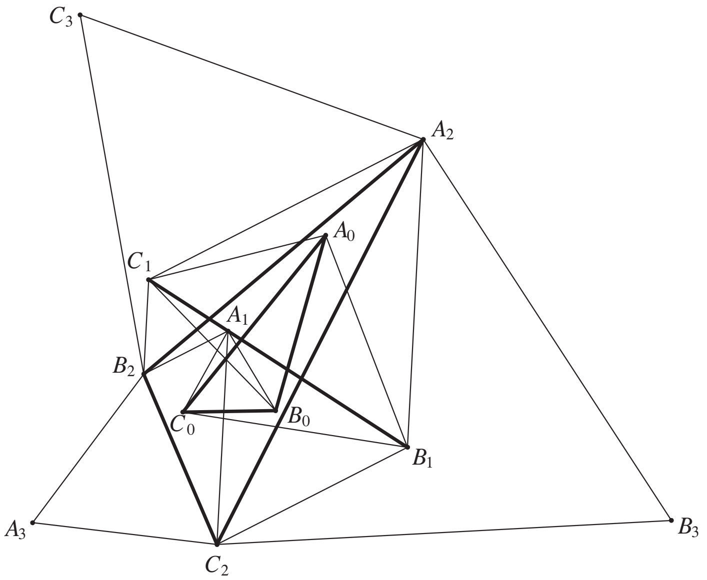

FIGURE 11

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oriented; and the equilateral triangles $A_{k+1}B_kC_k$, $A_kB_{k+1}C_k$ and $A_kB_kC_{k+1}$ are all negatively oriented.

As a final remark, if we fix B_0 and C_0, then $\text{area}(A_1B_1C_1) - \text{area}(A_0B_0C_0)$ can be regarded as a function of A_0, and as such, by Theorem 12, its level curves are concentric circles, centre A_1. Its minimum value is zero, achieved at $A_0 = A_1$, when also $B_0 = B_1$ and $C_0 = C_1$, so that triangles $A_0B_0C_0$ and $A_1B_1C_1$ are the same triangle.

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