A Note about Functions in Lip α

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This note gives a proof of the result:

A necessary and sufficient condition that a trigonometrical series $T(x)$ *be the Fourier series of a function* $f(x) \in \text{Lip}\,\alpha$ $(0 < \alpha < 1)$ *is that*

$$\sigma_n - \sigma_m = O(n^{-\alpha}) \text{ uniformly in } [0, 2\pi] \text{ for all } m > n,$$

where σ_n *is the* n^{th} $(C, 1)$ *mean of* $T(x)$.

Sufficiency. We have $\sigma_n - \sigma_m \to 0$ uniformly as $m, n \to \infty$ and thus there is an f such that $\sigma_n \to f$ uniformly as $n \to \infty$. Furthermore, since $\sigma_n - \sigma_m = O(n^{-\alpha})$ uniformly $(m > n)$, we have, on letting $m \to \infty$, $\sigma_n - f = O(n^{-\alpha})$ uniformly and so $f \in \text{Lip}\,\alpha$. This last follows from a theorem by de la Vallée Poussin which states that if there is a sequence of trigonometric polynomials $\{T_n(x)\}$ such that $f(x) - T_n(x) = O(n^{-\alpha})$ uniformly in $[0, 2\pi]$ then[1] $f \in \text{Lip}\,\alpha$. Since $\sigma_n \to f$ uniformly,

$$\int_0^{2\pi} f \begin{matrix}\cos \\ \sin\end{matrix} kx\,dx = \lim_{n \to \infty} \int_0^{2\pi} \sigma_n \begin{matrix}\cos \\ \sin\end{matrix} kx\,dx,$$

and it follows that the Fourier coefficients of f are the coefficients n $T(x)$.

Necessity. We have $f \in \text{Lip}\,\alpha$ and must show that $\sigma_n - \sigma_m = O(n^{-\alpha})$ for all $m \geqq n$ where $\{\sigma_n\}$ are the $(C, 1)$ means of the Fourier series of f. Now $\sigma_n - f = O(n^{-\alpha})$, by a theorem due to S. Bernstein[2] and so

$$|\sigma_n - \sigma_m| \leqq |\sigma_n - f| + |\sigma_m - f| = O(n^{-\alpha}) + O(m^{-\alpha}) = O(n^{-\alpha}).$$

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[1] C. de la Vallée Poussin, *Leçons sur l'approximation des fonctions* (Paris, 1919), § 41.

[2] Zygmund, *Trigonometrical Series* (1935), p. 62, No. 7.

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