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SOME RESULTS ON WEAK COVERING CONDITIONS

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1. Introduction. A space **X** is called *countably metacompact* (*countably* paracompact) if every countable open cover has a point finite (locally finite) open refinement. According to Hodel [**5**], a space X is called *countably subparacompact* if every countable open cover has a σ-discrete closed refinement. It is well-known (see Mansfield [**10**] and Dowker [**4**]) that in normal spaces all of the preceding notions are equivalent. Also, according to Hodel [**5**], a countably subparacompact space is countably metacompact and the reverse implication is false.

In Section 2 we define in a natural way the concept of a countably θ-refinable space and show that these spaces turn out to be exactly the countably metacompact spaces. In Section 3 we discuss $w\Delta$-spaces and show that every $w\Delta$-space is countably metacompact but not necessarily countably paracompact nor countably subparacompact. This result is compared with Ishii's result on wM-spaces in [**7**]. Finally, in Section 4 we give a new characterization of countably subparacompact spaces using σ-cushioned refinements.

Unless otherwise stated, no separation axioms are assumed; however normal spaces are assumed to be T_1. The set of positive integers is denoted by N.

2. Countably θ-refinable spaces. Let $\mathscr{U}$ be a collection in a space X and let $x \in X$. We mean by $\operatorname{ord}(x, \mathscr{U})$, the number of members of $\mathscr{U}$ which contain x.

A space X is *θ-refinable* [**13**] if, for every open cover $\mathscr{U}$ of X, there is a sequence $\{\mathscr{G}_n : n \in N\}$ of open refinements of $\mathscr{U}$ such that, if $x \in X$, there is an $n(x) \in N$ such that $\operatorname{ord}(x, \mathscr{G}_{n(x)})$ is finite. Such a sequence is called a *θ-refinement of $\mathscr{U}$*.

Definition 2.1. A space X is called *countably θ-refinable* if every countable open cover has a θ-refinement.

Clearly every countably metacompact space is countably θ-refinable and, as the following result shows, the reverse implication is also valid.

THEOREM 2.2. *For a space X, the following conditions are equivalent:*

(a) *X is countably metacompact.*

(b) *X is countably θ-refinable.*

(c) *If $\{F_n : n \in N\}$ is a decreasing sequence of closed subsets of X with*

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$\bigcap_{n=1}^{\infty} F_n = \emptyset$, there is a sequence $\{G_n : n \in N\}$ of G_δ-sets in X such that $G_n \supset F_n$ for all $n \in N$ and $\bigcap_{n=1}^{\infty} G_n = \emptyset$.

(d) *If $\{F_n : n \in N\}$ is a decreasing sequence of closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, there is a sequence $\{U_n : n \in N\}$ of open sets in X such that $U_n \supset F_n$ for all $n \in N$ and $\bigcap_{n=1}^{\infty} U_n = \emptyset$.*

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). Suppose X is countably θ-refinable and let $\{F_n : n \in N\}$ be a decreasing sequence of closed sets such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Let $\mathscr{U} = \{X - F_n : n \in N\}$. Then $\mathscr{U}$ is a countable open cover of X and hence has a θ-refinement $\{\mathscr{V}_n : n \in N\}$. Put $G_{nj} = \mathrm{St}(F_n, \mathscr{V}_j\}$ and $G_n = \bigcap_{j=1}^{\infty} G_{nj}$. Then each G_n is a G_δ-set and $G_n \supset F_n$. We assert that $\bigcap_{n=1}^{\infty} G_n = \emptyset$. If not, there is an $x \in X$ such that $x \in \bigcap_{n=1}^{\infty} G_n$. Choose j_0 such that $\mathrm{ord}(x, \mathscr{V}_{j_0}) < \infty$, say $\mathrm{ord}(x, \mathscr{V}_{j_0}) = k$. Then there are sets $V_1, \ldots, V_k \in \mathscr{V}_{j_0}$ such that $x \in V_i$ and $x \notin V \in \mathscr{V}_{j_0}$ for $V \neq V_i$, $i = 1, 2, \ldots k$. Now, for each i, there is an $n_i \in N$ such that $V_i \subset X - F_{n_i}$. If we put $n = \max\{n_1, \ldots, n_k\}$, then $V_i \subset X - F_n$ for $i = 1, 2, \ldots, k$. But $x \in G_{nj_0}$ and thus there exists a $V \in \mathscr{V}_{j_0}$ with $x \in V$ and $V \cap F_n \neq \emptyset$. Since $x \in V$, $V = V_i$ for some $i = 1, \ldots, k$ and thus $V \subset X - F_n$ which is a contradiction.

(c) $\Rightarrow$ (d). Let $\{F_n : n \in N\}$ be a decreasing sequence of closed subsets of X. By (b) there is a sequence $\{G_n : n \in N\}$ of G_δ-sets satisfying $G_n \supset F_n$ for all $n \in N$ and $\bigcap_{n=1}^{\infty} G_n = \emptyset$. Put $G_n = \bigcap_{j=1}^{\infty} G_{nj}$ where each G_{nj} is open in X. For $n \geqq 1$, define

$$U_n = \bigcap \{G_{ij} : 1 \leqq i \leqq n \text{ and } 1 \leqq j \leqq n\}.$$

Then clearly each U_n is open, $U_n \supset F_n$ and $\bigcap_{n=1}^{\infty} U_n = \emptyset$.

(d) $\Rightarrow$ (a) is due to Ishikawa [**8**].

Since every θ-refinable space is countably θ-refinable, we have:

COROLLARY 2.3. *Every θ-refinable space is countably metacompact.*

It is interesting to note that although the concepts of θ-refinability and metacompactness are equivalent when restricted to countable open covers this equivalence does not hold in general. Clearly every metacompact space is θ-refinable but there are many examples of non-metacompact, θ-refinable spaces. In fact, Bing's Example H is a normal subparacompact space (and thus a θ-refinable space) which is not metacompact. This example was noted by Burke in [**3**].

3. $w\Delta$-spaces and weak covering conditions. Let X be a space and $\{\mathscr{U}_n : n \in N\}$ a sequence of open covers of X subject to one of the following conditions:

(A) If $x_n \in \mathrm{St}(x, \mathscr{U}_n)$ for $n = 1, 2, \ldots$, then the sequence $\langle x_n \rangle$ has a cluster point.

(B) If $x_n \in \mathrm{St}^2(x, \mathscr{U}_n)$ for $n = 1, 2, \ldots,$ then the sequence $\langle x_n \rangle$ has a cluster point.

(C) If $x_n \in \mathrm{St}(x, \mathscr{U}_n)$ for $n = 1, 2, \ldots,$ then x is a cluster point of the sequence $\langle x_n \rangle$.

A space is called a *$w\Delta$-space* [**1**] if it satisfies (A) and a *wM-space* [**7**] if it satisfies (B). Clearly (C) is an equivalent formulation of developable spaces. It is immediate that every wM-space and every developable space is a $w\Delta$-space.

In [**7**] Ishii proved the following:

THEOREM 3.1. (1) *Every wM-space is countably paracompact.*

(2) *Every normal wM-space is collectionwise normal and countably paracompact.*

In this section we show that a $w\Delta$-space is countably metacompact but not necessarily countably paracompact. In light of Theorem 3.1 the following question seems interesting.

Question 3.2. Is there an example of a normal $w\Delta$-space which is not collectionwise normal?

THEOREM 3.3. *Every $w\Delta$-space is countably metacompact.*

Proof. Let $\{\mathscr{U}_n : n \in N\}$ be a sequence of open covers of X satisfying condition (A). We may assume $\mathscr{U}_{n+1} < \mathscr{U}_n$ for all $n \in N$. Let $\{F_n : n \in N\}$ be a decreasing collection of closed subsets of x such that $\bigcap_{n=1}^\infty F_n = \emptyset$. For each $n \in N$, put $G_n = \mathrm{St}(F_n, \mathscr{U}_n)$. Now clearly $G_n \supset F_n$ and each G_n is open in X. By Theorem 2.2 (d), we need only show $\bigcap_{n=1}^\infty G_n = \emptyset$. So assume there is an $x \in X$ such that $x \in \bigcap_{n=1}^\infty G_n$. But then, for each $n \in N$, there exists $U_n \in \mathscr{U}_n$ such that $x \in U_n$ and $U_n \cap F_n \neq \emptyset$. For each n, choose $x_n \in U_n \cap F_n$. Then $x_n \in \mathrm{St}(x, \mathscr{U}_n)$ and thus the sequence $\langle x_n \rangle$ has a cluster point x_0. But $x_0 \in \bigcap_{n=1}^\infty F_n$ and this is a contradiction.

We remark that the referee has informed us that Hodel [**6**] has recently shown that every β-space is countably metacompact. Clearly every $w\Delta$-space is a β-space.

Since, as was noted in the introduction, every normal countably metacompact space is countably paracompact, we have:

COROLLARY 3.4. *Every normal $w\Delta$-space is countably paracompact.*

Example 3.5. A $w\Delta$-space which is not countably paracompact:

Let ω be the first infinite ordinal and Ω the first uncountable ordinal. Let

$$X = [0, \omega] \times [0, \Omega] - (\omega, \Omega).$$

Ishii showed that X is a $w\Delta$-space; but Shiraki [**12**] proved that X is not countably paracompact. We also note that, according to Theorem 3.1, X is not a wM-space.

Example 3.6. A countably compact T_2-space (thus both a wM-space and a $w\Delta$-space) which is not countably subparacompact: Let $R = [0, \Omega]$, $S =$

$[0, \Omega)$ and $X = R \times S$. Then X is clearly a countably compact T_2-space; Kramer [9] has shown that this space is not countably subparacompact. In fact, if we let $H = \{(x, \Omega) : x \in S\}$ and $K = \{(x, x) : x \in S\}$ we have disjoint closed subsets of X. It can be shown that $\{X - H, X - K\}$ is an open cover which has no countable closed refinement. It follows from Theorem 4.1 (iv) that X is not countably subparacompact.

4. A new characterization of countably subparacompact spaces.

Let $\mathscr{B}$ be a cover of a space X. A cover $\mathscr{U}$ is said to be a *cushioned refinement* of $\mathscr{B}$ if to each $U \in \mathscr{U}$ we can assign a $B(U) \in \mathscr{B}$ such that

$$\overline{\bigcup\{U : U \in \mathscr{U}'\}} \subset \bigcup\{B(U) : U \in \mathscr{U}'\}$$

for every subcollection $\mathscr{U}'$ of $\mathscr{U}$.

In [3] Burke asks the following question: Is X subparacompact if every open cover of X has a σ-cushioned refinement? Although this question seems to remain open we show the corresponding result holds for countably subparacompact spaces.

THEOREM 4.1. *For a space X, the following are equivalent.*

(i) *Every countable open cover of X has a σ-discrete closed refinement* (i.e., X is countably subparacompact).

(ii) *Every countable open cover of X has a σ-locally finite closed refinement.*

(iii) *Every countable open cover of X has a σ-closure preserving closed refinement.*

(iv) *Every countable open cover of X has a countable closed refinement.*

(v) *Every countable open cover of X has a σ-cushioned refinement.*

Proof. That (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is obvious, as is (iv) $\Rightarrow$ (i). Thus it suffices to show that (v) $\Rightarrow$ (iv). Suppose $\mathscr{U} = \{U_n : n = 1, 2, \ldots\}$ is any countable open cover of X and let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$ be a σ-cushioned refinement of $\mathscr{U}$. Then, for each n, there exists a mapping $\phi_n : \mathscr{F}_n \rightarrow \mathscr{U}$ such that if $F \in \mathscr{F}_n$, $\phi_n(F) \in \mathscr{U}$, $F \subset \phi_n(F)$ and

$$\overline{\bigcup\{F : F \in \mathscr{F}_n'\}} \subset \bigcup\{\phi_n(F) : F \in \mathscr{F}_n'\}$$

for any $\mathscr{F}_n' \subset \mathscr{F}_n$. Define

$$G_{ij} = \bigcup\{F : F \in \mathscr{F}_i, \phi_i(F) = U_j\}.$$

Since $\mathscr{F}_i$ is a cushioned refinement, $\overline{G_{ij}} \subset U_j$. Thus $\{\overline{G_{ij}} : i = 1, 2, \ldots, j = 1, 2, \ldots\}$ is a countable closed refinement of $\mathscr{U}$ and the proof is complete.

We remark that the equivalence of (i) - (iv) is not new although (v) seems to be a new characterization. In fact, the equivalence of (i) - (iii) appears in [11] and the equivalence of (i) - (iv) is stated in [9].

Definition 4.2. A space X is called *countably σ-paracompact* if given a countable open cover $\mathscr{U}$ of X, there is a sequence $\{\mathscr{U}_n : n \in N\}$ of open covers of X such that, if $x \in X$, there is an $n(x) \in N$ and $U \in \mathscr{U}$ with $\mathrm{St}(x, \mathscr{U}_{n(x)}) \subset U$.

Kramer [**9**] introduced countably σ-paracompact spaces and proved the following.

THEOREM 4.3. *A space X is countably subparacompact if and only if X is countably σ-paracompact.*

It is worth noting that Burke [**2**] obtained the equivalence of (i), (ii), (iii) and Definition 4.2 for arbitrary open covers (i.e. for subparacompact spaces).

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