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A Truncated Integral of the Poisson Summation Formula

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Abstract. Let G be a reductive algebraic group defined over $\mathbb{Q}$, with anisotropic centre. Given a rational action of G on a finite-dimensional vector space V, we analyze the truncated integral of the theta series corresponding to a Schwartz-Bruhat function on $V(\mathbb{A})$. The Poisson summation formula then yields an identity of distributions on $V(\mathbb{A})$. The truncation used is due to Arthur.

0 Introduction

This paper is an extension of the previous paper [8]. In these two papers we extend some of the results about integrating the Poisson summation formula in Weil's famous paper [13], using methods developed by Arthur to deal with infinities in the trace formula. We will use many results from the paper [8], and that paper provides good preparation for the complicated geometric constructions in Section 3 and for the analysis in Section 4.

Suppose that V is a finite-dimensional vector space and that f is a Schwartz function on $V(\mathbb{A})$, the rational adelic points of V. The Poisson summation formula tells us that

$$\sum_{\gamma\in V(\mathbb{Q})} f(\gamma) = \sum_{\hat{\gamma}\in\hat{V}(\mathbb{Q})} \hat{f}(\hat{\gamma}),$$

where $\hat{V}$ is the vector space dual to V and $\hat{f}$ is the Fourier transform of f. Now suppose that a reductive algebraic group G defined over $\mathbb{Q}$ acts on V via a rational representation $\pi\colon G\to \mathrm{GL}(V)$. The Poisson summation formula can now be used to show that for every element g of $G(\mathbb{A})$,

$$\sum_{\gamma\in V(\mathbb{Q})} f\big(\pi(g^{-1})\gamma\big) = |\det\pi(g)| \sum_{\hat{\gamma}\in\hat{V}(\mathbb{Q})} \hat{f}\big(\tilde{\pi}(g^{-1})\hat{\gamma}\big)\,, \tag{0.1}$$

with $\tilde{\pi}\colon G\to\mathrm{GL}(\hat{V})$ the representation of G contragredient to π. Define the function $\phi_{f,\pi}$ on $G(\mathbb{A})$ by

$$\phi_{f,\pi}(g) = \sum_{\gamma\in V(\mathbb{Q})} f\big(\pi(g^{-1})\gamma\big)\,;$$

then the equality (0.1) gives a relation between the two functions $\phi_{f,\pi}$ and $\phi_{\hat{f},\tilde{\pi}}$ on $G(\mathbb{A})$. Notice that for any f and π, the function $\phi_{f,\pi}$ is left $G(\mathbb{Q})$-invariant.

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Weil [13] noticed that when the dimension of the representation is small compared to the rank of the group, the function $\phi_{f,\pi}$ is integrable. When this occurs, we may integrate the function $\phi_{f,\pi}$ over $G(\mathbb{Q}) \setminus G(\mathbb{A})$ and produce an equality of two $G(\mathbb{A})$-invariant distributions through the integration of the equality (0.1). One then easily obtains (see [8]) an equality of sums of orbital integrals on V and on $\hat{V}$. This equality is the basis of the Siegel-Weil formula. For general representations, however, the function $\phi_{f,\pi}$ will not be integrable over $G(\mathbb{Q}) \setminus G(\mathbb{A})$ and a more refined approach must be used. We must somehow regularize the functions $\phi_{f,\pi}$ and $\phi_{\hat{f},\tilde{\pi}}$ before integrating.

The regularization that we use is a truncation invented by Arthur, and described in [8]. The important properties of this truncation are reviewed in the next section. The problem is then to determine the behaviour of the integral of the truncation of $\phi_{f,\pi}$ as a function of the truncation parameter T. In the previous paper we dealt with reductive algebraic groups with rational rank at most two and with anisotropic centre. In this paper we deal with arbitrary reductive algebraic groups with anisotropic centre. Given a general reductive algebraic group G, the algebraic subgroup G^1 defined as the kernel of all rational characters of G is reductive with anisotropic centre, and G is the product of G^1 and the maximal split torus Z in the centre of G. The action of Z can be introduced by making a Shintani zeta function, but we will not discuss this further in this paper. See [14] for more about Shintani zeta functions.

We prove that the positive Weyl chamber is a finite union of sub-cones depending only of π satisfying the following property: within each sub-cone, the integral of the truncation of $\phi_{f,\pi}$ with respect to the point T asymptotically approaches the value of a finite sum of products of polynomials in T and exponentials of linear functionals of T. These functions do depend on the sub-cone.

Given a sub-cone $\mathcal{C}$, write $J_{\mathcal{C}}(f,\pi)$ for the constant term of the analytic function of the previous paragraph (see Section 4 for our definition of constant term). Then the basic form of the truncated Poisson summation formula is the statement

$$J_{\mathcal{C}}(f,\pi) = J_{\mathcal{C}}(\hat{f},\tilde{\pi}).$$

We can do better than this, as in the Selberg-Arthur trace formula, by breaking up both sides as sums over certain equivalence classes in the respective vector spaces; this is the way we present the material.

Notice that we obtain several truncated formulas this way. Because our proof does not explicitly produce the numbers $J_{\mathcal{C}}(f,\pi)$, it is not clear whether they are actually distinct for different sub-cones $\mathcal{C}$.

The truncated Poisson summation formula developed here is potentially useful both for producing a "regularized" Siegel-Weil theorem, extending results from [13] (see [7] for a different approach to this), and for new results about Shintani zeta functions.

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1 Preliminaries

Let G be a reductive algebraic group over $\mathbb{Q}$ with anisotropic centre. Let P_0 be a minimal parabolic subgroup of G, and M a Levi component and N the unipotent radical of P_0, all defined over $\mathbb{Q}$. These will be fixed throughout the paper. Write A for the maximal $\mathbb{Q}$-split torus of G contained in M and $\mathfrak{a}$ for the corresponding real vector space $\operatorname{Hom}\big(X(A)_{\mathbb{Q}}, \mathbb{R}\big)$. We will use the phrase "parabolic subgroup" to denote a standard parabolic subgroup, that is, a parabolic subgroup P of G defined over $\mathbb{Q}$ and containing P_0. Given a parabolic subgroup P we can define N_P, its unipotent radical, M_P the unique Levi component of P containing M, and A_P, the split component of the centre of M_P. We will use standard notation (see [8] or [1]) for roots, weights, and so forth. In particular, $\mathfrak{a}^+$ denotes the points in $\mathfrak{a}$ where all positive roots are positive.

Fix a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ that is admissible relative to M, as in [2]. With this choice we can define as in [1] a continuous map $H\colon G(\mathbb{A}) \to \mathfrak{a}$ invariant under right multiplication by K. Given a parabolic subgroup $P \subseteq G$, write H_P for the projection of H to $\mathfrak{a}_P$. Then H_P is left $P(\mathbb{Q})$-invariant, and so can be seen as a map from $P(\mathbb{Q}) \setminus G(\mathbb{A})$ to $\mathfrak{a}_P$.

Given $P \subseteq Q$ parabolic subgroups of G and X a point or subset of $\mathfrak{a}$ [resp. $\mathfrak{a}^*$], write X_P^Q for the projection of X to the subspace $\mathfrak{a}_P^Q$ of $\mathfrak{a}$ [resp. ${\mathfrak{a}^*}_P^Q$ of $\mathfrak{a}^*$], and $\exp(X_P^Q)$ for the pre-image of X_P^Q in $A_P^Q(\mathbb{R})^0$ under the map H_P. To simplify notation, we will remove all occurrences of P_0 as a subscript; for example the Levi component of P_0 is $M = M_{P_0}$. Notice that since the centre of G is anisotropic, $\mathfrak{a}_G = \mathbf{0}$ and so we can also eliminate all occurrences of G as a superscript. When it will cause no confusion we may also simplify expressions with subscripts and superscripts such as P_1 as follows: we will write $\mathfrak{a}_1^2$, for example, for $\mathfrak{a}_{P_1}^{P_2}$ where $P_1 \subseteq P_2$ are parabolic subgroups.

In the previous paper [8] we proved a result relating certain functions on quotients of $G(\mathbb{A})$ and certain functions on $\mathfrak{a}$. This result is the key geometric principle that will allow us to produce the truncated Poisson summation formula.

We first recall the functions involved. We fix throughout this paper a point T_1 in $-\mathfrak{a}^+$ and a compact subset ω of $N(\mathbb{A})M(\mathbb{A})^1$ such that

$$G(\mathbb{A}) = P(\mathbb{Q})\big\{\, pak \mid p \in \omega, a \in A(\mathbb{R})^0, k \in K, \alpha\big(H(a) - T_1\big) > 0 \text{ for all } \alpha \in \Delta^P \big\}$$

for every parabolic subgroup $P \subseteq G$. Then for a given $T \in \mathfrak{a}^+$, write $F^P(\cdot, T)$ for the characteristic function of the relatively compact set of all points $g \in P(\mathbb{Q}) \setminus G(\mathbb{A})$ with a representative in the set

$$\big\{\, pak \mid p \in \omega, a \in A(\mathbb{R})^0, k \in K, \alpha\big(H(a) - T_1\big) > 0 \text{ for all } \alpha \in \Delta^P,$$
$$\varpi\big(H(a) - T\big) \le 0 \text{ for all } \varpi \in \widehat{\Delta}^P \big\}.$$

Given $T \in \mathfrak{a}^+$, we also define $\Gamma(\cdot, T)$ by

$$\Gamma(X, T) = \sum_{R : P \subseteq R \subseteq Q} (-1)^{\dim(A_R/A_Q)} \tau_P^R(X) \hat{\tau}_R^Q(T - X), \quad X \in \mathfrak{a},$$

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and for parabolic subgroups $P \subseteq Q$ we define $\Gamma_P^Q(\cdot, T)$ by $\Gamma_P^Q(X, T) = \Gamma(X_P^Q, T_P^Q)$, $X \in \mathfrak{a}$. Lemma 3.4 of [8] states that the function $X \mapsto \Gamma_P^Q(X, T)$, $X \in \mathfrak{a}_P^Q$, is the characteristic function of a convex set whose closure is the convex hull of the points T_R, $P \subseteq R \subseteq Q$.

A relation between the functions F and Γ is given in Corollary 3.3 of [8] and runs as follows: Let $T_2 \in \mathfrak{a}^+$ be a fixed sufficiently regular point. Then for $T \in T_2 + \mathfrak{a}^+$, $S \in \mathfrak{a}^+$, and $g \in G(\mathbb{Q}) \setminus G(\mathbb{A})$,

(1.1)

$$F^G(g, T+S) - F^G(g, T) = \sum_{P \subseteq Q \subsetneq G} \sum_{\delta \in P(\mathbb{Q}) \setminus G(\mathbb{Q})} F^P(\delta g, T_2) \Gamma_P^Q\big(H_P(\delta g) - T_2, T - T_2\big) \Gamma_Q\big(H_Q(\delta g) - T, S\big).$$

Let π be a rational representation of G on a finite-dimensional vector space V. The following definitions were introduced in [8] and require only that G be a reductive algebraic group defined over $\mathbb{Q}$.

Define a *semisimple vector* in V to be one whose geometric orbit $\pi\big(G(\overline{\mathbb{Q}})\big)\gamma \subset V(\overline{\mathbb{Q}})$ is Zariski closed, and define a *nilpotent vector* in V to be one such that the origin is contained in the Zariski closure of its geometric orbit. If the representation π is the Adjoint representation of G on its Lie algebra $\mathfrak{g}$, then these definitions give the standard notions of semisimple and nilpotent elements of $\mathfrak{g}$. Since the properties of being semisimple and nilpotent are both invariant under the action of $G(\overline{\mathbb{Q}})$, we will call an orbit, either rational or geometric, *semisimple* or *nilpotent* if its elements are.

Let γ be an element of $V(\mathbb{Q})$. A standard argument in invariant theory shows that the Zariski closure of the geometric orbit $\pi\big(G(\overline{\mathbb{Q}})\big)\gamma$ of γ contains a unique closed geometric orbit in $V(\overline{\mathbb{Q}})$. The set of rational points in this closed $G(\overline{\mathbb{Q}})$-orbit is a union of semisimple $G(\mathbb{Q})$-orbits in $V(\mathbb{Q})$—a non-empty union by the rational Hilbert-Mumford Theorem [6]. We define the *geometric semisimple component* of the element $\gamma \in V(\mathbb{Q})$ to be this union. If the representation π is the Adjoint representation then the geometric semisimple component of an arbitrary element $\gamma \in V(\mathbb{Q})$ is the set of rational points in the geometric orbit of the semisimple part γ_s of γ given by the Jordan decomposition, which may be larger than the $G(\mathbb{Q})$-orbit of γ_s. Define a *geometric equivalence class* to be the set of elements in $V(\mathbb{Q})$ whose geometric semisimple component equals a given closed geometric orbit.

Remark For general representations, there is no known natural assignment of a single semisimple vector to each vector, extending the assignment to an element of its Lie algebra to its semisimple component in the Jordan decomposition, and such an assignment is unlikely to exist. (See [12] for an analysis of the case of direct sums of the Adjoint representation.) However, it may be possible to identify a canonical rational (rather than geometric, as done above) semisimple orbit to a given vector, by the following procedure, and that would be sufficient for our purposes. Define the Hilbert-Mumford closure of an orbit $\pi\big(G(\mathbb{Q})\big)\gamma$, $\gamma \in V(\mathbb{Q})$, to be the set of points in $V(\mathbb{Q})$ that can be expressed as

$$\lim_{t\to 0}\pi\big(p(t)g\big)\,\gamma$$

for some homomorphism $p\colon \mathbb{G}_m \to G$ defined over $\mathbb{Q}$ and some g in $G(\mathbb{Q})$. Luna's property (A) (see [10]) asserts that the topological closure of every orbit $\pi\big(G(\mathbb{R})\big)\,v$ of a point $v \in V(\mathbb{R})$ contains a unique $G(\mathbb{R})$ orbit of semisimple elements. Together with the rational Hilbert-Mumford theorem [6], property (A) implies that when the base field is $\mathbb{R}$ the Hilbert-Mumford closure of the rational orbit of a vector contains a unique rational semisimple orbit. This latter property trivially holds over algebraically closed fields, and is also known to hold over p-adic fields [11], but appears to be unknown in general over $\mathbb{Q}$. Some progress towards this question can be found in [9]. If it were true, then we would define the *semisimple component* of γ to be this unique semisimple orbit, and the *equivalence class* of γ to be the set of elements in $V(\mathbb{Q})$ whose semisimple component equals the semisimple component of γ. We note that if the Hilbert-Mumford closure of every $G(\mathbb{Q})$-orbit contains a unique semisimple orbit, then all the following results hold with equivalence classes rather than geometric equivalence classes.

The key property of the geometric equivalence classes is the following lemma, which generalizes Lemma 2.1 of [8] and is easily proven.

Lemma 1.1 *Let G be a reductive algebraic group defined over $\mathbb{Q}$, let π be a rational representation of G on a finite-dimensional vector space V, and let A be a torus in G that is split over $\mathbb{Q}$. Suppose that Λ_0 and Λ_+ are two sets of rational characters on A such that there exists a point $a \in A(\mathbb{Q})$ satisfying*

$$|\lambda(a)| = 1 \quad \text{for every } \lambda \in \Lambda_0,$$
$$|\lambda(a)| > 1 \quad \text{for every } \lambda \in \Lambda_+.$$

Write

$$V_0 = \bigoplus_{\lambda\in\Lambda_0} V^{\lambda}, \quad V_+ = \bigoplus_{\lambda\in\Lambda_+} V^{\lambda},$$

with V^{λ} the weight space in V corresponding to λ. Then for every geometric equivalence class $\mathfrak{o}$ in $V(\mathbb{Q})$ and subset $\mathcal{S}$ of $V_0(\mathbb{Q})$,

$$\big(S + V_+(\mathbb{Q})\big) \cap \mathfrak{o} = (\mathcal{S} \cap \mathfrak{o}) + V_+(\mathbb{Q}).$$

Recall that the weight space in V corresponding to $\lambda \in X^*(A)_{\mathbb{Q}}$ is the vector subspace $\{v \in V \mid \pi(a)v = \lambda(a)v \text{ for all } a \in A\}$ of V.

Let $\mathfrak{o}$ be a geometric equivalence class in V, and define the function

$$\phi_{\pi,\mathfrak{o}}(g, f) = \sum_{\gamma\in\mathfrak{o}} f\big(\pi(g^{-1})\gamma\big), \quad g \in G(\mathbb{A}),$$

a left $G(\mathbb{Q})$-invariant function of $G(\mathbb{A})$. We will eliminate the subscript π when the representation is understood. The Poisson summation formula implies that

$$\sum_{\mathfrak{o}\in\mathfrak{O}} \phi_{\pi,\mathfrak{o}}(g, f) = \sum_{\hat{\mathfrak{o}}\in\hat{\mathfrak{O}}} \phi_{\hat{\pi},\hat{\mathfrak{o}}}(g, \hat{f}), \tag{1.2}$$

where $\mathfrak{O}$ and $\tilde{\mathfrak{O}}$ are the collections of geometric equivalence classes with respect to the representations π and $\tilde{\pi}$, respectively—recall that we are assuming that the centre of G is anisotropic, so $|\det \pi(g)| = 1$.

Given a Schwartz-Bruhat function f on $V(\mathbb{A})$, a point $T \in \mathfrak{a}^+$, and a geometric equivalence class $\mathfrak{o} \in \mathfrak{O}$, define

$$J_{\mathfrak{o}}^T(f,\pi) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} F^G(g,T)\phi_{\pi,\mathfrak{o}}(g,f)\,dg.$$

As in [8] we will try to find asymptotic formulas for $J_{\mathfrak{o}}^T$ for T in certain cones (to be defined later) in $\mathfrak{a}^+$ by examining differences $J_{\mathfrak{o}}^{T+S}(f,\pi) - J_{\mathfrak{o}}^T(f,\pi)$ where T and S both lie in one cone and $\|S\| \le 1$. As on pages 1392–3 of [8], we can write $J_{\mathfrak{o}}(T+S) - J_{\mathfrak{o}}(T)$ as the finite sum over pairs (P,Q) of parabolic subgroups of G with $P \subseteq Q \subsetneq G$ of

$$\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \int_{A_P(\mathbb{R})^0} e^{-2\rho_P(H_P(a))}\phi_{\mathfrak{o}}(na, f^{M_P,K,T_2}) \times \Gamma_P^Q\big(H_P(a), T - T_2\big)\,\Gamma_Q\big(H_P(a) - T, S\big)\,da\,dn, \tag{1.3}$$

where

$$f^{M_P,K,T_2}(v) = e^{-2\rho_P((T_2)_P^Q)} \int_{\omega_{M_P}} F^P(m,T_2) \int_K f\Big(\pi\Big(\exp\big((T_2)_P^Q\big)\,mk\Big)^{-1} v\Big)\,dk\,dm,$$

with ω_{M_P} a fundamental domain of $M_P(\mathbb{Q}) \setminus M_P(\mathbb{A})^1$ and $T_2 \in \mathfrak{a}^+$ a fixed sufficiently regular point. Estimating each of the integrals (1.3) requires some lengthy preliminary constructions on the vector space V; these constructions of the subject of Section 3.

2 A Result of Brion-Vergne

In this section, we recall a result [4, Theorem 4.2] on Fourier transforms of convex polyhedra and give some consequences. The proof of our main theorem relies on the application of this result to certain polytopes to be constructed in Section 3. Before we can state Brion-Vergne's result, we must introduce some notation.

Suppose that V is a finite-dimensional real vector space. In Section 3, we will take $V = \mathfrak{a}_P$, so will write V^* instead of $\hat{V}$ for the dual of V. Suppose that $\mu_1,\dots,\mu_N$ are elements of V^*. Given $x = (x_1,\dots,x_N) \in \mathbb{R}^N$ we can define the convex polyhedron

$$P(x) := \{v \in V \mid \mu_i(v) + x_i \ge 0, 1 \le i \le N\}. \tag{2.1}$$

Assume that for some $x \in \mathbb{R}^N$, the polyhedron $P(x)$ is non-empty and contains no line; this implies that $\mu_1,\dots,\mu_N$ span V^*. Write $\mathcal{B}$ for the set of subsets σ of $\{1,\dots,N\}$ such that $\{\mu_i \mid i \notin \sigma\}$ is a basis of V^*. Given $\sigma \in \mathcal{B}$ we define the following three objects: $\{u_{i,\sigma}\}_{i\notin\sigma}$ is the basis of V dual to the basis $\{\mu_i \mid i \notin \sigma\}$ of V^*,

$s_\sigma \colon \mathbb{R}^N \to V$ is the linear map sending $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ to the unique point $v \in V$ with $\mu_i(v) + x_i = 0$, $i \notin \sigma$, and $C(\sigma)$ is the set $\{x \in \mathbb{R}^N \mid s_\sigma(x) \in P(x)\}$.

One can verify using an alternative description of $C(\sigma)$—the one given in [4, 3.1, 4.1]—that $C := \bigcup_{\sigma\in\mathcal{B}} C(\sigma)$ is the set of $x \in \mathbb{R}^N$ such that $P(x)$ is non-empty. To any set $\Sigma \subset \mathcal{B}$, write C_Σ for the intersection $\bigcap_{\sigma\in\Sigma} C(\sigma)$, and to any $x \in C$ assign the set $\Sigma_x = \{\sigma \in \mathcal{B} \mid x \in C(\sigma)\}$. Then each set C_{Σ_x}, $x \in C$, is a closed convex cone containing x, and we obtain finitely many cones this way. If we set $\mathcal{B}(\gamma) := \{\sigma \in \mathcal{B} \mid \gamma \subset C(\sigma)\}$ for any such cone γ, then $\gamma = C_{\mathcal{B}(\gamma)}$. If γ is maximal among these cones, we call it a *chamber* of C (in [4] it would be called the closure of a chamber).

The following result is Theorem 4.2 of [4].

Theorem 2.1 *Suppose that V is a finite-dimensional real vector space, that $\mu_1, \dots, \mu_N$ are in V^*, and that at some point in $\mathbb{R}^N$, (2.1) defines a non-empty polyhedron containing no line. Let γ be a chamber in C and x a point in γ.*

(a) The extreme points of $P(x)$ are the $s_\sigma(x)$, $\sigma \in \mathcal{B}(\gamma)$, with possible repetition.
(b) For generic

$$\mu \in \Big\{ \sum_{i=1}^{N} c_i\mu_i \mid c_i \geq 0 \Big\},$$

we have the identity

$$\int_{P(x)} e^{-\mu(v)}\, dv = \sum_{\sigma\in\mathcal{B}(\gamma)} e^{-\mu(s_\sigma(x))} \frac{\mathrm{vol}\{\sum_{i\notin\sigma} t_i u_{i,\sigma} \mid t_i \in [0,1]\}}{\prod_{i\notin\sigma} \mu(u_{i,\sigma})}, \tag{2.2}$$

where dv and vol *denote the same Lebesgue measure on V.*

Remark Notice that if (2.2) holds for a given μ, then the integral, as a function of $x \in \mathbb{R}^N$, is a finite linear combination of exponentials of linear functionals.

We will need to tweak somewhat the statement of this theorem to suit our needs.

Lemma 2.2 *Keep the assumptions in Theorem 2.1, and suppose also that for all $x \in C$, the set $P(x)$ is bounded. Then the set*

$$\Big\{ \sum_{i=1}^{N} c_i\mu_i \mid c_i \geq 0 \Big\}$$

is all of V^*.

Proof This follows from the theory of the polar. It follows immediately from [5, 6.1(a) and 9.1(b).] ■

Recall that the bounded polyhedra are just the polytopes, the convex hulls of finite sets of points.

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Recall from [8] that we call a function f on $\mathbb{R}^N$ *t-finite* if it is expressible (uniquely) as a finite sum $\sum p_\lambda(x)e^{\lambda(x)}$, with $\lambda \in (\mathbb{R}^N)^*$, $p_\lambda \in \mathbb{C}[x_1, \dots, x_N]$.

Lemma 2.3 *Keep the assumptions of Lemma 2.2. Then for any $\mu \in V^*$, the function*

$$x \mapsto F_\mu(x) := \int_{P(x)} e^{\mu(v)}\, dv, \quad x \in \gamma$$

is t-finite. In fact, each polynomial p_λ in its decomposition has degree at most n ($n-1$ for $\lambda \neq 0$).

Proof Let $\mu \in V^*$, and pick $\mu_0 \in V^*$ so that for every $t \in (0,1)$, $\mu + t\mu_0$ is generic. For any $x \in \gamma$ consider the function $t \in \mathbb{R} \mapsto F_{\mu+t\mu_0}(x)$. It is a continuous function whose value at $t = 0$ is $F_\mu(x)$. On the other hand, for $t \in (0,1)$, its value is given by (2.2). Expanding the exponentials in the right-hand side of (2.2) into a power series in t and letting $t \to 0$ gives the desired result. ■

Lemma 2.4 *Keep the assumptions of Lemma 2.2, and let $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ be points in $\mathbb{R}^N$. If for each extreme point v_x of $P(x)$, the set $\{v \in V \mid$ if $\mu_i(v_x) + x_i = 0$, then $\mu_i(v) + y_i = 0$, $i = 1, \dots, N\}$ contains a single point v_y that is extreme in $P(y)$, then $y \in \gamma$.*

Proof Fix $\sigma \in \mathcal{B}(\gamma)$. By Theorem 2.1, $s_\sigma(x)$ is an extreme point of $P(x)$. The extra hypothesis of the Lemma implies that $s_\sigma(y)$ is an extreme point of $P(y)$, in particular that $s_\sigma(y) \in P(y)$, so that $y \in C(\sigma)$. Therefore $y \in C_{\mathcal{B}(\gamma)} = \gamma$. ■

Remarks (1) The map from the extreme points of $P(x)$ to those of $P(y)$ given in the Lemma is surjective, by Theorem 2.1(a).

(2) There is a surjection from the power set of $\{1, \dots, N\}$ to the set of faces of $P(x)$, $x \in \gamma$, sending a set S to the set of points $v \in P(x)$ satisfying

$$\mu_i(v) + x_i = 0, \quad \text{for all } i \in S, \tag{2.3}$$

or equivalently, to the convex hull of those extreme points of $P(x)$ satisfying (2.3). Under the hypotheses of Lemma 2.4, this leads to a surjection from the set of faces of $P(x)$ to the set of faces of $P(y)$. If the hypotheses of the Lemma also hold with x and y reversed, then this map between faces is a bijection.

3 Geometry on $\mathfrak{a}$

As mentioned in the introduction, we will find that the integral $J_0^T(f,\pi)$ does not have simple asymptotic behaviour as T approaches infinity in $\mathfrak{a}^+$, but rather that there exist convex open cones in $\mathfrak{a}^+$ such that the function $J_0^T(f,\pi)$ is asymptotic to a t-finite function in T as T tends to infinity within each cone. Let us first describe these cones.

Let $\Psi_\pi \subset \mathfrak{a}^*$ be the union of the set Δ of simple roots of (G,A) and the set of non-zero weights of π with respect to A. Define a function d on $\mathfrak{a}$ as follows: for

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$X \in \mathfrak{a}$, $d(X)$ is the minimum, over all parabolic subgroups $P \subseteq Q \subsetneq G$ and subsets $\mathcal{S} \subseteq \Psi_{\pi,P}$ such that $\operatorname{span}(\mathcal{S}) \cap \mathfrak{a}_Q^*$ does not equal the trivial subspace $\mathbf{0}$, of the distance

$$\operatorname{dist}\big(\ker \mathcal{S}, \operatorname{cvx}(X_R)_{P\subseteq R\subseteq Q}\big)$$

between the kernel $\ker \mathcal{S} \subset \mathfrak{a}_P$ of the set $\mathcal{S}$ and the convex hull $\operatorname{cvx}(X_R)_{P\subseteq R\subseteq Q} \subset \mathfrak{a}_P \subseteq \mathfrak{a}$ of the projections of X to $\mathfrak{a}_R$, $P \subseteq R \subseteq Q$. Notice that for $t \in \mathbb{R}^+$ and $X \in \mathfrak{a}$, $d(tX) = td(X)$. This function d is the correct analogue to this situation of the functions denoted similarly in [3] and [8].

This definition may appear strange at first glance. The restriction on $\mathcal{S}$ comes about for the following reason: for any parabolic subgroups $P \subseteq Q$, and any $\mathcal{S} \subset \mathfrak{a}_P^*$, if $\operatorname{span} \mathcal{S} \cap \mathfrak{a}_Q^* = \mathbf{0}$ then there must be points in $\ker \mathcal{S} \cap \operatorname{cvx}(X_R)_{P\subseteq R\subseteq Q}$ for any $X \in \mathfrak{a}$. The converse of this holds for any $\mathcal{S} \subset \Psi_{\pi,P}$ if $d(X) \neq 0$. Consider as an example the case where $G = \mathrm{SL}(4)$, π is the irreducible representation of G whose highest weight is three times the sum of the fundamental weights, $P = P_0$, Q satisfies $\Delta^Q = \{\alpha_3\}$, and $\mathcal{S} = \{\alpha_1 - \alpha_3, \alpha_2 - \alpha_3\}$. Then $\operatorname{span} \mathcal{S} \cap \mathfrak{a}_Q^* \neq \mathbf{0}$ since it contains $-3(\alpha_1-\alpha_3)+2(\alpha_2-\alpha_3) = (-3\alpha_1+2\alpha_2)_Q$, and $\operatorname{dist}\big(\ker \mathcal{S}, \operatorname{cvx}(X_R)_{P\subseteq R\subseteq Q}\big)$ measures the distance between the line $\alpha_1 = \alpha_2 = \alpha_3$ and the line segment $\operatorname{cvx}(X, X_Q)$ in $\mathfrak{a} \cong \mathbb{R}^3$.

The set of zeros of d is a finite union of closed convex cones, each corresponding to a single choice of P, Q and $\mathcal{S}$. Each of these cones has positive codimension; this is because $\ker \mathcal{S} \cap \operatorname{cvx}(X_R)_{P\subseteq R\subseteq Q}$ is contained in $\ker \mathcal{S} \cap (X_Q + \mathfrak{a}_P^Q)$ and this set is non-empty only if X_Q lies in $\ker\big((\operatorname{span} \mathcal{S}) \cap \mathfrak{a}_Q^*\big)$, a non-trivial condition by the non-triviality of $(\operatorname{span} \mathcal{S}) \cap \mathfrak{a}_Q^*$. If we choose $P = Q$ and $\mathcal{S}$ to be a singleton $\{\lambda_P\}$, $\lambda \in \Psi_\pi$, the subspace $\ker\big((\operatorname{span} \mathcal{S}) \cap \mathfrak{a}_Q^*\big)$ is the hyperplane $\ker \lambda_P$, so d maps each of these hyperplanes to 0. We will see that the function $J_{\mathfrak{o}}^T(f,\pi)$ is asymptotic to a real-analytic function as T tends to infinity in each convex open cone in the complement in $\mathfrak{a}^+$ of the set of zeros of d.

Notice that the above facts about the zeros of d trivially imply that the complement to $d = 0$ in $\mathfrak{a}^+$ can be expressed as a finite union of convex open cones that may intersect. Call the cones appearing in such a decomposition the π-dependent cones in $\mathfrak{a}^+$. The asymptotics of the function $J_{\mathfrak{o}}^T(f,\pi)$ as T varies in the different π-dependent cones in $\mathfrak{a}^+$ will in general be different; it seems likely however that their constant terms (when defined correctly, see the comments before Theorem 4.5) will often be the same. If π is the Adjoint representation, then d is the function given in [3], and there is only one π-dependent cone, namely the whole of $\mathfrak{a}^+$.

Remark The complement of the set of zeros of d in $\mathfrak{a}^+$ decomposes as a finite union of *disjoint* convex open cones in $\mathfrak{a}^+$. We will not require this fact in the paper, so we will not prove it here.

Choose a non-empty convex open cone in the complement in $\mathfrak{a}^+$ of the set of zeros of d and call it $\mathcal{C}$. The cone $\mathcal{C}$ will be fixed throughout this section. Given $\varepsilon > 0$, we write $\mathcal{C}_\varepsilon$ for the set of X in $\mathcal{C}$ satisfying $d(X) > \varepsilon\|X\|$. Fix for the remainder of this section $\varepsilon > 0$ sufficiently small that $\mathcal{C}_\varepsilon$ is a non-empty convex open cone in $\mathfrak{a}^+$. We also write $\mathcal{C}_\varepsilon(1)$ for the set of $X \in \mathcal{C}_\varepsilon$ with $\|X\| \leq 1$.

The inner integral in (1.3) is over the set of $a \in A_P(\mathbb{R})^0$ such that

$$\Gamma_P^Q\big(H_P(a), T - T_2\big)\,\Gamma_Q\big(H_P(a) - T, S\big) = 1,$$

that is, $H_P(a)$ lies in a polytope in real Euclidean space $\mathfrak{a}_P$. The key to estimating (1.3) is breaking up this polytope into pieces on which the integral is much easier to evaluate.

Fix for the remainder of this section two parabolic subgroups $P \subseteq Q \subsetneq G$. Define $R_P^Q(T, S)$ as the support of the function $X \mapsto \Gamma_P^Q(X, T)\Gamma_Q(X - T, S)$, $X \in \mathfrak{a}_P$, so that the integral in (1.3) can be taken over a with $H_P(a) \in R_P^Q\big(T - (T_2)_P^Q, S\big)$. We will break up $R_P^Q(T, S)$ in a complicated way depending on the hyperplanes $\ker\lambda$, $\lambda \in \Psi_{\pi,P}$ that intersect it.

For $T \in \mathfrak{a}^+$, define ${R'}_P^Q(T) \subset T_Q + \mathfrak{a}_P^Q$ as the convex hull $\operatorname{cvx}(T_R)_{P\subseteq R\subseteq Q}$ of the projections of T to $\mathfrak{a}_R$; this is the support in $T_Q + \mathfrak{a}_P^Q$ of the function $X \mapsto \Gamma_P(X, T)$. Then for $T, S \in \mathfrak{a}^+$,

$$R_P^Q(T, S) = {R'}_P^Q(T) + {R'}_Q(S) \supseteq {R'}_P^Q(T),$$

so if $\|S\| \leq 1$, every point in $R_P^Q(T, S)$ is within unit distance of ${R'}_P^Q(T)$.

We say that a hyperplane H in an affine space is a boundary hyperplane of a convex polytope with non-empty interior if H is the affine span of a codimension one face (a facet) of the polytope, and that a half-space is a boundary half-space of a polytope it contains if its bounding hyperplane is a boundary hyperplane. Let the relative interior, the relative boundary, and a relative boundary hyperplane of a region be, respectively, its interior, its boundary, and a boundary hyperplane of the region considered as an object *in its affine span*. The dimension of a polytope means the dimension of its affine span.

Lemma 3.1 *For T in $\mathfrak{a}^+$, the non-empty faces of ${R'}_P^Q(T)$ are exactly the convex hulls*

$$cvx(T_R)_{P_1\subseteq R\subseteq P_2} = {R'}_1^2(T),$$

where P_1 and P_2 are parabolic subgroups satisfying $P \subseteq P_1 \subseteq P_2 \subseteq Q$. Furthermore, the dimension of ${R'}_1^2$ is $\dim \mathfrak{a}_1^2$.

Proof Notice that the affine span of ${R'}_P^Q$ is $T_Q + \mathfrak{a}_P^Q$. A face of ${R'}_P^Q$ in the intersection of ${R'}_P^Q$ with some of its relative boundary hyperplanes. Lemma 3.4 of [8] implies that the relative boundary hyperplanes of ${R'}_P^Q$ are

$$\{X \in T_Q + \mathfrak{a}^Q \mid \alpha(X) = 0\}, \tag{3.1}$$

for roots $\alpha \in \Delta_P^Q$, and

$$\{X \in T_Q + \mathfrak{a}_P^Q \mid \varpi(X) = \varpi(T)\}, \tag{3.2}$$

for weights $\varpi \in \widehat{\Delta}_P^Q$. Consider a collection of these hyperplanes in $T_Q + \mathfrak{a}_P^Q$ whose intersection intersects ${R'}_P^Q$ non-trivially. Choose parabolic subgroups $P_1, P_2, P \subseteq P_1$, $P_2 \subseteq Q$, so that Δ_P^1 is the set of roots whose corresponding hyperplane (3.1) is in the collection and $\widehat{\Delta}_2^Q$ is the set of weights whose hyperplane (3.2) is in the collection.

Suppose that there were a root $\alpha \in \Delta_P^1$ whose dual weight ϖ_α in $(\mathfrak{a}_P^Q)^*$ lay in $\widehat{\Delta}_2^Q$. Since ϖ_α can be written in the form

$$\varpi_\alpha = c\alpha + \sum_{\substack{\varpi \in \widehat{\Delta}_P^Q \\ \varpi \neq \varpi_\alpha}} d_\varpi \varpi$$

with c positive and all d nonnegative, we see that for every X in the intersection of the given collection of hyperplanes,

$$\sum_{\substack{\varpi \in \widehat{\Delta}_P^Q \\ \varpi \neq \varpi_\alpha}} d_\varpi \varpi(X) = \varpi_\alpha(X) = \varpi_\alpha(T) = c\alpha(T) + \sum_{\substack{\varpi \in \widehat{\Delta}_P^Q \\ \varpi \neq \varpi_\alpha}} d_\varpi \varpi(T).$$

Since $c\alpha(T) > 0$ and all d_ϖ are non-negative, it must be that $\varpi(X) > \varpi(T)$ for some $\varpi \in \widehat{\Delta}_P^Q$. Therefore the intersection of ${R'}_P^Q$ with the collection of hyperplanes is trivial. This gives a contradiction, so no such α can exist. This proves that $P_1 \subseteq P_2$.

Therefore every non-empty face of ${R'}_P^Q$ is the set of points X in $T_{P_2} + \mathfrak{a}_1^2$ such that

$$\alpha(X) \geq 0 \quad \text{for every } \alpha \in \Delta_P^Q,$$
$$\varpi(X) \leq \varpi(T) \quad \text{for every } \varpi \in \widehat{\Delta}_P^Q.$$

This trivially equals the set of points X in $T_{P_2} + \mathfrak{a}_1^2$ such that

$$\begin{aligned} &\alpha(X) \geq 0 \quad \text{for every } \alpha \in \Delta_1^Q, \\ &\varpi(X) \leq \varpi(T) \quad \text{for every } \varpi \in \widehat{\Delta}_P^2. \end{aligned} \tag{3.3}$$

Consider the smaller collection of inequalities

$$\begin{aligned} &\alpha(X) \geq 0, \quad \text{for every } \alpha \in \Delta_1^2 \\ &\varpi(X) \leq \varpi(T), \quad \text{for every } \varpi \in \widehat{\Delta}_1^2, \end{aligned} \tag{3.4}$$

for $X \in T_{P_2} + \mathfrak{a}_1^2$. Since every weight $\varpi \in \widehat{\Delta}_P^2 \setminus \widehat{\Delta}_1^2$ can be written in the form

$$\varpi = \varpi^1 + \sum_{\varpi' \in \widehat{\Delta}_1^2} d_{\varpi'} \varpi'$$

with ϖ^1 in $\widehat{\Delta}_P^1$ and all $d_{\varpi'}$ non-negative, if $X \in T_{P_2} + \mathfrak{a}_1^2$ satisfies the inequalities (3.4), it satisfies all the inequalities in the second line of (3.3). Since every root $\alpha \in \Delta_1^Q \setminus \Delta_1^2$ can be written in the form

$$\alpha = \alpha_2 - \sum_{\varpi \in \widehat{\Delta}_1^2} d_\varpi \varpi$$

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with α_2 in Δ_2 and all d_ϖ non-negative, if $X \in T_{P_2} + \mathfrak{a}_1^2$ satisfies the inequalities (3.4), then

$$\begin{aligned}\alpha(X) &= \alpha_2(X) - \sum_{\varpi \in \widehat{\Delta}_1^2} d_\varpi \varpi(X) \\ &\geq \alpha_2(T) - \sum_{\varpi \in \widehat{\Delta}_1^2} d_\varpi \varpi(T) = \alpha(T) > 0,\end{aligned}$$

for any $\alpha \in \Delta_1^Q \backslash \Delta_1^2$, so that the inequalities (3.3) and (3.4) are equivalent on $T_{P_2} + \mathfrak{a}_1^2$. By Lemma 3.4 of [8], the set of $X \in T_{P_2} + \mathfrak{a}_1^2$ satisfying (3.4) is exactly $\operatorname{cvx}(T_R)_{P_1 \subseteq R \subseteq P_2}$. This proves the first statement of the Lemma. The second statement is trivial, as T lies in $\mathfrak{a}^+$. ∎

The following lemma points out the key properties of the region $\mathcal{C}$.

Lemma 3.2

(i) *The collection of subsets* $\mathcal{S}$ *of* $\Psi_{\pi,P}$ *such that* $\ker \mathcal{S}$ *intersects* $R'^Q_P(T)$ *is independent of* $T \in \mathcal{C}$.
(ii) *Given* $T \in \mathcal{C}_\varepsilon$*, parabolic subgroups* $P \subseteq Q \subsetneq G$ *and a subset* $\mathcal{S}$ *of* $\Psi_{\pi,P}$*, if the distance from* $\ker \mathcal{S}$ *to* $R'^Q_P(T)$ *is less than* $\varepsilon\|T\|$*, then* $\ker \mathcal{S}$ *must actually intersect* $R'^Q_P(T)$.

Proof (i) Suppose that $\ker \mathcal{S}$ intersects $R'^Q_P(T)$ but not $R'^Q_P(T')$, with $\mathcal{S} \subseteq \Psi_{\pi,P}$, and $T, T' \in \mathcal{C}$. Consider a minimal face of $R'^Q_P(T)$ that intersects $\ker \mathcal{S}$, so that the intersection is a single point in the relative interior of the face. By Lemma 3.1 this face is of the form $R'^2_1(T)$ for two parabolic subgroups P_1, P_2, with $P \subseteq P_1 \subseteq P_2 \subseteq Q$, and hence $\ker \mathcal{S}$ intersects the affine space $T_{P_2} + \mathfrak{a}_1^2 = \operatorname{affspan} R'^2_1(T)$ in a point. If we write $\mathcal{S}_1$ for the set of projections of weights in $\mathcal{S}$ to $\mathfrak{a}_1^*$, this implies that

$$\operatorname{span}(\mathcal{S}_1, \mathfrak{a}_2^*) = \mathfrak{a}_1^*. \tag{3.5}$$

Now, by our assumptions, $\ker \mathcal{S}$ intersects $R'^2_1(T)$ but not $R'^2_1(T')$. Therefore for some point $T'' \in \mathcal{C}$ on the line segment joining T and T', $\ker \mathcal{S}$ intersects only the relative boundary of $R'^2_1(T'')$, not its relative interior. The intersection $(\ker \mathcal{S}) \cap R'^2_1(T'') = (\ker \mathcal{S}_1) \cap R'^2_1(T'')$ is again a single point, and it must lie on a relative boundary hyperplane of $R'^2_1(T'')$. First suppose that this relative boundary hyperplane is of the form $(\ker \alpha_1) \cap (T_2 + \mathfrak{a}_1^2)$, for some $\alpha_1 \in \Delta_1^2$. By (3.5) we know that α_1 is in the span of $\mathcal{S}_1$ and $\mathfrak{a}_2^*$, so that

$$\operatorname{span}(\mathcal{S}_1 \cup \{\alpha_1\}) \cap \mathfrak{a}_2^* \neq \mathbf{0}.$$

But this contradicts the facts that $\ker(\mathcal{S}_1 \cup \{\alpha_1\})$ intersects $R'^2_1(T'')$ and that $T'' \in \mathcal{C}$. Next suppose that the relative boundary hyperplane of $R'^2_1(T'')$ is of the form

$$\{X \in T_2 + \mathfrak{a}_1^2 \mid \varpi(X) = \varpi(T)\}$$

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for some $\varpi \in \widehat{\Delta}_1$. Again, by (3.5) we know that ϖ is in the span of $\mathcal{S}_1$ and $\mathfrak{a}_2^*$, so if we choose $P_3 \subsetneq P_2$ to be the parabolic subgroup with $\widehat{\Delta}_3 = \widehat{\Delta}_2 \cup \{\varpi\}$, we have

$$\operatorname{span}(\mathcal{S}_1) \cap \mathfrak{a}_3^* \neq \mathbf{0}.$$

But this contradicts the facts that $\ker \mathcal{S}_1$ intersects $R'^3_1(T'')$ and that $T'' \in \mathcal{C}$. We have proven part (i).

(ii) Consider a minimal face $R'^2_1(T)$ of $R'^Q_P(T)$ such that

$$\operatorname{dist}\big(\ker \mathcal{S}, R'^2_1(T)\big) = \operatorname{dist}\big(\ker \mathcal{S}, R'^Q_P(T)\big) \leq \varepsilon \|T\|.$$

Then, since T lies in $\mathcal{C}_\varepsilon$, we know that $\operatorname{span}(\mathcal{S}_1) \cap \mathfrak{a}_2^* = 0$, so that $\ker \mathcal{S}$ intersects $T_1 + \mathfrak{a}_1^2$. Clearly, given a convex set A and an affine subspace S that intersects the affine span of A but not A itself, the points on A closest to S lie on the relative boundary of A. But then the minimality of $R'^2_1(T)$ implies that $\ker \mathcal{R}$ intersects $R'^2_1(T) \subseteq R'^Q_P(T)$, which proves (ii). ■

Given a functional $\lambda \in \mathfrak{a}_P^*$, write H_λ for the hyperplane $\ker \lambda$ of $\mathfrak{a}_P$. Given in addition a positive real number b, define a "thickened hyperplane" $H_\lambda(b)$ to be the set of points $X \in \mathfrak{a}_P$ such that $|\lambda(X)| \leq b$; it depends on λ, not just the hyperplane. We will say that $T \in \mathcal{C}$ is sufficiently large if $\|T\|$ is sufficiently large. We will also say (T, S) is well-situated if $S, T \in \mathcal{C}_\varepsilon$, T is sufficiently large, and $\|S\| \leq 1$. Recall that $\varepsilon > 0$ is fixed throughout this section.

Remark Since the largest angle between two vectors in the convex cone $\mathcal{C}_\varepsilon \subset \mathfrak{a}^+$ is less than $\pi/2$, it is valid to conclude that if (T, S) and (T', S') are both well-situated, then so is every point (T'', S'') on the line segment in $\mathfrak{a} \times \mathfrak{a}$ joining them.

Lemma 3.3 *There exists $B \in \mathfrak{a}^*$, positive on $\mathcal{C}$, such that for every parabolic subgroup $P \subseteq Q$, every subset $\mathcal{S}$ of $\Psi_{\pi,P}$, and all well-situated (T, S), the intersection*

$$\bigcap_{\lambda \in \mathcal{S}} H_\lambda\big(B(T)\big) \cap R^Q_P(T, S)$$

is non-empty if and only if the intersection

$$\bigcap_{\lambda \in \mathcal{S}} H_\lambda \cap R'^Q_P(T) = (\ker \mathcal{S}) \cap R'^Q_P(T)$$

is non-empty.

Proof We first observe that since Ψ_π is finite there exists a positive constant $\kappa > 1$ such that for any $b > 0$, parabolic subgroup P, and $\mathcal{S} \subseteq \Psi_{\pi,P}$, if $X \in \mathfrak{a}_P$ lies in $\bigcap_{\lambda \in \mathcal{S}} H_\lambda(b)$, then $\operatorname{dist}(X, \ker \mathcal{S}) < \kappa b$. This follows from repeated applications of the following: given a subspace $\mathbf{S}$ of $\mathfrak{a}_P$ not contained in a hyperplane $H \subset \mathfrak{a}_P$ through 0, a point within distance B of both $\mathbf{S}$ and H is within $B/\sin(\theta/2)$ of their intersection, where θ is the minimum angle between the projections of $\mathbf{S}$ and H to $\mathfrak{a}_P/(\mathbf{S}\cap H)$. (Notice that the projection of $\mathbf{S}$ is one-dimensional and not contained in the projection of H, so the angle θ is well-defined and positive. The distance from the given point to $\mathbf{S}\cap H$ is the length of the diagonal through $\mathbf{S}\cap H$ of a parallelogram in $\mathfrak{a}_P/(\mathbf{S}\cap H)$ with one side in the projection of $\mathbf{S}$, an adjacent side in the projection of H, and all side lengths at most B. This length is at most $B/\sin(\theta/2)$, the length of the long diagonal of a rhombus with inner angles $\theta \le \pi-\theta$ whose parallel sides are distance B apart.)

Fix a parabolic subgroup $Q \supseteq P$. Since $R_P^Q(T,S)$ contains ${R'}_P^Q(T)$, $\ker\mathcal{S}\cap {R'}_P^Q(T)\neq\varnothing$ implies

$$\Bigl(\bigcap_{\lambda\in\mathcal{S}} H_\lambda(b)\Bigr)\cap R_P^Q(T,S)\neq\varnothing,$$

for any $b\ge 0$. On the other hand, if an intersection $\bigcap_{\lambda\in\mathcal{S}} H_\lambda(b)$ intersects $R_P^Q(T,S)$, then since every point in $R_P^Q(T,S)$ is within unit distance of ${R'}_P^Q(T)$,

$$\operatorname{dist}\Bigl({R'}_P^Q(T),\bigcap_{\lambda\in\mathcal{S}} H_\lambda\Bigr)\le \operatorname{dist}\Bigl(R_P^Q(T,S),\bigcap_{\lambda\in\mathcal{S}} H_\lambda\Bigr)+1\le \kappa b+1.$$

For $b\le \frac{\varepsilon}{2\kappa}\|T\|$ and T sufficiently large, this implies that the distance between ${R'}_P^Q(T)=\operatorname{cvx}(T_R)_{P\subseteq R\subseteq Q}$ and $\bigcap_{\lambda\in\mathcal{S}} H_\lambda$ is less than $\varepsilon\|T\|$. Since $T\in\mathcal{C}_\varepsilon$, Lemma 3.2 lets us conclude that $\bigcap_{\lambda\in\mathcal{S}} H_\lambda$ intersects ${R'}_P^Q(T)$.

We therefore need only find $B\in\mathfrak{a}^*$ such that $0<B(T)\le\frac{\varepsilon}{2\kappa}\|T\|$ for all $T\in\mathcal{C}$. The choice $B=\frac{\varepsilon}{2\kappa\|\alpha\|}\alpha$, α any simple root in Δ, satisfies this condition. ∎

Let B be as in Lemma 3.3, and let us return to the fixed parabolic subgroups $P\subseteq Q\subsetneq G$. For the remainder of this section we will work in $\mathfrak{a}_P$ so that by the kernel of a linear functional we will mean the kernel in $\mathfrak{a}_P$, and by the interior of a region in $\mathfrak{a}_P$, its interior in the topology of $\mathfrak{a}_P$. Write $\Pi=\Pi_P\in\mathfrak{a}_P^*$ for the set of non-zero weights of π with respect to the torus A_P. Given any subset Π^+ of Π, write $R_P^Q(\Pi^+,T,S)$ for the closure of the set of $X\in R_P^Q(T,S)\subset\mathfrak{a}_P$ such that for $\lambda\in\Pi$, $\lambda(X)>0$ exactly when $\lambda\in\Pi^+$. Clearly each non-empty set $R_P^Q(\Pi^+,T,S)$ is a closed convex polytope with non-empty interior in $\mathfrak{a}_P$, and for (T,S) fixed, any two have disjoint interiors, and their (finite) union over $\Pi^+\subseteq\Pi$ is $R_P^Q(T,S)$.

Let $\Pi^+\subset\Pi$ be a set such that $R_P^Q(\Pi^+,T,S)$ is non-empty. Given any subset $\Lambda\subseteq\Psi_{\pi,P}=\Pi\cup\Delta_P$, define $\Lambda^+=\Lambda\cap\Pi^+$, $\Lambda^-=\Lambda\setminus\Pi^+$, and define the sign $\operatorname{sgn}(\lambda)$ of a weight $\lambda\in\Pi$ to be 1 if $\lambda\in\Pi^+$ and -1 if $\lambda\in\Pi^-$. For any functional $\lambda\in\Delta_P\cup\Delta_Q\cup\hat{\Delta}_Q\cup\hat{\Delta}_P^Q$, define $\operatorname{sgn}\lambda=1$. We will now decompose $R_P^Q(\Pi^+,T,S)$ as a union of closed convex polytopes in $\mathfrak{a}_P$ with disjoint non-empty interiors; in the following section we will evaluate the contribution to (1.3) of the integral over each of these latter polytopes. The construction of these polytopes requires a lengthy recursion.

Let $\delta=1/|\Pi|$. Then for any $\Pi^+\subseteq\Pi$, any positive number b, any subset $\mathcal{S}$ of Π, any linear combination

$$\mu=\sum_{\lambda\in\mathcal{S}} d_\lambda\lambda,$$

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and any $X \in \mathfrak{a}_P^+$ such that

$$(\operatorname{sgn}\lambda)\lambda(X) > 0, \quad \text{for all } \lambda \in \Pi,$$
$$\mu(X) > b,$$

some $\lambda \in \mathcal{S}$ with $d_\lambda(\operatorname{sgn}\lambda) > 0$ must satisfy

$$\lambda(X) > \delta b / d_\lambda. \tag{3.6}$$

The 0-th step of the recursion proceeds as follows: for each subset Λ_0 of Π, let $R_P^Q(\Lambda_0; \Pi^+, T, S)$ be the closure of the set of $X \in R_P^Q(\Pi^+, T, S)$ such that for $\lambda \in \Pi$, $(\operatorname{sgn}\lambda)\lambda(X) \geq B(T)$ exactly for $\lambda \in \Lambda_0$. Notice that each non-empty set $R_P^Q(\Lambda_0; \Pi^+, T, S)$ is a closed convex polytope in $\mathfrak{a}_P$, that the non-empty sets corresponding to different Λ_0 have disjoint nonempty interiors, and that their union over all $\Lambda_0 \subset \Pi$ equals $R_P^Q(\Pi^+, T, S)$.

Now suppose that we have constructed, at the k-th step, a non-empty region

$$R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S),$$

with $\Lambda_0, \dots, \Lambda_k$ disjoint subsets of Π. Our "inductive hypothesis" is that the region $R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S)$ has non-empty interior, and that $X \in \mathfrak{a}_P$ belongs to $R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S)$ exactly when X satisfies the following inequalities:

$$\begin{aligned}
(\operatorname{sgn}\lambda)\lambda(X) &\geq 0 && \text{for all } \lambda \in \Big(\Pi \setminus \bigcup_{i=0}^{k} \Lambda_i\Big) \cup \Delta_P, \\
(\operatorname{sgn}\lambda)\lambda(X) &\geq \delta_i B(T) && \text{for all } \lambda \in \Lambda_i, i = 0, \dots, k, \\
(\operatorname{sgn}\lambda)\lambda(X) &\leq \delta_i B(T) && \text{for all } \lambda \in \Lambda_{i+1}, \text{ if } i = 0, \dots, k-1, \\
& && \text{and all } \lambda \in \Pi \setminus \bigcup_{j=0}^{k} \Lambda_j, \text{ if } i = k,
\end{aligned} \tag{3.7}$$

the inequalities of immediate concern to us, and

$$\begin{aligned}
\varpi(X) &\leq \varpi(T), && \text{for all } \varpi \in \widehat{\Delta}_P^Q, \\
\alpha(X) &\geq \alpha(T), && \text{for all } \alpha \in \Delta_Q \\
\varpi(X) &\leq \varpi(T+S), && \text{for all } \varpi \in \widehat{\Delta}_Q.
\end{aligned} \tag{3.8}$$

The constants δ_i are all positive, with $\delta_0 = 1$. We will now express $R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S)$ as a union of regions with disjoint non-empty interiors.

If the intersection

$$\ker\Big(\Pi \setminus \bigcup_{i=0}^{k} \Lambda_i\Big) \cap R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S)$$

is non-empty, we end the recursion and do not break up the set $R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S)$ any further. Otherwise, there exists an assignment a of constants $c_a(\lambda)$ (independent of T, S) for each functional $\lambda \in \bigcup_{i=0}^k \Lambda_i \cup \Delta_P$ such that the linear combination

$$\mu_a = \sum_{\lambda \in \bigcup \Lambda_i \cup \Delta_P} c_a(\lambda)\lambda \in \operatorname{span}\Big(\bigcup_{i=0}^{k} \Lambda_i \cup \Delta_P\Big)$$

lies in $\operatorname{span}(\Pi \setminus \bigcup_{i=0}^k \Lambda_i)$, and the weighted sum of certain of the inequalities (3.7) is an inequality of the form $\mu_a(X) \geq c_a B(T)$ with the number c_a strictly positive—this follows from a very special case of the Krein-Milman theorem together with Lemmas 3.2 and 3.3. Pick one such assignment a and write μ_a as a linear combination of elements of $\operatorname{span}(\Pi \setminus \bigcup_{i=0}^k \Lambda_i)$:

$$\mu_a = \sum_{\lambda \in \Pi \setminus \bigcup_{i=0}^k \Lambda_i} d_\lambda \lambda.$$

Let D be the maximum of $|d_\lambda|$, $\lambda \in \Pi \setminus \bigcup_{i=0}^k \Lambda_i$. By (3.6) we can conclude that for all X in $R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S)$, at least one $\lambda \in \Pi \setminus \bigcup_{i=0}^k \Lambda_i$ satisfies

$$(\operatorname{sgn} \lambda)\lambda(X) \geq \frac{\delta c_a}{D} B(T).$$

Let δ_{k+1} equal $\delta c_a / D$, and for each subset Λ_{k+1} of $\Pi \setminus \bigcup_{i=0}^k \Lambda_i$ define

$$R_P^Q(\Lambda_0, \dots, \Lambda_{k+1}; \Pi^+, T, S)$$

to be the closure of the set of points X satisfying the strict inequality of each inequality in (3.7) and (3.8) and also

$$(\operatorname{sgn} \lambda)\lambda(X) > \delta_{k+1} B(T) \quad \text{for all } \lambda \in \Lambda_{k+1}$$

$$(\operatorname{sgn} \lambda)\lambda(X) < \delta_{k+1} B(T) \quad \text{for all } \lambda \in \Pi \setminus \bigcup_{i=0}^{k+1} \Lambda_i.$$

The non-empty sets $R(\Lambda_0, \dots, \Lambda_{k+1}; \Pi^+, T, S)$, are the regions constructed at the $(k+1)$-st step. Notice that these regions are determined by inequalities of the form (3.7) and (3.8), that their union over all non-empty Λ_{k+1} is $R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S)$, and that their interiors are pairwise disjoint.

Since Π is finite and each Λ_i, $i \geq 1$, is non-empty in a given non-empty region $R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S)$, we can write $R_P^Q(T, S)$ as a *finite* union

$$\bigcup_{(\Lambda_0, \dots, \Lambda_k; \Pi^+) \in I(T,S)} R_P^Q(\Lambda_0, \dots, \Lambda_k; \Pi^+, T, S) \tag{3.9}$$

of convex polytopes with disjoint non-empty interiors, indexed by a (finite) set $I(T, S)$ of ordered tuples of varying size, such that the region corresponding to each tuple

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$(\Lambda_0,\ldots,\Lambda_k;\Pi^+)\in I(T,S)$ is not broken up by the above algorithm. We will use this decomposition of $R_P^Q(T,S)$ in the next section, by estimating an integral of the form of (1.3) over each of these regions. Notice that by Lemmas 3.2 and 3.3, the index set $I(T,S)$ and the implicit constants δ_i and other implicit choices made for each element of $I(T,S)$, can be chosen independently of $S\in\mathcal{C}_\varepsilon(1)$ and sufficiently large $T\in\mathcal{C}_\varepsilon$. We can therefore denote this set simply as I; it depends on P, Q, and, of course, π. The set I is in no way canonical, but is sufficient for our purposes.

Consider a region $R_i(T,S)=R_P^Q(\Lambda_0,\ldots,\Lambda_k;\Pi^+,T,S)$ corresponding to

$$i=(\Lambda_0,\ldots,\Lambda_k;\Pi^+)\in I(T,S),$$

and write

$$\Pi_0=\Pi\setminus\bigcup_{i=0}^k\Lambda_i,\quad \Pi_+=\bigcup_{i=0}^k\Lambda_i^+\subseteq\Pi^+,\quad \Pi_-=\bigcup_{i=0}^k\Lambda_i^-\subseteq\Pi^-.$$

Notice that by the construction of $R_i(T,S)$, the set of weights of π that vanish on $\ker\Pi_0\cap R_i(T,S)$ equals Π_0, so in particular $(\operatorname{span}\Pi_0)\cap\Pi=\Pi_0$. The region $R_i(T,S)$ is a convex polytope bounded by the inequalities (3.7) and (3.8), so that each boundary hyperplanes of $R_i(T,S)$ is given by one of the following equations:

$$\lambda(X)=0,\quad \lambda\in\Big(\Pi\setminus\bigcup_{i=0}^k\Lambda_i\Big)\cup\Delta_P,$$

$$(3.7)'\qquad (\operatorname{sgn}\lambda)\lambda(X)=\delta_iB(T),\quad \lambda\in\Lambda_i,\, i=0,\ldots,k,$$

$$(\operatorname{sgn}\lambda)\lambda(X)=\delta_iB(T),\qquad \lambda\in\Lambda_{i+1},\text{ if } i=0,\ldots,k-1,\text{ and all } \lambda\in\Pi\setminus\textstyle\bigcup_{j=0}^k\Lambda_j,\text{ if } i=k,$$

and

$$\varpi(X)=\varpi(T),\quad \varpi\in\widehat{\Delta}_P^Q,$$

$$(3.8)'\qquad \alpha(X)=\alpha(T),\quad \alpha\in\Delta_Q$$

$$\varpi(X)=\varpi(T+S),\quad \varpi\in\widehat{\Delta}_Q.$$

Let Y be an extreme point of $R_i(T,S)$, let $\mathcal{H}_1(T,S)$ and $\mathcal{H}_2(T,S)$ be the set of boundary hyperplanes of $R_i(T,S)$ of the form $(3.7)'$ and $(3.8)'$, respectively, that contain Y. Each of the hyperplanes in $\mathcal{H}_k(T,S)$, $k=1,2$, depends explicitly on T and S through the corresponding equality in $(3.7)'$ or $(3.8)'$, so that given any $T',S'\in\mathfrak{a}$, we can naturally define the sets $\mathcal{H}_k(T',S')$.

Lemma 3.4 *Suppose that we are given (T,S) well-situated, $i\in I(T,S)$, and Y an extreme point of $R_i(T,S)$. Let $\mathcal{H}_1$, $\mathcal{H}_2$ send elements of $\mathfrak{a}\times\mathfrak{a}$ to sets of hyperplanes in $\mathfrak{a}_P$ as described above. If (T',S') is also well-situated, then the intersection*

$$(3.10)\qquad \bigcap_{H\in\mathcal{H}_1(T',S')\cup\mathcal{H}_2(T',S')}H$$

has exactly one element. Call this element $Y(T',S')$. Then $Y(T',S')$ is an extreme point of $R_i(T',S')$.

Proof Let $\mathbf{S}_{(k)}$ be $\bigcap_{H\in\mathcal{H}_k(T,S)} H$, for $k = 1, 2$. Let $\mathcal{S} \subset \Psi_{\pi,P}$ be the set of functionals λ such that the point Y satisfies an equality in $(3.7)'$ involving λ, and let $R \subseteq Q$ be the parabolic subgroup containing P so that $\widehat{\Delta}_R^Q$ is the set of $\varpi \in \widehat{\Delta}_P^Q$ whose hyperplane $\varpi(X) = \varpi(T)$ appears in $\mathcal{H}_2(T,S)$. Then Lemma 3.3 and the definition of d imply that $\operatorname{span}(\mathcal{S}) \cap \mathfrak{a}_R^* = \mathbf{0}$. Since we also know that $\mathbf{S}_{(1)} \cap \mathbf{S}_{(2)} = \{Y\}$, we can conclude that

$$\mathfrak{a}_P^* = \operatorname{span}\mathcal{S} \oplus \mathfrak{a}_R^*, \tag{3.11}$$

and that the intersection of the hyperplanes H_R, $H \in \mathcal{H}_2(T,S)$, of $\mathfrak{a}_R$ is the point Y_R. Since Y lies in $R_i(T,S)$, $Y_Q - T_Q$ must lie in $R'_Q(S)$, and projecting the preceding sentence to $\mathfrak{a}_Q$, we find that the point $Y_Q - T_Q$ is extreme in $R'_Q(S)$. Lemma 3.1 then says that for some parabolic subgroup $P_1 \supseteq Q$, $\{Y_Q - T_Q\} = {R'}_1^1 = \{S_1\}$, with S_1 the projection of S to $\mathfrak{a}_1$, so $Y_Q = T_Q + S_1$. The definition of R tells us that $Y_R^Q = T_R^Q$, so that $Y_R = T_R + S_1$. Putting all this together, we see that $\mathbf{S}_{(2)} = T_R + S_1 + \mathfrak{a}_P^R$.

Going now to T' and S', notice that the hyperplanes in $\mathcal{H}_1(T',S')$ are of all the form

$$\lambda(X) = kB(T'),$$

where the constants k and functionals $\lambda \in \Psi_{\pi,P}$ are determined by Y. The intersection $\bigcap_{H\in\mathcal{H}_1(T',S')} H$ equals $\frac{B(T')}{B(T)}\mathbf{S}_{(1)} = \frac{B(T')}{B(T)}Y + \ker\mathcal{S}$. Also, the previous paragraph implies the equality

$$\bigcap_{H\in\mathcal{H}_2(T',S')} H = T'_R + S'_1 + \ker\mathfrak{a}_R^*. \tag{3.12}$$

We conclude from (3.11) that the set (3.10) contains a single point $Y(T',S')$. This proves the first statement of the Lemma.

Since $Y(T',S')$ lies on a collection of boundary hyperplanes of $R_i(T',S')$ that intersect in a point, it suffices to prove simply that $Y(T',S')$ lies in $R_i(T',S')$, that is, that $Y(T',S')$ satisfies all the inequalities in (3.7) and (3.8).

We will first consider the inequalities in (3.8). The inequalities on the last two lines of (3.8) follow immediately from (3.12). The inequalities on the first line of (3.8) hold for $\varpi \in \widehat{\Delta}_R^Q$ and are actually equalities in the case, again by (3.12). Lastly, let $\varpi \in \widehat{\Delta}_P^Q \setminus \widehat{\Delta}_R^Q$. Since $\mathfrak{a}_P^* = \operatorname{span}\mathcal{S} \oplus \mathfrak{a}_R^*$, we know that

$$\operatorname{span}\mathcal{S} \cap \mathfrak{a}_{R'}^* \neq \mathbf{0}, \tag{3.13}$$

where $R' \subseteq Q$ is the parabolic subgroup satisfying $\Delta_{R'}^Q = \Delta_R^Q \cup \{\varpi\}$, and that $\varpi(Y) \leq \varpi(T)$. If $\varpi\big(Y(T',S')\big) > \varpi(T')$, then for some (T'',S'') on the line segment joining (T,S) and (T',S') (and by an earlier remark, necessarily well-situated), the point $Y(T'',S'')$ would satisfy $\varpi\big(Y(T'',S'')\big) = \varpi(T'')$. But then (3.13) and Lemma 3.3 would yield a contradiction.

We next show that $Y(T',S')$ lies in $R_P^R(T',S')$. Given the above paragraph, this is equivalent to showing that $\alpha\big(Y(T',S')\big) \geq 0$ for all $\alpha \in \Delta_P^R$. Assume otherwise, and let (T'',S'') be the point on the line segment joining (T,S) and (T',S') such that

- $\alpha\big(Y(T'', S'')\big) \geq 0$ for all $\alpha \in \Delta_P^R$,
- $\beta\big(Y(T'', S'')\big) = 0$ for some $\beta \in \Delta_P^R$ with $\beta\big(Y(T', S')\big) < 0$.

Then the point $Y(T'', S'')$ lies in

$$\bigcap_{\lambda \in \mathcal{S} \cup \{\beta\}} H_\lambda\big(B(T'')\big) \cap R_P^R(T'', S''),$$

so by Lemma 3.3, the set $\ker(\mathcal{S} \cup \{\beta\}) \cap {R'}_P^R(T'')$ is non-empty, and since T'' lies in $\mathcal{C}_\varepsilon$, we see that $\operatorname{span}(\mathcal{S} \cup \{\beta\}) \cap \mathfrak{a}_R^* = \mathbf{0}$. By (3.11) this implies that $\beta \in \operatorname{span} \mathcal{S}$. But then $\beta\big(Y(T', S')\big)$ and $\beta\big(Y(T'', S'')\big) = 0$ can be explicitly given as $cB(T')$ and $cB(T'')$, respectively, for some constant c. The constant c must be zero, and we obtain a contradiction to the definition of β. Therefore $Y(T', S')$ does in fact lie in $R_P^R(T', S')$.

Now, let μ be any linear functional in $\Psi_{\pi,P}$. We must show that the inequalities in (3.7) that mention μ hold for the point $Y(T', S')$ (and T' instead of T). If $\mu \in \operatorname{span} \mathcal{S}$, then $\mu(Y)$ and $\mu\big(Y(T', S')\big)$ are explicitly given as $cB(T)$ and $cB(T')$, respectively, for some constant c. Since Y satisfies all the inequalities in (3.7) (with T), $Y(T', S')$ must satisfy those inequalities from (3.7) that mention μ (with T'). Next, suppose that $\mu \notin \operatorname{span} \mathcal{S}$. Because of (3.11) we must have that $\operatorname{span}(\mathcal{S} \cup \{\mu\}) \cap \mathfrak{a}_R^* \neq \varnothing$, so since $T \in \mathcal{C}_\varepsilon$, $\operatorname{dist}\big(\ker(\mathcal{S} \cup \{\mu\}), {R'}_P^R\big) > \varepsilon \|T\|$. Lemma 3.3 then says that given a point $X \in R_P^R(T, S)$, $|\lambda(X)| > B(T)$ for some $\lambda \in \mathcal{S} \cup \{\mu\}$. Now, $Y \in R_P^R(T, S)$ and $|\lambda(Y)| \leq B(T)$ for all $\lambda \in \mathcal{S}$, so $(\operatorname{sgn} \mu)\mu(Y) > B(T)$. The inequalities (3.7) must be consistent with this, so since each $\delta_k \leq 1$, the only inequality in (3.7) that mentions μ must be of the form $(\operatorname{sgn} \mu)\mu \geq B(T)$. On the other hand, $Y(T', S') \in R_P^R(T', S')$, so that we can similarly obtain the inequality $(\operatorname{sgn} \mu)\mu\big(Y(T', S')\big) > B(T')$. This finishes our proof that $Y(T', S')$ lies in, and hence is an extreme point of, $R_i(T', S')$. ■

In the course of estimating an integral over $R_i(T, S)$ in the next section, we will express the integrand as a sum of terms corresponding to certain subsets of Π_0, and to estimate the integral of the piece of the integrand corresponding to one subset Π_1 of Π_0, we will need to further manipulate the set $R_i(T, S)$. The necessary constructions form the remainder of this section.

Let $\Pi_1 \subseteq \Pi_0$ be a subset of Π_0 satisfying

$$\Pi_1 = \big\{ \lambda \in \Pi \mid \lambda\big((\ker \Pi_1) \cap R_i(T, S)\big) = 0 \big\},$$

so that in particular $(\operatorname{span} \Pi_1) \cap \Pi_0 = \Pi_1$. (Note that both $\{0\} \cap \Pi_0$ and Π_0 satisfy this property.) Let $\mathcal{B} \subseteq \Pi_1$ be a basis and let d be the dimension of $\operatorname{span} \Pi_1$. We want to examine the dependence of the convex polytope

$$(X + \ker \Pi_1) \cap R_i(T, S) \tag{3.14}$$

on T, S and $X \in R_i(T, S)$.

Write $\overline{\mathfrak{a}_P}$ for the quotient space $\mathfrak{a}_P / \ker \Pi_1$, and make the natural identification of the dual space of $\overline{\mathfrak{a}_P}$ with $\operatorname{span} \Pi_1$. The projection map $\mathfrak{a}_P \to \overline{\mathfrak{a}_P}$ sending $X \in \mathfrak{a}_P$ to its projection, to be denoted $\overline{X}$, in $\overline{\mathfrak{a}_P}$ sends polytopes to polytopes, so the projection of $R_i(T,S)$ is a polytope $\overline{R_i(T,S)}$.

The dependence of the set (3.14) on X is clearly through $\overline{X}$. To simplify matters, we will consider only points $\overline{X} \in \overline{R_i(T,S)}$ close to zero, in a sense to be defined presently.

The point $\overline{0} = \ker \Pi_1$ lies in $\overline{R_i(T,S)}$ since $\ker \Pi_1 \supseteq \ker \Pi_0$ intersects $R_i(T,S)$, and it is an extreme point because of the inequalities $(\operatorname{sgn}\lambda)\lambda \geq 0$, $\lambda \in \Pi_1$, that hold on $R_i(T,S)$. The polytope $\overline{R_i(T,S)}$ has finitely many facets $\overline{F}$ through $\overline{0}$; the boundary half-space corresponding to one such facet $\overline{F}$ is given by an inequality of the form

$$\sum_{\lambda\in\mathcal{B}} c_\lambda^{\overline{F}} \lambda \geq 0, \tag{3.15}$$

for some numbers $c_\lambda^{\overline{F}}$, $\lambda \in \mathcal{B}$.

Lemma 3.5 *For any well-situated (T',S'), the inequality (3.15) defines a boundary half-space of $\overline{R_i(T',S')}$.*

Proof That the half-space defined by (3.15) is a boundary half-space of $\overline{R_i(T,S)}$ is equivalent to the following statement: The point $\sum_{\lambda\in\mathcal{B}} c_\lambda^{\overline{F}}\lambda$ is among those $\mu \in \operatorname{span}\Pi_1$ such that $\mu(X) \geq 0$ for all $X \in R_i(T,S)$ and is not the non-trivial convex combination of non-proportional elements of this set. We must prove that if we replace (T,S) with (T',S'), this statement is still true.

By the theory of the polar [5, Theorem 6.4] and the extremality of $\sum_{\lambda\in\mathcal{B}} c_\lambda^{\overline{F}}\lambda$, the inequality (3.15) can be written as a linear combination of the inequalities (3.7) and (3.8). Furthermore, if we take $X \in R_i(T,S)$ a pre-image of $\overline{0} \in \overline{F}$, then each inequality that appears in the above linear combination with non-zero coefficient is actually an equality at X. Let Λ [resp. W] be the set of functionals appearing in equalities from $(3.7)'$ [resp. $(3.8)'$] that hold at X. Then we have an equality

$$\sum_{\lambda\in\Lambda} d_\lambda\lambda + \sum_{\varpi\in W} d_\varpi \varpi = \sum_{\lambda\in\mathcal{B}} c_\lambda^{\overline{F}}\lambda,$$

for some constants d_λ, d_ϖ. The set $\operatorname{span}(W) + \mathfrak{a}_Q^*$ is of the form $\mathfrak{a}_R^*$ for some parabolic subgroup $R \subseteq Q$. The point X lies in $R_P^R(T,S)$ and in $\bigcap_{\lambda\in\Lambda\cup\mathcal{B}} H_\lambda\big(B(T)\big)$, so by Lemma 3.3 and the well-situatedness of (T,S), $\operatorname{span}(\Lambda\cup\mathcal{B}) \cap \mathfrak{a}_R^* = \varnothing$, and so all the constants d_ϖ must be 0. Therefore the inequality (3.15) is a linear combination just of inequalities from (3.7), and so takes the form

$$\sum_{\lambda\in\mathcal{B}} c_\lambda^{\overline{F}}\lambda \geq cB(T),$$

for some constant c. The constant c must clearly be 0.

Going now to $R_i(T',S')$, we see that taking the same linear combination of the inequalities (3.7) (with T' instead of T), we find that the inequality (3.15) holds also on $R_i(T',S')$. If the point $\sum_{\lambda\in\mathcal{B}} c_\lambda^{\overline{F}}\lambda$ were a non-trivial convex combination of non-proportional functionals $\mu \in \operatorname{span}\Pi_1$ that are non-negative on all of $R_i(T',S')$, then

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we could reason as above to find that these functionals are also non-negative on all of $R_i(T,S)$, and so obtain a contradiction to the above-stated extremality of $\sum c_\lambda^{\overline{F}}\lambda$. This completes our proof. ■

The functional

$$\lambda_{\mathcal{B}} = \sum_{\lambda\in\mathcal{B}} (\operatorname{sgn}\lambda)\lambda$$

is non-negative on all $R_i(T,S)$ (and vanishes on the non-empty set $(\ker\Pi_1)\cap R_i(T,S)$) for any well-situated (T,S). The proof of Lemma 3.5 implies that the value of $\lambda_{\mathcal{B}}$ at each extreme point $Y(T,S) = \bigcap_{H\in\mathcal{H}_1(T,S)\cup\mathcal{H}_2(T,S)} H$ of $R_i(T,S)$ is determined from the equalities in $(3.7)'$ that define hyperplanes $\mathcal{H}_1(T,S)$, so that there exists a constant c_Y independent of T and S such that for any well-situated (T',S'), we have $\lambda_{\mathcal{B}}\big(Y(T',S')\big) = c_y B(T')$, where $Y(T',S')$ is the extreme point of $R_i(T',S')$ given in Lemma 3.4. Therefore, for well-situated (T,S), the minimal non-zero value of $\lambda_{\mathcal{B}}$ on the extreme points of $R_i(T,S)$ is given by $2\delta' B(T)$ for some non-zero constant δ' independent of T and S. Since the non-zero extreme points of $\overline{R_i(T,S)}$ are projections of those extreme points of $R_i(T,S)$ where $\lambda_{\mathcal{B}}$ is non-zero, and $\lambda_{\mathcal{B}}$ is left invariant by projection to $\overline{\mathfrak{a}}_P$, the hyperplane $\{X\in\overline{\mathfrak{a}}_P \mid \lambda_{\mathcal{B}}(X) = \delta' B(T)\}$ of $\overline{\mathfrak{a}}_P$ separates $\overline{0}$ from the other extreme points of $\overline{R_i(T,S)}$.

Let $R_i(\delta',T,S)$ be the set of X in $R_i(T,S)$ satisfying

$$\lambda_{\mathcal{B}}(X) = \sum_{\lambda\in\mathcal{B}} (\operatorname{sgn}\lambda)\lambda(x) \le \delta' B(T). \tag{3.16}$$

The boundary hyperplanes of $R_i(\delta',T,S)$ are exactly the boundary hyperplanes of $R_i(T,S)$ that intersect $(\ker\Pi_1)\cap R_i(T,S)$, and the hyperplane $\lambda_{\mathcal{B}} = \delta' B(T)$ from (3.16). This is because every face of $R_i(T,S)$ that does not intersect $\ker\Pi_1$ is the convex hull of a collection of extreme points of $R_i(T,S)$ not in $\ker\Pi_1$, and so has no points satisfying (3.16). This argument also shows that the projection $\overline{R_i(\delta',T,S)}$ of $R_i(\delta',T,S)$ to $\overline{\mathfrak{a}_{\mathfrak{P}}}$ is a pyramid with apex $\overline{0}$ whose boundary half-spaces are exactly those given by (3.15) and (3.16).

We want to examine the dependence of (3.14) on $\overline{X}$, T, S, with (T,S) well-situated, and $\overline{X}$ in the interior of $\overline{R_i(\delta',T,S)}$. A basic example is the intersection of translates of the line $y=z=0$ with the octahedron $R\subset\mathbb{R}^3$ given by

$$0 \le y+z,\, y-z,\, x+y-z,\, x+y+z \le B(T)$$

(the polytope $0\le y+z, y-z \le \delta B(T)$, $\delta B(T) \le x+y-z, x+y+z \le B(T)$ is similar); this models the points of the intersection $(X+\ker\Pi_1)\cap R_i(T,S)$ in $T_P + \mathfrak{a}_P^Q$ when $\dim\mathfrak{a}_P^Q = 3$. The intersection

$$\big((x_0,y_0,z_0)+L\big)\cap R, \quad (x_0,y_0,z_0)\in\operatorname{Int}(R),$$

is a line segment and each of its two endpoints lies on a boundary hyperplane of R. The two hyperplanes on which the end points lie are $x+y-z=0$, $x+y+z=B(T)$

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if $z_0 \geq 0$ and are $x + y + z = 0$, $x + y - z = B(T)$ if $z_0 \leq 0$. Therefore, the extreme points of

$$\big((x_0, y_0, z_0) + L\big) \cap R$$

are linear in T and in (x_0, y_0, z_0) on a specified side of $z_0 = 0$. We will see that the general situation is similar.

Notice that $\lambda_{\mathcal{B}}(X + \ker \Pi_1) = \lambda_{\mathcal{B}}(X)$, so that for any point X in $R_i(\delta', T, S)$, the polytope (3.14) equals $(X + \ker \Pi_1) \cap R_i(\delta', T, S)$.

Given X in the interior $\operatorname{Int} R_i(\delta', T, S)$ of $R_i(\delta', T, S)$ and an extreme point Y of the polytope (3.14), there is a face F of $R_i(\delta', T, S)$ such that

$$(X + \ker \Pi_1) \cap \operatorname{affspan} F = \{Y\}. \tag{3.17}$$

Since $X \in \operatorname{Int} R_i(\delta', T, S)$, $\lambda_{\mathcal{B}}(Y) < \delta' B(T)$, so F is not contained in the hyperplane $\lambda_{\mathcal{B}} = \delta' B(T)$. Therefore F intersects $\ker \Pi_1$, so that $\operatorname{affspan} F \cap R_i(T, S)$ is a face of $R_i(T, S)$ of dimension $\dim F$. Since (3.17) contains a unique point, elementary linear algebra implies that

$$|(Z + \ker \Pi_1) \cap (\operatorname{affspan} F)| \leq 1 \text{ for all } Z \in \mathfrak{a}_P. \tag{3.18}$$

Suppose that (T', S') is also well-situated, that F is a face of $R_i(\delta', T, S)$ satisfying (3.18), and that F' is the corresponding face of $R_i(\delta', T', S')$ (more precisely, the intersection with $R_i(\delta', T', S')$ of the face of $R_i(T', S')$ corresponding, by Lemma 3.4 and the remark after Lemma 2.4, to $R_i(T, S) \cap \operatorname{affspan} F$; F' clearly also satisfies (3.18)). Suppose also that $X \in \operatorname{Int} R_i(\delta', T, S)$, $X' \in \operatorname{Int} R_i(\delta', T', S')$ satisfy

- $(X + \ker \Pi_1) \cap \operatorname{affspan} F$ contains a (unique) point, which is extreme in $(X + \ker \Pi_1) \cap R_i(\delta', T, S)$,
- $(X' + \ker \Pi_1) \cap \operatorname{affspan} F'$ contains a (unique) point, which is not extreme in $(X' + \ker \Pi_1) \cap R_i(\delta', T', S')$.

Then for some point $(X'', T'', S'') \in \mathfrak{a}_P \times \mathfrak{a} \times \mathfrak{a}$ on the line segment joining (X, T, S) and (X', T', S') (so that (T'', S'') is well-situated and X'' lies in $\operatorname{Int} R_i(\delta', T'', S'')$), $X'' + \ker \Pi_1$ intersects a face $\tilde{F}$ of $R_i(\delta', T'', S'')$ strictly contained in F'', the face corresponding for F. Notice also that since $F'' \supsetneq \tilde{F}$ satisfies (3.18), there must exist points Z arbitrarily close to X'' such that $Z + \ker \Pi_1$ does not intersect $\tilde{F}$ (or even $\operatorname{affspan} \tilde{F}$). Let us examine these faces $\tilde{F}$.

Let $\tilde{F}$ be a maximal face of $R_i(\delta', T, S)$ such that for some $X \in \operatorname{Int} R_i(\delta', T, S)$ the intersection $(X + \ker \Pi_1) \cap \operatorname{affspan} F$ is non-empty, but that there exist points in any neighbourhood of X such that the corresponding intersection is empty. We will call all such faces *problematic*. The face $\tilde{F}$ must intersect $\ker \Pi_1$, so the projection $\overline{H}$ of $\operatorname{affspan} F$ to $\overline{\mathfrak{a}_{\mathfrak{P}}}$ is a subspace. In fact, maximality of $\tilde{F}$ is easily seen to imply that $\overline{H}$ is actually a hyperplane through $\overline{0}$ in $\overline{\mathfrak{a}_{\mathfrak{P}}}$, and so can be given by an equality

$$\sum_{\lambda \in \mathcal{B}} d_\lambda^{\tilde{F}} \lambda = 0. \tag{3.19}$$

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An argument similar to that in Lemmas 3.4 and 3.5 shows that given a problematic face $\tilde{F}$ of $R_i(\delta', T, S)$, and any well-situated (T', S'), the corresponding face $\tilde{F}'$ of $R_i(\delta', T', S')$ is also problematic and its projection to $\overline{\mathfrak{a}_{\mathfrak{P}}}$ is a hyperplane given by the equation (3.19). The complement in $R_i(\delta', T, S)$ [resp. $\overline{R_i(\delta', T, S)}$] of the union over all problematic faces $\tilde{F}$ of $R_i(\delta', T, S)$, of the hyperplanes in $\mathfrak{a}_P$ [resp. $\overline{\mathfrak{a}_{\mathfrak{P}}}$] determined by (3.19), is a finite, disjoint union of convex open polytopes, each of whose closures is given as the set of X in $R_i(\delta', T, S)$ [resp. $\overline{R_i(\delta', T, S)}$] satisfying

$$j(F) \sum_{\lambda \in \mathcal{B}} d_\lambda^{\tilde{F}} \lambda(X) \geq 0, \quad \text{for each problematic face } \tilde{F}, \tag{3.20}$$

with j an assignment of ± 1 to each problematic face of $R_i(\delta', T, S)$. Let J_i be the set of assignments j such that the set of X in $\overline{R_i(\delta', T, S)}$ satisfying (3.20) has non-empty interior in $\overline{\mathfrak{a}_{\mathfrak{P}}}$. Then J_i is independent of well-situated (T, S).

Definition 3.5

(a) Given (T, S) well-situated, $i \in I$, $j \in J_i$, define $R_{i,j}(\delta', T, S)$ [resp. $\overline{R_{i,j}(\delta', T, S)}$] to be the set of X in $R_i(\delta', T, S)$ [resp. $\overline{R_i(\delta', T, S)}$] satisfying the inequalities (3.20).

(b) Given $X \in \mathfrak{a}_P$, $i \in I$, define $R_i(T, S)_X$ to be the polytope

$$\big((X + \ker \Pi_1) \cap R_i(T, S)\big) - X \subset \ker \Pi_1.$$

We have proven the following Lemma.

Lemma 3.6 *Fix $i \in I$, $j \in J_i$. Suppose that we are given (T, S), (T', S') well-situated, and $X \in R_{i,j}(\delta', T, S)$, $X' \in R_{i,j}(\delta', T', S')$. If Y is an extreme point of (3.14), let F be a face of $R_i(T, S)$ satisfying (3.17), and let F' be the corresponding face of $R_i(T', S')$. Then $(X' + \ker \Pi_1) \cap F'$ is an extreme point of $(X' + \ker \Pi_1) \cap R_i(T', S')$.*

Corollary 3.7 *Fix $i \in I$, $j \in J_i$, $\mu \in \mathfrak{a}^*$. Then the integral*

$$\int_{R_i(T,S)_X} e^{\mu(H)}\, dH, \quad (T, S) \text{ well-situated, } X \in R_{i,j}(\delta', T, S),$$

is t-finite in X, T, S.

Proof This is immediate from Lemmas 3.6, 2.3, 2.4. ∎

Notice lastly that for every j, the polytope $\overline{R_{i,j}(\delta', T, S)}$ is again a pyramid with apex $\overline{0}$, whose bounding half-spaces are given by the inequalities from (3.15), (3.20) and (3.16) (giving the base), and so is independent of S.

Write $\overline{R_{i,j}}$ for the polyhedron bounded only by the inequalities from (3.15) and (3.20); the penultimate step in the proof of our main theorem (3.3) will be noticing that $\overline{R_{i,j}}$ does not depend on T or S.

4 The Main Theorem

In [8], we saw that if the rank of G is at most two, then the truncated integral $J_{\mathfrak{o}}^T(f,\pi)$ was asymptotic, as T approached infinity in certain sub-cones of $\mathfrak{a}^+$, to a t-finite function of T. In this section we prove this for general G.

Theorem 4.1 *Let G be a rational reductive group with anisotropic centre, and let π be a rational representation of G on a finite-dimensional vector space V. For each π-dependent cone $\mathcal{C}$ in $\mathfrak{a}^+$, each geometric equivalence class $\mathfrak{o} \in \mathfrak{O}$, and each Schwartz-Bruhat function f on $V(\mathbb{A})$, there exists a unique t-finite function $P_{\mathfrak{o},\mathcal{C}}$ on $\mathfrak{a}$ such that for every sufficiently small $\varepsilon > 0$ and every $c > 0$ there exists a continuous seminorm $\|\cdot\| = \|\cdot\|_{\varepsilon,c}$ on the space of Schwartz-Bruhat functions on $V(\mathbb{A})$ such that*

$$\sum_{\mathfrak{o}\in\mathfrak{O}} |J_{\mathfrak{o}}^T(f,\pi) - P_{\mathfrak{o},\mathcal{C}}(T)| < \|f\| e^{-c\|T\|},$$

for all T in $\mathcal{C}_\varepsilon$. The function

$$P_{\mathcal{C}}(T) = \sum_{\mathfrak{o}\in\mathfrak{O}} P_{\mathfrak{o},\mathcal{C}}(T)$$

is also t-finite.

The basic outline of our proof of this theorem is the same as that of Theorem 6.1 of [8], however there are additional complications, arising from the constructions of Section 3 and the need to watch the dependence on f. As in [8] we prove the Theorem by examining differences $J_{\mathfrak{o}}(T+S) - J_{\mathfrak{o}}(T)$ and then applying Lemma 4.2, whose proof is easily adapted from the proof of Lemma 4.2 of [8].

Lemma 4.2 *Suppose that $\{J_{\mathfrak{o}}\}_{\mathfrak{o}\in\mathfrak{O}}$ is a collection of continuous functions on an open cone $\mathcal{C}$ in $\mathfrak{a}^+$, and that there exists a collection, indexed by $\mathfrak{o} \in \mathfrak{O}$, of t-finite functions $\{p_{\mathfrak{o}}\}$ on $\mathfrak{a} \times \mathfrak{a}$ and a constant b such that for every $c > 0$ there exists a constant C_c such that*

$$\sum_{\mathfrak{o}\in\mathfrak{O}} |J_{\mathfrak{o}}(T+S) - J_{\mathfrak{o}}(T) - p_{\mathfrak{o}}(S,T)| < C_c e^{-c\|T\|},$$

for all S in $\mathcal{C}(1)$ and every T in the cone with $\|T\| \geq b$. Then for each $\mathfrak{o} \in \mathfrak{O}$ there exists a unique t-finite function $P_{\mathfrak{o}}$ and for every $c > 0$ there exists a constant d_c depending only on c such that

$$\sum_{\mathfrak{o}\in\mathfrak{O}} |J_{\mathfrak{o}}(T) - P_{\mathfrak{o}}(T)| < d_c C_c e^{-c\|T\|}$$

for every T in the cone.

We used reduction theory to reduce the difference $J_{\mathfrak{o}}(T+S) - J_{\mathfrak{o}}(T)$ to expressions of the form (1.3). The idea now is to use the Poisson summation formula and the constructions of Section 3 to reduce the problem to an application of Corollary 3.7. A simplified example to keep in mind is the following: for f a Schwartz function on $\mathbb{R}$, the integral over x in the interval from 0 to T of $\sum_{n\in\mathbb{Z}} f(e^x n)$ approaches

$$(e^{-T} - 1)\hat{f}(0) + \int_0^\infty \sum_{n\neq 0} e^x \hat{f}(e^x n)\,dx$$

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as T tends to infinity in the cone $\mathbb{R}^-$ of negative reals and approaches

$$Tf(0) + \int_0^\infty \sum_{n\neq 0} f(e^x n)\,dx$$

as T tends to infinity in the cone $\mathbb{R}^+$ of positive reals. The t-finite functions T and $e^{-T} - 1$ are integrals of the form of Corollary 3.7.

Proof of Theorem 4.1 Fix a geometric equivalence class $\mathfrak{o} \in \mathfrak{O}$ and an $\varepsilon > 0$ sufficiently small that the set $\mathcal{C}_\varepsilon$ is non-empty. Clearly, if for each sufficiently small $\varepsilon > 0$ there exists a t-finite function satisfying the above estimate, then it must be unique and independent of ε. We need therefore only find an approximation on the cone $\mathcal{C}_\varepsilon$ for a fixed $\varepsilon > 0$. Let us also fix $c > 0$.

To apply Lemma 4.2, we must estimate the difference $J_{\mathfrak{o}}^{T+S}(f,\pi) - J_{\mathfrak{o}}^{T}(f,\pi)$. Notice that we can require $\|T\|$ to be larger than any constant that is independent of f. Throughout this proof this is what we will mean by choosing T sufficiently large. Take T, S as in Lemma 4.2, with T sufficiently large. For the remainder of the proof we will assume without mention that (T,S) are well-situated, so that T is sufficiently large and lies in $\mathcal{C}_\varepsilon$, and that S lies in $\mathcal{C}_\varepsilon(1)$. By a fixed constant we mean one that is independent of T, S, $\mathfrak{o}$, and f.

Equation (1.3) states that

$$J_{\mathfrak{o}}^{T+S}(f,\pi) - J_{\mathfrak{o}}^{T}(f,\pi) = \sum_{P\subseteq Q\subsetneq G} \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \int_{A_P(\mathbb{R})^0} e^{-2\rho_P(H_P(a))} \phi_{\mathfrak{o}}(na, f^{P,K,T_2}) \times \Gamma_P^Q\big(H_P(a), T - T_2\big)\,\Gamma_Q\big(H_P(a) - T, S\big)\,da\,dn.$$

Since the outer sum is finite, we need consider only the term corresponding to a fixed choice of $P \subseteq Q \subsetneq G$. Since the function f^{P,K,T_2} is Schwartz-Bruhat and is independent of T and S, we can reduce the problem to constructing a t-finite function that approximates the integral

$$\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \int_{A_P(\mathbb{R})^0} e^{-2\rho_P(H_P(a))} \phi_{\mathfrak{o}}(na, f) \times \Gamma_P^Q\big(H_P(a), T - T_2\big)\,\Gamma_Q\big(H_P(a) - T, S\big)\,da\,dn, \tag{4.1}$$

for a Schwartz-Bruhat function f on $V(\mathbb{A})$.

Recall that the set $R_{P,Q}(T,S)$ defined in the previous section is the support in $\mathfrak{a}_P$ of the characteristic function sending X to

$$\Gamma_P^Q(X,T)\Gamma_Q(X - T, S).$$

Therefore the integral over a in (4.1) can be seen as an integral over the set of a such that $H_P(a)$ lies in $R_{P,Q}\big(T - (T_2)_P^Q, S\big)$. We spent a lot of effort in the previous section

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producing a decomposition of $R_{P,Q}(T,S)$; let us now make use of it. Consider a closed region $R = R_i\big(T-(T_2)_P^Q, S\big)$ in the decomposition

$$R_{P,Q}\big(T-(T_2)_P^Q, S\big) = \bigcup_{i\in I} R_i\big(T-(T_2)_P^Q, S\big)$$

of $R_{P,Q}\big(T-(T_2)_P^Q, S\big)$ given in the previous section. Since $R_{P,Q}\big(T-(T_2)_P^Q, S\big)$ is the disjoint (modulo boundaries) union of these regions, and since there are only finitely many of the regions, it is sufficient to estimate the integral for $H_P(a)$ in this one region R. Recall that together with the region R we associated a disjoint decomposition of the set Π of non-zero weights of π with respect to A_P:

$$\Pi = \Pi_- \cup \Pi_0 \cup \Pi_+;$$

define the weight spaces

$$V_- = \bigoplus_{\lambda\in\Pi_-} V^\lambda, \quad V_0 = \bigoplus_{\lambda\in\Pi_0\cup\{0\}} V^\lambda, \quad V_+ = \bigoplus_{\lambda\in\Pi_+} V^\lambda.$$

Pick a basis of $V(\mathbb{Q})$ so that each basis element is in some V^λ. By setting the basis to be orthonormal, we obtain an inner product $v\cdot w$ and a norm $\|v\|$ on $V(\mathbb{Q})$ and on $V(\mathbb{R})$—notice that for $v, w \in V(\mathbb{Q})$, $v\cdot w$ is rational. We also set $V(\mathbb{Z})$ to be the set of integral linear combinations of the elements of this basis, and similarly with $V(\frac{1}{N}\mathbb{Z})$ for any positive integer N.

Replace the integral over $N_P(\mathbb{Q}) \setminus N_P(\mathbb{A})$ by one over ω_P, a relatively compact convex fundamental domain of $N_P(\mathbb{Q}) \setminus N_P(\mathbb{A})$ containing the identity. With this substitution, we need not worry about $N_P(\mathbb{Q})$-invariance of our expressions.

We must prove that there exists a t-finite approximation of the integral

$$\int_{\omega_P}\int_{\exp R} e^{-2\rho_P(H_P(a))}\phi_{\mathfrak{o}}(na, f)\, da\, dn,$$

where $\exp R = \{a \in A_P(\mathbb{R})^0 \mid H_P(a) \in R\}$.

Notice that because of Lemma 1.1, we have the following equality:

$$\begin{aligned}\phi_{\mathfrak{o}}(na, f) &= \sum_{\gamma\in\mathfrak{o}} f\big(\pi(na)^{-1}\gamma\big)\\ &= \sum_{\gamma\in(\mathfrak{o}\cap V_0(\mathbb{Q}))+V_+(\mathbb{Q})} f\big(\pi(na)^{-1}\gamma\big) + \sum_{\gamma\in\mathfrak{o}\cap V(\mathbb{Q})'} f\big(\pi(na)^{-1}\gamma\big),\end{aligned} \tag{4.2}$$

where $V(\mathbb{Q})'$ is the set of $\gamma \in V(\mathbb{Q})$ with a nonzero component in V_-.

We claim that the second term of the right-hand side of (4.2) is an error term. More precisely, we claim that there exists a fixed continuous seminorm $\|\cdot\|_1$ on the space fixed of Schwartz-Bruhat functions on $V(\mathbb{A})$ such that the expression

$$\int_{\exp R} e^{-2\rho_P(H_P(a))}\int_{\omega_P}\sum_{\gamma\in V(\mathbb{Q})'} \big|f\big(\pi(na)^{-1}\gamma\big)\big|\, dn\, da, \tag{4.3}$$

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can be bounded by $\|f\|_1 e^{-c\|T\|}$ for T sufficiently large.

We prove this as follows. Define a function $\overline{f}$ on $V(\mathbb{A})$ by

$$\overline{f}(v) = \sup_{n \in \omega_P} \left| f\big(\pi(n^{-1})v\big) \right|.$$

Now, for $a \in A_P(\mathbb{R})^0$ with $H_P(a)$ in $\mathfrak{a}_P^+$, the set $a^{-1}\omega_P a$ is contained in ω_P, and so, given any function h on $N_P(\mathbb{A})$,

$$\int_{\omega_P} h(a^{-1}na)\,dn \le \sup_{n \in \omega_N} |h(n)|. \tag{4.4}$$

(Notice that ω_P has volume one.) Therefore, (4.3) is bounded by

$$\begin{aligned}
&\int_{\exp R} e^{-2\rho_P(H_P(a))} \int_{\omega_P} \sum_{\gamma \in V(\mathbb{Q})'} \left| f\Big(\pi(a^{-1}na)^{-1}\big(\pi(a^{-1})\gamma\big)\Big) \right| da \\
&\quad \le \int_{\exp R} e^{-2\rho_P(H_P(a))} \sum_{\gamma \in V(\mathbb{Q})'} \overline{f}\big(\pi(a^{-1})\gamma\big)\, da.
\end{aligned} \tag{4.5}$$

Since ω_P is relatively compact, the function $\overline{f}$ is continuous and rapidly decreasing, that is, $\overline{f}$ is a finite sum of functions of the form $\prod_v f_v$, with each f_v a continuous function on $V(\mathbb{Q}_v)$ that is compactly supported if v is finite, and decreases faster than the inverse of any polynomial if v is infinite. In particular there is an integer

$$N_1(f) = \prod_p p^{n_p(f)}$$

determined by the support of $\overline{f}$ such that the sum in (4.5) can be taken on $V(\frac{1}{N_1(f)}\mathbb{Z})' = V(\mathbb{Q})' \cap V(\frac{1}{N_1(f)}\mathbb{Z})$ instead of $V(\mathbb{Q})'$.

By the definition of R we know that for a in $\exp R$ and $\lambda \in \Lambda_i^-$, we have

$$\lambda\big(H_P(a)\big) < -\delta_i B(T).$$

Since B is positive on $\mathcal{C}$, B is larger on all $\mathcal{C}_\varepsilon$ than some fixed multiple, depending on ε, of the norm. This implies the existence of a fixed positive constant k depending on ε such that

$$\lambda\big(H_P(a)\big) \le -k\|T\|, \quad \text{for all } a \in \exp R, \lambda \in \Pi_-. \tag{4.6}$$

There is also a fixed positive constant k', also depending on ε, so that

$$\lambda\big(H_P(a)\big) \ge -k'\|T\|, \quad \text{for all } a \in \exp R, \lambda \in \Pi_0 \cup \Pi_+.$$

The inequality (4.6) implies that we can force all the points $\pi(a^{-1})\gamma$, $a \in \exp R$, $\gamma \in V(\frac{1}{N_1(f)}\mathbb{Z})'$, to lie outside any fixed compact set, by choosing T sufficiently large.

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Given a vector $v \in V(\mathbb{A})$ and a weight $\lambda \in \Pi$, write v^λ for the component of v in V^λ and $v_{\mathbb{R}}$ for the component of v in $V(\mathbb{R})$. Let μ be a weight in Π_-. Since $\overline{f}$ is rapidly decreasing the previous paragraph implies that we can bound

$$\begin{aligned}\bigl|\overline{f}\bigl(\pi(a^{-1})\gamma\bigr)\bigr| &\le |||f|||_1\, \|\pi(a^{-1})\gamma^\mu_{\mathbb{R}}\|^{-\ell} \prod_{\lambda\in\Pi\setminus\mu} \bigl(1+\|\pi(a^{-1})\gamma^\lambda_{\mathbb{R}}\|^n\bigr)^{-1} \\ &= |||f|||_1 e^{\ell\mu(H_P(a))}\|\gamma^\mu_{\mathbb{R}}\|^{-\ell} \prod_{\lambda\in\Pi\setminus\mu} (1+e^{-n\lambda(H_P(a))}\|\gamma^\lambda_{\mathbb{R}}\|^n)^{-1}\end{aligned} \tag{4.7}$$

for all $\gamma \in V(\frac{1}{N_1(f)}\mathbb{Z})'$ with γ^μ nonzero, by choosing $T \in \mathcal{C}_\varepsilon$ sufficiently large, where $\ell \ge n$ are arbitrarily large fixed integers. The norm $|||\cdot|||_1$ is given by

$$|||f|||_1 = \sup_{\substack{\mu\in\Pi_- \\ v\in V(\mathbb{A}),\, v^\mu_{\mathbb{R}}\neq 0}} \Bigl(\overline{f}(v)\|v^\mu_{\mathbb{R}}\|^\ell \prod_{\lambda\in\Pi\setminus\mu}(1+\|v^\lambda_{\mathbb{R}}\|^n)\Bigr),$$

and is continuous with respect to the topology on the space of Schwartz-Bruhat functions on $V(\mathbb{A})$.

For $\lambda \in \Pi_-$ and x a non-negative real number we can bound

$$(1+e^{-n\lambda(H_P(a))}x)^{-1} \le (1+x)^{-1}.$$

For $\lambda \in \Pi_0 \cup \Pi_+$ and x a non-negative real number we can bound

$$\begin{aligned}1+e^{-n\lambda(H_P(a))}x &\ge \min(1, e^{n\lambda(H_P(a))})(1+x) \\ &\ge e^{-nk'\|T\|}(1+x).\end{aligned}$$

Also, since μ lies in Π_-,

$$e^{\ell\mu(H_P(a))} \le e^{-\ell k\|T\|}.$$

Putting together the above inequalities, we obtain that $\overline{f}\bigl(\pi(a^{-1})\gamma\bigr)$ is bounded by

$$|||f|||_1 \exp\bigl(-(\ell k - |\Pi_0\cup\Pi_+|nk')\|T\|\bigr)\, \|\gamma^\mu_{\mathbb{R}}\|^{-\ell} \prod_{\lambda\in\Pi\setminus\mu}(1+\|\gamma^\lambda_{\mathbb{R}}\|^n)^{-1},$$

where $|\Pi_0 \cup \Pi_+|$ denotes the cardinality of the set. Since $\|\gamma^\mu_{\mathbb{R}}\| \ge 1/N_1(f)$ with $N_1(f)$ a positive integer, and $\ell \ge n$, we have

$$\|\gamma^\mu_{\mathbb{R}}\|^{-\ell} = \|\gamma^\mu_{\mathbb{R}}\|^{-(\ell-n)}\, \|\gamma^\mu_{\mathbb{R}}\|^{-n} \le N_1(f)^{\ell-n}\frac{\bigl(N_1(f)^n+1\bigr)}{1+\|\gamma^\mu_{\mathbb{R}}\|^n} \le 2N_1(f)^\ell(1+\|\gamma^\mu_{\mathbb{R}}\|^n)^{-1},$$

and so we obtain the bound

$$2N_1(f)^\ell |||f|||_1 \exp\bigl(-(\ell k - |\Pi_0\cup\Pi_+|nk')\|T\|\bigr) \prod_{\lambda\in\Pi}(1+\|\gamma^\lambda_{\mathbb{R}}\|^n)^{-1}$$

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of $\overline{f}\big(\pi(a^{-1})\gamma\big)$; this bound is independent of the weight μ, and hence is valid for all $\gamma \in V(\frac{1}{N_1(f)}\mathbb{Z})'$. Choose n so large that for every $\lambda \in \Pi$, the sum

$$\sum_{\gamma \in V^\lambda(\mathbb{Z})} (1 + \|\gamma_{\mathbb{R}}\|^n)^{-1}$$

converges, and write C_λ for its value. Then for every natural number N and every $\lambda \in \Pi$, the sum

$$\sum_{\gamma \in V^\lambda(\frac{1}{N}\mathbb{Z})} (1 + \|\gamma_{\mathbb{R}}\|^n)^{-1}$$

is bounded by $C_\lambda N^n$. Choose $\ell \geq n$ so that $\ell k - |\Pi_0 \cup \Pi_+| nk' \geq c + 1$. Then (4.5) is bounded by

$$\begin{aligned}
2N_1(f)^\ell |||f|||_1 e^{-(c+1)\|T\|} &\left(\int_{\exp R} e^{-2\rho_P(H_P(a))}\, da \right) \sum_{\gamma \in V(\frac{1}{N_1(f)}\mathbb{Z})} \prod_{\lambda \in \Pi} (1 + \|\gamma_{\mathbb{R}}^\lambda\|^n)^{-1} \\
&\leq 2N_1(f)^\ell |||f|||_1 e^{-(c+1)\|T\|} \operatorname{vol}(R) \prod_{\lambda \in \Pi} \sum_{\gamma \in V^\lambda(\frac{1}{N_1(f)}\mathbb{Z})} (1 + \|\gamma_{\mathbb{R}}\|^n)^{-1} \\
&\leq C_1 N_1(f)^{\ell + |\Pi| n} |||f|||_1 e^{-(c+1)\|T\|} \operatorname{vol}(R),
\end{aligned}$$

where

$$C_1 = 2 \prod_{\lambda \in \Pi} C_\lambda$$

is a fixed constant. By Lemma 3.4, the extreme points of the region R are linear in $T - (T_2)_P^Q$ and S for all sufficiently large $T \in \mathcal{C}_\varepsilon$ and all $S \in \mathcal{C}_\varepsilon(1)$, so the volume of R is a polynomial in T and S. Therefore

$$C_1 N_1(f)^{\ell + |\Pi| n} |||f|||_1 e^{-(c+1)\|T\|} \operatorname{vol}(R) \leq C_1' N_1(f)^{\ell + |\Pi| n} |||f|||_1 e^{-c\|T\|}$$

for some fixed constant C_1'. This completes our bound of (4.3), since the norm $\|\cdot\|_1$ given by $\|f\|_1 = C_1' N_1(f)^{\ell + |\Pi| n} |||f|||_1$ is a fixed continuous seminorm.

Rewrite the first term of (4.2), with our fixed geometric equivalence class $\mathfrak{o}$, as

$$\sum_{\gamma_0 \in \mathfrak{o} \cap V_0(\mathbb{Q})} \sum_{\gamma_+ \in V_+(\mathbb{Q})} f\big(\pi(na)^{-1}(\gamma_0 + \gamma_+)\big)$$

$$\begin{aligned}
&= \sum_{\gamma_0 \in \mathfrak{o} \cap V_0(\mathbb{Q})} \sum_{\gamma_+ \in V_+(\mathbb{Q})} \int_{V_+(\mathbb{A})} f\big(\pi(na)^{-1}(\gamma_0 + v)\big) \cdot \psi(\gamma_+ \cdot v)\, dv \\
&= \sum_{\gamma_0 \in \mathfrak{o} \cap V_0(\mathbb{Q})} \int_{V_+(\mathbb{A})} f\big(\pi(na)^{-1}(\gamma_0 + v)\big)\, dv \\
&\quad + \sum_{\gamma_0 \in \mathfrak{o} \cap V_0(\mathbb{Q})} \sum_{\gamma_+ \in V_+(\mathbb{Q}) \setminus \{0\}} \int_{V_+(\mathbb{A})} f\big(\pi(na)^{-1}(\gamma_0 + v)\big) \psi(\gamma_+ \cdot v)\, dv;
\end{aligned} \tag{4.8}$$

the first step was Poisson summation on $V_+(\mathbb{Q})$. We have chosen here ψ to be the standard additive character on $\mathbb{A}$ given in Tate's thesis.

The following expression dominates the sum over all geometric equivalence classes $\mathfrak{o}$ of the second term of the right-hand side of (4.8):

$$\sum_{\gamma_0\in V_0(\mathbb{Q})}\ \sum_{\gamma_+\in V_+(\mathbb{Q})\setminus\{0\}}\left|\int_{V_+(\mathbb{A})} f\big(\pi(na)^{-1}(\gamma_0+\nu)\big)\,\psi(\gamma_+\cdot\nu)\,d\nu\right|. \tag{4.9}$$

We claim that the integral over a and n of (4.9) is another error term. We prove this as follows.

Define the function

$$f_{a,n}(\gamma_0,\gamma_+)=\int_{V_+(\mathbb{A})} f\big(\pi(an)^{-1}(\gamma_0+\nu)\big)\,\psi(\gamma_+\cdot\nu)\,d\nu,\qquad \gamma_0\in V_0(\mathbb{A}),\gamma_+\in V_+(\mathbb{A}).$$

For $\gamma_0\in V(\mathbb{Q})$, $\gamma_+\in V_+\setminus\{0\}$, $|f_{a,n}(\gamma_0,\gamma_+)|$ is the summand in (4.9) corresponding to γ_0 and γ_+. By (4.4), we have the inequality

$$\int_{\omega_P}|f_{a,n}(\gamma_0,\gamma_+)|\,dn\le\sup_{n\in\omega_P}|f_{a,n}(\gamma_0,\gamma_+)|.$$

Notice that

$$\begin{aligned} f_{a,n}(\gamma_0,\gamma_+)&=\int_{V_+(\mathbb{A})} f\big(\pi(n^{-1})\pi(a^{-1})(\gamma_0+\nu)\big)\,\psi(\gamma_+\cdot\nu)\,d\nu\\ &=\int_{V_+(\mathbb{A})} f\Big(\pi(n^{-1})\big(\pi(a^{-1})\gamma_0+\pi(a^{-1})\nu\big)\Big)\,\psi\big(\pi(a)\gamma_+\cdot\pi(a^{-1})\nu\big)\,d\nu\\ &=e^{2\rho_+(H_P(a))}\int_{V_+(\mathbb{A})} f\Big(\pi(n^{-1})\big(\pi(a^{-1})\gamma_0+\nu\big)\Big)\,\psi\big(\pi(a)\gamma_+\cdot\nu\big)\,d\nu\\ &=e^{2\rho_+(H_P(a))}f_{1,n}\big(\pi(a^{-1})\gamma_0,\pi(a)\gamma_+\big), \end{aligned}$$

where $e^{2\rho_+(H_P(a))}$ is the Jacobian of the change of variables

$$\nu\mapsto\pi(a)\nu,\qquad \nu\in V_+(\mathbb{A})$$

on $V_+(\mathbb{A})$, for $a\in A_P(\mathbb{R})$. Now, the function $f_{1,n}$ is Schwartz-Bruhat for each $n\in\omega_P$ and continuous with respect to n. Since ω_P is relatively compact the function f_N on $V_0(\mathbb{A})\times V_+(\mathbb{A})$ defined by

$$f_N(\nu_0,\nu_+)=\sup_{n\in\omega_P}|f_{1,n}(\nu_0,\nu_+)|,\qquad \nu_0\in V_0(\mathbb{A}),\qquad \nu_+\in V_+(\mathbb{A})$$

is continuous and rapidly decreasing.

The integral over a and n of (4.9) is therefore bounded by

$$\int_{\exp R} e^{(2\rho_+-2\rho_P)(H_P(a))}\sum_{\gamma_0\in V_0(\mathbb{Q})}\ \sum_{\gamma_+\in V_+(\mathbb{Q})\setminus\{0\}} f_N\big(\pi(a^{-1})\gamma_0,\pi(a)\gamma_+\big)\,da, \tag{4.10}$$

where, since the integrand is positive and the expression clearly converges, Fubini's theorem allowed the free interchange of integrals. Fubini's theorem will trivially apply through (4.23) because all integrals will be over compact sets of continuous functions and its use will not be mentioned. The function f_N is rapidly decreasing, $a \in \exp R$ is real, so the sums in (4.10) can be taken in $V(\frac{1}{N_2(f)}\mathbb{Z})$ instead of $V(\mathbb{Q})$, for some integer $N_2(f)$ depending only on f.

Pick new fixed positive constants k, k' (depending on ε) so that

$$\lambda\big(H_P(a)\big) \geq k\|T\|, \quad \text{for every } a \in \exp R,\ \lambda \in \Pi_+,$$
$$\lambda\big(H_P(a)\big) \leq k'\|T\|, \quad \text{for every } a \in \exp R,\ \lambda \in \Pi_0;$$

this again is possible because of the inequalities in (3.7). Let $a \in \exp R$ and $\gamma_+ \in V_+(\frac{1}{N_2(f)}\mathbb{Z}) \setminus \{0\}$, be arbitrary, and let $\mu \in \Pi^+$ be any weight in Π_+ such that γ_+^μ is non-zero. Then

$$\left\| \big(\pi(a)\gamma_+\big)_{\mathbb{R}} \right\| \geq \|\pi(a)(\gamma_+^\mu)_{\mathbb{R}}\| = e^{\mu(H_P(a))}\|(\gamma_+^\mu)_{\mathbb{R}}\| \geq \frac{1}{N_2(f)} e^{k\|T\|}.$$

Since f_N is rapidly decreasing, the following inequality holds for arbitrarily large integers $\ell \geq n$, $a \in \exp R$, $\gamma_0 \in V_0(\frac{1}{N_2(f)}\mathbb{Z})$, and $\gamma_+ \in V_+(\frac{1}{N_2(f)}\mathbb{Z}) \setminus \{0\}$, if $T \in \mathcal{C}_\varepsilon$ is sufficiently large:

$$\begin{aligned}
&\left| f_N\big(\pi(a^{-1})\gamma_0, \pi(a)\gamma_+\big) \right| \\
&\quad \leq |||f|||_2 \|\pi(a)(\gamma_+)_{\mathbb{R}}^\mu\|^{-\ell} \prod_{\lambda\in\Pi_0} \big(1 + \|\pi(a^{-1})(\gamma_0)_{\mathbb{R}}^\lambda\|^n\big)^{-1} \\
&\qquad\quad \cdot \prod_{\lambda\in\Pi_+\setminus\mu} \big(1 + \|\pi(a)(\gamma_+)_{\mathbb{R}}^\lambda\|^n\big)^{-1} \\
&\quad \leq 2N_2(f)^\ell |||f|||_2 \exp\big(-\|T\|(\ell k - |\Pi_0| n k')\big) \prod_{\lambda\in\Pi_0\cup\Pi_+} \big(1 + \|(\gamma_0+\gamma_+)_{\mathbb{R}}^\lambda\|^n\big)^{-1},
\end{aligned}$$

where $|||\cdot|||_2$ is a fixed continuous seminorm. Write $\ell' = \ell k - |\Pi_0| n k'$. We have bounded the summand in (4.10) independently of the choice of μ, so (4.10) is bounded by

$$\begin{aligned}
&2N_2(f)^\ell |||f|||_2 e^{-\ell'\|T\|} \left(\int_{\exp R} e^{(2\rho_+ - 2\rho_P)(H_P(a))}\, da \right) \\
&\qquad \cdot \sum_{\gamma\in(V_0\oplus V_+)(\frac{1}{N_2(f)}\mathbb{Z})} \prod_{\lambda\in\Pi_0\cup\Pi_+} (1 + \|\gamma_{\mathbb{R}}^\lambda\|^n)^{-1} \\
&\qquad\qquad = 2N_2(f)^\ell |||f|||_2 e^{-\ell'\|T\|} \left(\int_{\exp R} e^{2\rho_+ - 2\rho_P)(H_P(a))}\, da \right) \\
&\qquad\qquad\qquad \cdot \prod_{\lambda\in\Pi_0\cup\Pi_+} \sum_{\gamma^\lambda \in V^\lambda(\frac{1}{N_2(f)}\mathbb{Z})} (1 + \|\gamma_{\mathbb{R}}^\lambda\|^n)^{-1}.
\end{aligned}$$

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Choose n so large that the sum

$$\sum_{\gamma^\lambda \in V^\lambda(\mathbb{Z})} (1 + \|\gamma^\lambda_{\mathbb{R}}\|^n)^{-1}$$

converges for every $\lambda \in \Pi_0 \cup \Pi_+$, so that (4.10) is bounded above by

$$C_2 N_2(f)^{\ell+|\Pi_0\cup\Pi_+|n} |||f|||_2 e^{-\ell'\|T\|} \left(\int_{\exp R} e^{(2\rho_+ - 2\rho_P)(H_P(a))}\, da \right), \tag{4.11}$$

for some fixed constant C_2. Since the extreme points of R are linear in $T - (T_2)_P^Q$ and S, the expression in parentheses in (4.11) is bounded by the exponential of some fixed multiple of $\|T\|$. By choosing ℓ sufficiently large we obtain that (4.10) is bounded by $C_2' N_2(f)^{\ell+|\Pi_0\cup\Pi_+|n} |||f|||_2 e^{-c\|T\|}$ for all sufficiently large T, where C_2' is some fixed constant. The seminorm $\|\cdot\|_2$ given by $\|f\|_2 = C_2' N_2(f)^{\ell+|\Pi_0\cup\Pi_+|n} |||f|||_2$ is continuous, so that this term, too, is an error term.

We can now deal with the integral of the first summand of (4.8),

$$\int_{\exp R} e^{-2\rho_P(H_P(a))} \int_{\omega_P} \sum_{\gamma \in \mathfrak{o}\cap V_0(\mathbb{Q})} \int_{V_+(\mathbb{A})} f\big(\pi(na)^{-1}(\gamma + \nu)\big)\, d\nu\, dn\, da. \tag{4.12}$$

We are trying to prove that this integral has a t-finite approximation.

Given a subset $\mathcal{S}$ of Π_0, write $\mathfrak{s}(\mathcal{S})$ for the set of weights in Π that vanish on all

$$(\ker \mathcal{S}) \cap R = \ker\Big(\sum_{\lambda \in \mathcal{S}} (\operatorname{sgn} \lambda)\lambda\Big) \cap R;$$

by (3.7) we have $\mathfrak{s}(\mathcal{S}) \subseteq \Pi_0$.

Given $\nu \in V$, write $\operatorname{supp} \nu$ for the set of those weights λ such that ν has a non-zero component in the weight space V^λ. Given a subset Π' of Π_0, write

$$W_0(\Pi') = \bigoplus_{\lambda \in \Pi'} V^\lambda, \quad W_+(\Pi') = \bigoplus_{\lambda \in \Pi_0^+ \setminus \Pi'} V^\lambda,$$

$$W_0'(\Pi') = \big\{ \nu \in W_0(\Pi') \mid \mathfrak{s}\big(\operatorname{supp}(\nu) \cap \Pi_0^-\big) = \Pi' \big\}.$$

Notice that $W_0'(\Pi')$ is empty if $\mathfrak{s}(\Pi') \neq \Pi'$. By Lemma 1.1, we know that

$$\mathfrak{o} \cap \big(W_0'(\Pi')(\mathbb{Q}) + W_+(\Pi')(\mathbb{Q})\big) = \big(\mathfrak{o} \cap W_0'(\Pi')(\mathbb{Q})\big) + W_+(\Pi')'(\mathbb{Q}).$$

We can write

$$\sum_{\gamma \in \mathfrak{o}\cap V_0(\mathbb{Q})} F(\gamma) = \sum_{\substack{\Pi' \subseteq \Pi_0 \\ \mathfrak{s}(\Pi') = \Pi'}} \sum_{\gamma \in \mathfrak{o}\cap W_0'(\Pi')(\mathbb{Q})} \sum_{\gamma_+ \in W_+(\Pi')(\mathbb{Q})} F(\gamma + \gamma_+),$$

for any function F on $V_0(\mathbb{Q})$; a vector $\gamma \in \sigma \cap V_0(\mathbb{Q})$ appears in the summand corresponding to $\Pi' = \mathfrak{s}(\operatorname{supp} \gamma \cap \Pi_0^-)$.

There are only finitely many choices for Π'. Make one and write $W_0 = W_0(\Pi')$, $W_+ = W_+(\Pi')$, and $W_0' = W_0'(\Pi')$. Consider the summand in (4.12) corresponding to Π'. It is

(4.13)
$$\int_{\exp R} e^{-2\rho_P(H_P(a))} \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in W_+(\mathbb{Q})} \int_{V_+(\mathbb{A})} f\big(\pi(na)^{-1}(\gamma + \gamma_+ + \nu)\big)\, d\nu\, dn\, da$$
$$= \int_{\exp R} e^{-2\rho_P(H_P(a))} \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in W_+(\mathbb{Q})} \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(na)^{-1}(\gamma + \nu)\big)\, \psi(\gamma_+ \cdot \nu)\, d\nu\, dn\, da.$$

We will next break up the sum over γ_+ in (4.13).

Given a subset $\mathcal{S}$ of Π_0^+, write $\mathfrak{b}(\mathcal{S})$ for the set of weights in Π_0^+ that vanish on all

$$\ker(\Pi' \cup \mathcal{S}) \cap R = \ker\Big(\sum_{\lambda \in \Pi' \cup \mathcal{S}} (\operatorname{sgn} \lambda)\lambda\Big) \cap R.$$

Given a subset Π'' of Π_0^+, write

$$U_0(\Pi'') = \bigoplus_{\lambda \in \Pi'' \setminus \Pi'} V^\lambda, \quad U_+(\Pi'') = V_+ \oplus \bigoplus_{\lambda \in \Pi_0^+ \setminus \Pi''} V^\lambda,$$
$$U_0'(\Pi'') = \{\nu \in U_0(\Pi'') \mid \mathfrak{b}(\operatorname{supp} \nu) = \Pi''\}.$$

We can write
$$\sum_{\gamma_+ \in W_+(\mathbb{Q})} F(\gamma_+) = \sum_{\substack{\Pi'' \subset \Pi_0^+ \\ \mathfrak{b}(\Pi'') = \Pi''}} \sum_{\gamma_+ \in U_0'(\Pi'')(\mathbb{Q})} F(\gamma_+),$$
for any function F on $W_+(\mathbb{Q})$; a vector $\gamma_+ \in W_+(\mathbb{Q})$ appears in the summand corresponding to $\Pi'' = \mathfrak{b}(\operatorname{supp} \gamma_+)$.

There are only finitely many choices for Π''. Make one and write $U_0 = U_0(\Pi'')$, $U_+ = U_+(\Pi'')$, and $U_0' = U_0'(\Pi'')$. The summand of (4.13) corresponding to it is

$$\int_{\exp R} e^{-2\rho_P(H(a))} \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in U_0'(\mathbb{Q})} \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(na)^{-1}(\gamma + \nu)\big)\, \psi(\gamma_+ \cdot \nu)\, d\nu\, dn\, da. \tag{4.14}$$

Let $\Pi_1 \subseteq \Pi_0$ be the set of weights in Π that vanish on all $\ker(\Pi' \cup \Pi'') \cap R$. This latter polytope equals

$$\ker\Big(\sum_{\lambda \in \operatorname{supp}(\gamma + \gamma_+)} (\operatorname{sgn} \lambda)\lambda\Big) \cap R. \tag{4.15}$$

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for any $\gamma \in W_0'$, $\gamma_+ \in U_0'$. Then

$$\ker \Pi_1 \cap R = \ker(\Pi' \cup \Pi'') \cap R,$$

so Π_1 also equals the set of weights in Π_0 that vanish on $\ker \Pi_1 \cap R$. Recall that in the previous section we defined a number δ' and a decomposition

$$R_i\big(\delta', T - (T_2)_P^Q, S\big) = \bigcup_{j \in J} R_{i,j}\big(\delta', T - (T_2)_P^Q, S\big)$$

of part of the region $R = R_i\big(T - (T_2)_P^Q, S\big)$, that depended on Π_1 so that the regions $\overline{R_X} = \overline{R_i(T,S)_X}$ defined in Lemma 3.5 behaved well for X in any given $R_{i,j}(\delta', T, S)$. We will soon use this decomposition.

First, we break up the integral over a as follows. Let $\overline{A}_P(\mathbb{R})^0 \subset A_P(\mathbb{R})^0$ be a complement to the subgroup $\exp(\ker \Pi_1)$ so that the natural projection $p \colon \overline{A}_P(\mathbb{R})^0 \mapsto \overline{\mathfrak{a}_{\mathfrak{P}}} = \mathfrak{a}_P / \ker \Pi_1$ is an isomorphism, and normalize Haar measures on these two subgroups so that their product is da. We do this independently of f, T, S. Write $a = a_0 \overline{a}$, $a_0 \in \exp(\ker \Pi_1)$, $\overline{a} \in \overline{A}_P(\mathbb{R})^0$, for the canonical decomposition and write $\exp \overline{R} = \{\overline{a} \mid H_P(a) \in R\}$. Then the integral (4.14) becomes

$$(4.16) \quad \int_{\exp \overline{R}} e^{-2\rho_P(H_P(\overline{a}))} \int_{\exp R_{H_P(\overline{a})}} e^{-2\rho_P(H_P(a_0))} \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in U_0'(\mathbb{Q})} \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(n a_0 \overline{a})^{-1}(\gamma + v)\big)\, \psi(\gamma_+ \cdot v)\, dv\, dn\, da_0\, d\overline{a}.$$

where $\exp R_{H_P(\overline{a})}$ is the exponential of the set $R_{H_P(\overline{a})} = R_i\big(T - (T_2)_P^Q, S\big)_{H(\overline{a})}$ defined in Lemma 3.5. We will see that the dependence of the innermost integral on a_0 is particularly simple.

We first show that N_P preserves both $U_+(\mathbb{A})$ and $W_0 \oplus W_+ \oplus V_+$. The two facts are proven similarly, so we will consider only the second. It is clearly sufficient to prove that $\pi(n)V^\lambda$ lies in $W_0 \oplus W_+ \oplus V_+ = \bigoplus_{\mu \in \Pi' \cup \Pi_0^+ \cup \Pi_+} V^\mu$ for every vector $v^\lambda \in V^\lambda$, $\lambda \in \Pi' \cup \Pi_0^+ \cup \Pi_+$. A weight μ in $\operatorname{supp}\big(\pi(n)v^\lambda\big)$ is the sum of λ and a non-negative linear combination of $\alpha \in \Delta_P$, and so by the inequalities (3.7), must be at least as large as λ on R. If μ is 0 or $\operatorname{sgn}\mu = 1$ we have nothing to show, while if $\operatorname{sgn}\mu = -1$, then for any $X \in (\ker \Pi') \cap R$, $\mu(X)$ must be both at most (since $\mu \in \Pi^-$) and at least (since $\mu(X) \geq \lambda(X)$) zero, so that μ lies in Π', proving the fact.

We now prove that for $a_0 \in \exp(\ker \Pi_1)$, $n \in N_P(\mathbb{A})$, and $w \in W_0(\mathbb{A}) \oplus W_+(\mathbb{A}) \oplus V_+(\mathbb{A}) = W_0(\mathbb{A}) \oplus U_0(\mathbb{A}) \oplus U_+(\mathbb{A})$,

$$(4.17) \qquad \pi(n a_0 n^{-1})w - w \text{ lies in } U_+(\mathbb{A}).$$

We have already shown that $\pi(n^{-1})w$ can be written in the form $\pi(n^{-1})w = w_0 + w_+$, with $w_0 \in W_0(\mathbb{A}) + U_0(\mathbb{A})$, $w_+ \in U_+(\mathbb{A})$. The action of a_0 on $W_0 \oplus U_0$ is trivial, so

$$\pi(a_0 n^{-1})w = \pi(a_0)(w_0 + w_+) = w_0 + \pi(a_0)w_+,$$

and hence

$$\pi(a_0n^{-1})w - \pi(n^{-1})w = \pi(a_0)w_+ - w_+ \in U_+(\mathbb{A}).$$

Since the action of N_P preserves $U_+(\mathbb{A})$, we have proven (4.17) which implies that the change of variables

$$\nu \mapsto \pi(na_0n^{-1})\nu + \big(\pi(na_0n^{-1})\gamma - \gamma\big)$$

is an isomorphism on $W_+(\mathbb{A}) \oplus V_+(\mathbb{A})$ that does not change $\gamma_+ \cdot \nu$ for any $\gamma_+ \in U_0'(\mathbb{Q})$; its Jacobian is $e^{2\rho_+'(H_P(a_0))}$, where $2\rho_+'$ is the sum of all weights in $\Pi^+ \setminus \Pi'$, including multiplicities. The integral (4.16) therefore equals

$$\int_{\exp \overline{R}} e^{-2\rho_P(H_P(\overline{a}))} \left(\int_{R_{H_P(\overline{a})}} e^{(2\rho_+' - 2\rho_P)(Y)} dY \right) \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in U_0'(\mathbb{Q})} \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(n\overline{a})^{-1}(\gamma + \nu)\big)\, \psi(\gamma_+ \cdot \nu)\, d\nu\, dn\, d\overline{a}.$$

The region $R = R_i\big(T - (T_2)_P^Q, S\big)$ breaks up as

$$R = \bigcup_{j \in J_i} R_{i,j}\big(\delta', T - (T_2)_P^Q, S\big) \cup \Big(R \setminus R_i\big(\delta', T - (T_2)_P^Q, S\big)\Big),$$

where the sets on the right are disjoint modulo boundary; this gives a similar decomposition of $\exp \overline{R}$. We must estimate the contribution to (4.16) of the integral over $\overline{a}$ in each piece of the decomposition of $\exp \overline{R}$.

We first claim that the last piece in this decomposition gives an error term, that is, that the integral over $\overline{a}$ and Y with $H_P(\overline{a}) + Y \in \Big(R \setminus R_i\big(\delta', T - (T_2)_P^Q, S\big)\Big)$ of $e^{-2\rho_P(H_P(\overline{a}))} \cdot e^{(2\rho_+' - 2\rho_P)(Y)}$ times

$$(4.18) \quad \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in U_0'(\mathbb{Q})} \left| \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(n\overline{a})^{-1}(\gamma + \nu)\big)\, \psi(\gamma_+ \cdot \nu)\, d\nu \right| dn$$

can be bounded by $\|f\|_3 e^{-c\|T\|}$ for some fixed continuous seminorm $\|\ \|_3$. As with (4.3) and (4.9), we can bound (4.18) by an expression of the form

$$e^{2\rho_P'(H_P(\overline{a}))} \sum_{\gamma \in W_0'(\frac{1}{N_3(f)}\mathbb{Z})} \sum_{\gamma_+ \in U_0'(\frac{1}{N_3(f)}\mathbb{Z})} f_N\big(\pi(\overline{a})^{-1}\gamma, \pi(\overline{a})\gamma_+\big),$$

with f_N a continuous, rapidly decreasing function on $W_0(\mathbb{A}) \times U_0(\mathbb{A})$, and $N_3(f)$ an integer determined by the support of f. Also, there exists a fixed constant k'' such that

$$e^{(2\rho_+' - 2\rho_P)(H_P(\overline{a}))} \int_{R_P(\overline{a})} e^{(2\rho_+' - 2\rho_P)Y}\, dY$$

is bounded by $e^{k''\|T\|}$ for any $\overline{a} \in \exp \overline{R}$, for all sufficiently large $T \in \mathcal{C}_\varepsilon$ (recall that $R_{H_P(\overline{a})}$ is a slice in $R = R_i\big(T - (T_2)_P^Q, S\big)$).

Points X in $R \setminus R_i\big(\delta', T - (T_2)_P^Q, S\big)$ satisfy

$$(4.19) \qquad \sum_{\lambda \in \mathcal{B}} (\operatorname{sgn} \lambda) \lambda(X) \geq \delta' B\big(T - (T_2)_P^Q\big),$$

with $\mathcal{B} \subset \Pi_1$ a previously selected basis of $\operatorname{span} \Pi_1$. Since Π_0 is finite and T is sufficiently large, there exists a new fixed (and hence independent of Π_1) constant k, depending on ε, so that for each such X, some $\mu \in \Pi_1$ satisfies $(\operatorname{sgn} \mu)\mu(X) \geq k\|T\|$.

Now, let $X \in R \setminus R_i\big(\delta', T - (T_2)_P^Q, S\big)$, μ as in the previous paragraph, and γ be any vector in $W_0'(\mathbb{Q})$, γ_+ in $U_0'(\mathbb{Q})$. Since μ lies in Π_1, it vanishes on (4.15). The boundary hyperplanes of R that contain (4.15) are all of the form $(3.7)'$ or $(3.8)'$; let $\lambda_1, \ldots, \lambda_k$ be corresponding functionals (so that each λ_i lies in $\Pi \cup \Delta_P \cup \Delta_Q \cup \widehat{\Delta}_Q \cup \widehat{\Delta}_P^Q$). The theory of the polar (see [5, Theorem 6.4]) implies that

$$(4.20) \qquad \sum_{\lambda \in \operatorname{supp}(\gamma+\gamma_+)} (\operatorname{sgn} \lambda)\lambda$$

is a positive linear composition of $(\operatorname{sgn} \lambda_i)\lambda_i$, and that $(\operatorname{sgn} \mu)\mu$ is a non-negative linear combination of $(\operatorname{sgn} \lambda_i)\lambda_i$. All the constants involved in these linear combinations can be chosen independently of T and S, since all the functionals lie in the finite set $\Pi \cup \Delta_P \cup \Delta_Q \cup \widehat{\Delta}_Q \cup \widehat{\Delta}_P^Q$. Therefore, $(\operatorname{sgn} \mu)\mu$ is at most a fixed multiple of (4.20) and so there exists a new fixed constant k', depending on ε, such that some $\lambda' \in \operatorname{supp}(\gamma + \gamma_+)$ satisfies

$$(\operatorname{sgn} \lambda')\lambda'(X) \geq k'\|T\|.$$

At this point, we continue as with (4.10) to prove our claimed bound of (4.18). Therefore we need only estimate the contribution to (4.16) of the integral over $\overline{a}$ of the regions

$$(4.21) \qquad \exp\Big(\overline{R_{i,j}\big(\delta', T - (T_2)_P^Q, S\big)}\Big) = \big\{\overline{a} \mid H_P(a) \in R_{i,j}\big(\delta', T - (T_2)_P^Q, S\big)\big\}$$

for each $j \in J_i$.

Fix $j \in J_i$, and write $\exp \overline{R}_j(T)$ for the set (4.21), where we include the T to remind ourselves of the dependence on T. The contribution of $\exp \overline{R}_j(T)$ is

$$(4.22) \qquad \int_{\exp \overline{R}_j(T)} e^{-2\rho_P(H_P(\overline{a}))} \Big(\int_{R_{H_P(\overline{a})}} e^{(2\rho_+' - 2\rho_P)(Y)}\, dY\Big) \int_{\omega_P} \sum_{\gamma \in \mathfrak{v} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in U_0'(\mathbb{Q})} \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(n\overline{a})^{-1}(\gamma + \nu)\big)\, \psi(\gamma_+ \cdot \nu)\, d\nu\, dn\, d\overline{a},$$

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Corollary 3.7 says that the function Lemma 3.5 says that for $\overline{a} \in \exp \overline{R}_j(T)$, the extreme points of $R_{H_P(\overline{a})}$ are linear in $H_P(\overline{a})$, $T - (T_2)_P^Q$, and S. Lemma 4.2 then implies that the integral

$$X \mapsto \int_{R_X} e^{(2\rho'_+ - 2\rho_P)(Y)}\, dY, \quad X \in R_{i,j}\big(\delta', T - (T_2)_P^Q, S\big),$$

is a fixed t-finite function of X, $T - (T_2)_P^Q$, and S, for all well-situated (T, S).

We can therefore write (4.22) as

$$(4.23) \quad \int_{\exp \overline{R}(T)} e^{-2\rho_P(H)p(\overline{a}))} \nu\big(H_P(\overline{a}), T, S\big) \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in U_0'(\mathbb{Q})} \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(n\overline{a})^{-1}(\gamma + \nu)\big)\, \psi(\gamma_+ \cdot \nu)\, d\nu\, dn\, d\overline{a},$$

for a fixed t-finite function ν on $\mathfrak{a}_P \times \mathfrak{a} \times \mathfrak{a}$. This is still the integral of a continuous function on a compact set and so converges absolutely. At this point, we are almost done.

Write $\exp \overline{R}$ for the set $\{\overline{a} \mid \overline{H_P(a)} \in \overline{R}_{i,j}\}$, where $\overline{R}_{i,j}$ is as at the end of Section 3. The sum over all geometric equivalence classes of the absolute value of

$$(4.24) \quad \int_{\exp(\overline{R}_{i,j} \setminus \overline{R}(T))} e^{-2\rho_P(H_P(\overline{a}))} \nu\big(H_P(\overline{a}), T, S\big) \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in U_0'(\mathbb{Q})} \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(n\overline{a})^{-1}(\gamma + \nu)\big)\, \psi(\gamma_+ \cdot \nu)\, d\nu\, dn\, d\overline{a}$$

converges absolutely and can be shown to be an error term, since every point X in $\overline{R} \setminus \overline{R}(T)$ satisfies (4.19), so that the argument following (4.19) again applies.

However, the sum of (4.23) and (4.24) is the (absolutely convergent) integral

$$(4.25) \quad \int_{\exp \overline{R}_{i,j}} e^{-2\rho_P(H_P(a))} \nu\big(H_P(\overline{a}), T, S\big) \int_{\omega_P} \sum_{\gamma \in \mathfrak{o} \cap W_0'(\mathbb{Q})} \sum_{\gamma_+ \in U_0'(\mathbb{Q})} \int_{W_+(\mathbb{A}) \oplus V_+(\mathbb{A})} f\big(\pi(n\overline{a})^{-1}(\gamma + \nu)\big)\, \psi(\gamma_+ \cdot \nu)\, d\nu\, dn\, d\overline{a},$$

which is a t-finite function in T and S, since its dependence on them arises only through the function $\nu(\cdot, T, S)$. The seminorm $\|\cdot\|$ needed in the statement of the theorem can be chosen to be the sum of all the seminorms that appeared when bounding each error term.

The sum of (4.25) over all geometric equivalence classes converges, and is again t-finite, as the function ν does not depend on $\mathfrak{o}$. This completes the proof of the Theorem. ■

Remark The proof of theorem implies (just as in [8]) that the integral

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\gamma \in V(\mathbb{Q})} f\big(\pi(g^{-1})\gamma\big)$$

converges if and only if the linear functional

$$\sum_{\lambda \in \Pi} \max(m_\lambda \lambda, 0) - \sum_{\alpha \in \Sigma} \max(m_\alpha \alpha, 0)$$

is negative on $\mathfrak{a} \setminus \{0\}$, where m_λ and m_α denote the multiplicity of the weight in the representations π and Ad, respectively. The sufficiency of this condition for convergence of the integral is due to Weil [13]. Its necessity was apparently also known, and is due to Igusa.

Fix one π-dependent cone $\mathcal{C}$. The above theorem shows that on this cone, the functions $J_{\mathfrak{o}}^T(f,\pi)$ approximate t-finite functions. As in [8], the proof of the theorem allows us to explicitly produce the non-constant terms of each of the t-finite functions $P_{\mathfrak{o},\mathcal{C}}$, so all we need to completely determine the functions $P_{\mathfrak{o},\mathcal{C}}$ is the constant term with respect to any point in $\mathfrak{a}$. Let T_0 be the unique point in $\mathfrak{a}$ such that

$$H(w_s^{-1}) + s^{-1}T_0 = T_0$$

for every element s of the Weyl group of (G,A), where w_s is any representative of s in $G(\mathbb{Q})$; the existence of T_0 is the statement of Lemma 1.1 of [2]. Write $P_{\mathfrak{o},\mathcal{C}}(T)$ as a finite linear combination of functions $e^{\lambda(T-T_0)}(T-T_0)^n$, $\lambda \in \mathfrak{a}^*$, n a nonnegative integer, and set $J_{\mathfrak{o},\mathcal{C}}$ to be the constant term, that is the term where both λ and n equal zero. Then the basic form of the truncated Poisson summation formula for the representation π of G on V and the function f on $V(\mathbb{A})$ is the following theorem, proven exactly as in [8].

Theorem 4.3

$$\sum_{\mathfrak{o} \in \mathfrak{o}} J_{\mathfrak{o},\mathcal{C}}(f,\pi) = \sum_{\tilde{\mathfrak{o}} \in \tilde{\mathfrak{O}}} J_{\tilde{\mathfrak{o}},\mathcal{C}}(\hat{f},\tilde{\pi}).$$

Remark Notice that because the weights of $\tilde{\pi}$ are the negatives of the weights of π, that the cones $\mathcal{C}$ determined by π and $\tilde{\pi}$ are the same.

The definition of $J_{\mathfrak{o},\mathcal{C}}(f,\pi)$ depended on a number of choices. The methods of [8] (based on those in [2]) show that the distributions $J_{\mathfrak{o},\mathcal{C}}$ are independent of ω and T_1, and that if s is an element of the Weyl group of (G,A) and J' denotes the constant term of the truncated integral with respect to the non-standard minimal parabolic subgroup $w_s^{-1}P_0w_s$, then

$$J_{\mathfrak{o},\mathcal{C}}(f,\pi) = J'_{\mathfrak{o},s^{-1}\mathcal{C}}(f,\pi).$$

If the representation π is the Adjoint representation, then this formula does depend on our choice of K, but for other representations it need not.

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