Canad. J. Math. Vol. **60** (2), 2008 pp. 313–333

Asymptotic Properties for Increments of l^∞-Valued Gaussian Random Fields

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Abstract. This paper establishes general theorems which contain both moduli of continuity and large incremental results for l^∞-valued Gaussian random fields indexed by a multidimensional parameter under explicit conditions.

1 Introduction and Results

Initial studies on the asymptotics of increments of Wiener and related processes, partial sum and empirical processes were integrated and furthered as well in [10]. Since then, various limit theories on moduli of continuity and large incremental results have been developed for l^p-valued, $1 \le p < \infty$, or finite dimensional space-valued Gaussian and related stochastic processes [4, 6, 7, 11, 12, 19, 23], and for renewal processes [32]. Moreover, Csörgő, Lin, and Shao [8] obtained moduli of continuity results for l^∞-valued one-parameter Gaussian and Ornstein–Uhlenbeck processes.

For illustration of the latter and further reference, we introduce one of the inspiring results of [8] (see Theorem 1.1 below). Let $\{X_k(t), -\infty < t < \infty\}_{k=1}^\infty$ be a sequence of centered continuous Gaussian processes with stationary increments $\sigma_k^2(h) := E\{X_k(t+h) - X_k(t)\}^2$, where $\sigma_k(h)$ are nondecreasing in $h > 0$.

We recall that a function $Q(x)$ is said to be *quasi-increasing* on (a, b) if there exists a constant $c > 0$ such that $Q(x) \le cQ(y)$ for $a < x < y < b$.

Put $\sigma_*^2(h) = \max_{k\ge1} \sigma_k^2(h)$, and suppose that $\sigma_*^2(h)/h^\alpha$ is quasi-increasing for some $\alpha > 0$.

We quote the following result from [8].

Theorem 1.1 *Let $X_k(\cdot)$ and $\sigma_k(\cdot)$ be as above. Suppose that there exist positive numbers A and h_0 such that*

$$\sum_{k=1}^{\infty} \sigma_k^A(h_0) < \infty. \tag{1.1}$$

Received by the editors July 4, 2005; revised September 6, 2006.

The first author was supported by NSERC Canada grants of Miklós Csörgő and Barbara Szyszkowicz at Carleton University, Ottawa, and by KRF-2006-C00052.The second author was supported by NSERC Canada grants at Carleton University, Ottawa.

AMS subject classification: Primary: 60F15; secondary: 60G15, 60G60.

Keywords: l^∞-valued Gaussian random field, modulus of continuity, regularly varying function, large deviation probability.

https://doi.org/10.4153/CJM-2008-015-4 Published online by Cambridge University Press

Let y_h be the solution of the equation

$$\sum_{k=1}^{\infty}(hy_h)^{\sigma_*^2(h)/\sigma_k^2(h)} = h. \tag{1.2}$$

If, in addition, $X_k(\cdot)$, $k = 1, 2, \dots,$ are independent and for $0 \le t_1 < t_2 \le t_3 < t_4$,

$$E\big\{\big(X_k(t_2) - X_k(t_1)\big)\big(X_k(t_4) - X_k(t_3)\big)\big\} \le 0, \tag{1.3}$$

then

$$\lim_{h\downarrow 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s) - X_k(t)|}{\sigma_*(h)\big\{2\log(1/(hy_h))\big\}^{1/2}} = 1 \quad \textit{a.s.} \tag{1.4}$$

Further, if condition (1.1) *is replaced by conditions for $0 < h \le h_0$ so that*

$$\inf_{0<s\le h}\frac{\sigma_*(s)}{\sigma_k(s)} \ge c_1\frac{\sigma_*(h)}{\sigma_k(h)} \text{ for some } c_1 > 0 \text{ and every } k \ge 1 \tag{1.5}$$

and

$$\sum_{k=1}^{\infty} h^{\sigma_*^2(h)/\sigma_k^2(h)} < \infty, \tag{1.6}$$

then (1.4) *remains true with $y_h = 1$, that is,*

$$\lim_{h\downarrow 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s) - X_k(t)|}{\sigma_*(h)\big\{2\log(1/h)\big\}^{1/2}} = 1 \quad \textit{a.s.} \tag{1.7}$$

In the proof of Theorem 1.1 we can find that $0 < y_h \le 1$, condition (1.1) implies condition (1.6), and that the latter condition guarantees that the solution of equation (1.2) exists and is unique. Thus we conclude that conditions (1.3) and (1.6) are essential to get (1.7) which is a modulus of continuity for l^∞-valued one-parameter Gaussian processes.

As an analogue of (1.7), Lin and Quin [26] obtained a large incremental result for l^∞-valued one-parameter Gaussian processes under similar conditions to those of Theorem 1.1.

Various aspects of infinite dimensional Gaussian processes have been extensively studied in the literature since the appearance of Dawson [13]. Their importance is based on their natural roles in many different areas of pure and applied mathematics. In particular, infinite dimensional Ornstein–Uhlenbeck processes have played a prominent role in the study of stochastic differential equations [1, 14, 18, 29, 34]. They also appeared in constructive quantum field theory [3, 16], in the study of infinite particle systems [17] and of infinite dimensional diffusions [18, 20, 28, 30, 33]. The papers listed in the first paragraph of this section also deal with various path properties of some of the processes that are found in the just mentioned works (*cf.* [7, 34]).

The object of this paper is to establish general theorems which contain both moduli of continuity and large incremental results for l^∞-valued Gaussian random fields indexed by a multidimensional parameter under explicit conditions as in (1.8) below and (1.11) of Theorem 1.3, in place of conditions (1.1)–(1.5) of Theorem 1.1.

Given the nonpositive condition (1.3) of Theorem 1.1, it is in general easy enough to prove liminf results to get (1.4) and (1.7) by simply applying Slepian's lemma (see Lemma 2.4 below). However, since condition (1.11) of Theorem 1.3 yields a positive (or nonpositive) covariance function of increments, as opposed to the restricted condition (1.3), the proofs of Theorems 1.3, 1.6 and 1.8 below are accomplished with new techniques built on several lemmas that are of interest on their own.

We assume that the realizations of random fields $\{X_k(\mathbf{t}), \mathbf{t} \in [0,\infty)^N\}_{k=1}^\infty$ indexed by N-dimensional parameter $\mathbf{t} := (t_1, \dots, t_N)$ are different objects and that the choice of coordinates of the parameter is not necessarily limited to length and time. Any appropriate scale of measurement might be involved.

We first introduce some notations and conditions that will be used throughout. Let $\mathbf{t} = (t_1, \dots, t_N)$ and $\mathbf{s} = (s_1, \dots, s_N)$ be vectors in $[0,\infty)^N$. Define:

$$\mathbf{0} = (0, \dots, 0) \quad \text{and} \quad \mathbf{1} = (1, \dots, 1) \text{ in } [0,\infty)^N;$$
$$(\mathbf{t}, \mathbf{s}) = (t_1, \dots, t_N, s_1, \dots, s_N) \in [0,\infty)^{2N};$$
$$\mathbf{t} \le \mathbf{s} \quad \text{if } t_i \le s_i \text{ for all integers } 1 \le i \le N;$$
$$\mathbf{t} \pm \mathbf{s} = (t_1 \pm s_1, \dots, t_N \pm s_N); \quad \mathbf{ts} = (t_1 s_1, \dots, t_N s_N);$$
$$a\mathbf{t} = (at_1, \dots, at_N) \quad \text{for a real number } a.$$

Let $\mathbb{D} = \{\mathbf{t} : \mathbf{t} := (t_1, \dots, t_N) \in [0,\infty)^N\}$ be an N-dimensional parameter space with the Euclidean norm $\|\cdot\|$ such that $\|\mathbf{t}\| = \left(\sum_{i=1}^N t_i^2\right)^{1/2}$. Let $\{X_k(\mathbf{t}), \mathbf{t} \in \mathbb{D}\}_{k=1}^\infty$ be a sequence of real-valued continuous and centered Gaussian random fields with $X_k(\mathbf{0}) = 0$ and stationary increments

$$\sigma_k(\|\mathbf{h}\|) := \sqrt{E\{X_k(\mathbf{t}+\mathbf{h}) - X_k(\mathbf{t})\}^2},$$

where $\sigma_k(h)$ are nondecreasing continuous functions of $h > 0$.

A positive function $f(h)$ of $h > 0$ is said to be *regularly varying* with exponent $\alpha > 0$ at $b \ge 0$ if $\lim_{h\to b}\{f(xh)/f(h)\} = x^\alpha$ for $x > 0$.

Put $\sigma_*(h) = \sup_{k\ge1} \sigma_k(h)$ and assume that $\sigma_*(h)$ is a regularly varying function with exponent α at $b \ge 0$ for some $0 < \alpha < 1$. Suppose that either

$$\text{(1.8)} \qquad \sigma_*(h)/\sigma_k(h) \ge \sqrt{1+\log k}, \quad k \ge 1 \quad \text{or} \quad \sum_{k=1}^\infty h^{\sigma_*^2(h)/\sigma_k^2(h)} < \infty.$$

For instance, take $\sigma_k(h) = h^\alpha/\sqrt{1 + 2\log k}$, $h > 0$. Then it is clear that condition (1.1) in Theorem 1.1 is not satisfied but condition (1.8) is satisfied; in the latter case of (1.8) one can, for example, take $h = h_n = e^{-n}$, $n > 1$.

We note that condition (1.8) is essential and used only for getting large deviation probabilities for l^∞-valued Gaussian random fields as in Lemmas 2.1 and 2.2, and that it is an explicit condition as well.

Let $\{\mathbf{X}(\mathbf{t}) := (X_1(\mathbf{t}), X_2(\mathbf{t}), \dots), \mathbf{t} \in \mathbb{D}\}$ be a Gaussian random field taking values in l^∞-space with l^∞-norm $\|\cdot\|_\infty$ defined by $\|\mathbf{X}(\mathbf{t})\|_\infty = \sup_{k\ge 1} |X_k(\mathbf{t})|$.

For each $i = 1, 2, \dots, N$, let $a_i(\mathbf{T})$ and $b_i(\mathbf{T})$ be positive continuous functions of $\mathbf{T} = (T_1, \dots, T_N) > \mathbf{0}$ such that $a_i(\mathbf{T}) \le b_i(\mathbf{T})$, and denote:

$$\mathbf{a_T} = (a_1(\mathbf{T}), \dots, a_N(\mathbf{T})), \quad \mathbf{b_T} = (b_1(\mathbf{T}), \dots, b_N(\mathbf{T})),$$

$$\beta_1(\mathbf{T}) = \Big\{ 2\big(N \log(\|\mathbf{b_T}\|/\|\mathbf{a_T}\|) + \log|\log\|\mathbf{b_T}\||\big) \Big\}^{1/2},$$

$$\beta_2(\mathbf{T}) = \Big\{ 2\big(N \log(\|\mathbf{b_T}\|/\|\mathbf{a_T}\|) + \log_\theta|\log\|\mathbf{b_T}\||\big) \Big\}^{1/2},$$

$$\beta_3(\mathbf{T}) = \Big\{ 2N \log(\|\mathbf{b_T}\|/\|\mathbf{a_T}\|) \Big\}^{1/2},$$

where $\log x = \ln(\max\{x, 1\})$ and $1 < \theta < e$.

In the above context, in case $k = N = 1$, if we put $\mathbf{X}(\mathbf{t}) = X(t)$, $\sigma_k(\cdot) = \sigma(\cdot)$, $\mathbf{a_T} = a_T$, and $\mathbf{b_T} = b_T$, then we can see that the results in this paper generalize some main theorems related to moduli of continuity and large incremental results for 1-dimensional one parameter Wiener and further Gaussian and related stochastic processes in [4, 7, 9, 10, 25, 27, 35]. For example, if we put $a_T = 1/T = h$ $(0 < h < 1)$ and $b_T = 1$ in Corollary 1.9, we obtain the modulus of continuity [7, (3.11)]; if we put $b_T = T$ in Corollary 1.9 as well, we obtain the large incremental result [7, (3.17)].

The main results are as follows.

Theorem 1.2 *Let $\mathbf{X}(\mathbf{t})$ and $\sigma_*(\cdot)$ be as above. For each $i = 1, 2, \dots, N$, let $a_i(\mathbf{T})$ and $b_i(\mathbf{T})$ be positive continuous functions on $(0,\infty)^N$ such that*

$$\frac{\|\mathbf{b_T}\|}{\|\mathbf{a_T}\|} + \|\mathbf{a_T}\| \to \infty \quad \text{as } \|\mathbf{T}\| \to \infty. \tag{1.9}$$

Then we have

$$\limsup_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_1(\mathbf{T})} \le 1 \quad a.s. \tag{1.10}$$

Note that $\{\mathbf{s} : \mathbf{0} \le \mathbf{s} \le \mathbf{h}\} \subset \{\mathbf{s} : \|\mathbf{s}\| \le \|\mathbf{h}\|\}$. Hence we prefer to take suprema via Euclidean norms of vectors.

Condition (1.9) implies that $\mathbf{a_T}$ and $\mathbf{b_T}$ may be many kinds of functions. However, in order to obtain the opposite inequality of (1.10), the conditions on $\mathbf{a_T}$, $\mathbf{b_T}$ and $\sigma_*(\cdot)$ are a little bit restricted as in the following theorem.

Theorem 1.3 *Let $\mathbf{X}(\mathbf{t})$ and $\sigma_*(\cdot)$ be as in Theorem 1.2. For each $i = 1, 2, \dots, N$, let $a_i(\mathbf{T})$ and $b_i(\mathbf{T})$ be positive increasing and continuous functions on $(0,\infty)^N$ such that*

$\lim_{\|\mathbf{T}\|\to\infty} b_i(\mathbf{T}) = \infty$ *and* $b_i(\mathbf{T})/a_i(\mathbf{T})\ (>1)$ *is increasing. If, in addition,* $X_k(\cdot)$, $k = 1, 2, \dots,$ *are independent and there are positive constants* c_1 *and* c_2 *such that, for* $h > 0$,

$$(1.11)\qquad \Big|\frac{d\sigma_*^2(h)}{dh}\Big| \le c_1\frac{\sigma_*^2(h)}{h} \quad \text{and} \quad \Big|\frac{d^2\sigma_*^2(h)}{dh^2}\Big| \le c_2\frac{\sigma_*^2(h)}{h^2},$$

then we have

$$\limsup_{\|\mathbf{T}\|\to\infty} \frac{\|\mathbf{X}(\mathbf{b_T}+\mathbf{a_T})-\mathbf{X}(\mathbf{b_T})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_1(\mathbf{T})} \ge 1 \quad \text{a.s.}$$

The class of variance functions $\sigma_*^2(\cdot)$ satisfying condition (1.11) contains all concave functions with $0 < \alpha \le 1/2$ (*e.g.*, $\sigma_*^2(h) = \sqrt{h}$) and convex functions with $1/2 < \alpha < 1$ (see [19]). We recall that the correlation function on increments of a stochastic process with stationary increments is nonpositive (positive) if and only if its variance function is nearly concave (convex). In this regard, compare condition (1.3) of Theorem 1.1 with the paragraph following (2.6) below; see also the nonpositive conditions [7, (3.10), (4.2)].

The proofs of Theorems 1.3 and 1.6 under condition (1.11) as above are accomplished via several lemmas, because we must compute correlation functions for increments of the Gaussian random field (see (2.6) and (2.18)).

From Theorems 1.2 and 1.3 we obtain the following lim sup result.

Corollary 1.4 *Under the assumptions of Theorem* 1.3, *we have*

$$\limsup_{\|\mathbf{T}\|\to\infty}\ \sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|}\ \sup_{\|\mathbf{s}\|\le\|\mathbf{a_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_1(\mathbf{T})} = 1 \quad \text{a.s.},$$
$$\limsup_{\|\mathbf{T}\|\to\infty} \frac{\|\mathbf{X}(\mathbf{b_T}+\mathbf{a_T})-\mathbf{X}(\mathbf{b_T})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_1(\mathbf{T})} = 1 \quad \text{a.s.}$$

From now on, we are to show that lim inf results differ from their corresponding lim sup results under the additional condition (1.12) of the next theorem.

Theorem 1.5 *Let* $\mathbf{X}(\mathbf{t})$ *and* $\sigma_*(\cdot)$ *be as in Theorem* 1.2. *For each* $i = 1, 2, \dots, N$, *let* $a_i(\mathbf{T})$ *and* $b_i(\mathbf{T})$ *be positive continuous functions on* $(0,\infty)^N$ *such that as* $\|\mathbf{T}\|\to\infty$,

$$(1.12)\qquad \|\mathbf{b_T}\| \to \infty \text{ or } 0 \quad \text{and} \quad \frac{\log\big(\|\mathbf{b_T}\|/\|\mathbf{a_T}\|\big)}{\log_\theta|\log\|\mathbf{b_T}\||} \to r, \quad 0 \le r \le \infty,$$

where θ *is as in* $\beta_2(\mathbf{T})$. *Then we have*

$$\liminf_{\|\mathbf{T}\|\to\infty}\ \sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|}\ \sup_{\|\mathbf{s}\|\le\|\mathbf{a_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_2(\mathbf{T})} \le \Big(\frac{rN}{1+rN}\Big)^{1/2} \quad \text{a.s.}$$

Theorem 1.6 *Let* $\mathbf{X}(\mathbf{t})$ *and* $\sigma_*(\cdot)$ *be as in Theorem* 1.5. *Assume that conditions* (1.11) *and* (1.12) *are satisfied. Then we have*

$$\liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a}_\mathbf{T})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\beta_2(\mathbf{T})} \ge \Big(\frac{rN}{1+rN}\Big)^{1/2} \quad a.s. \tag{1.13}$$

Condition (1.12) guarantees that the class of vector functions $\mathbf{a}_\mathbf{T}$ and $\mathbf{b}_\mathbf{T}$ contains many functions [2]. As a consequence of Theorems 1.2, 1.3, 1.5, 1.6, and 1.8, we can obtain moduli of continuity as well as large incremental results as in Examples 1.10 and 1.11. We note also that condition (1.12) and $\beta_2(T)$ are tools that can be used to show various deviations between lim sup and lim inf results in other random fields, as well as in this paper.

Combining Theorems 1.5 and 1.6, we obtain the following lim inf result, which deviates from Corollary 1.4.

Corollary 1.7 *Under the assumptions of Theorem* 1.6, *we have*

$$\begin{aligned}\liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_\mathbf{T}\|} & \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\beta_2(\mathbf{T})} \\ &= \liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a}_\mathbf{T})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\beta_2(\mathbf{T})} \\ &= \Big(\frac{rN}{1+rN}\Big)^{1/2} \quad a.s.\end{aligned}$$

and, equivalently, by (1.12),

$$\begin{aligned}\liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_\mathbf{T}\|} & \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\beta_1(\mathbf{T})} \\ &= \liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a}_\mathbf{T})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\beta_1(\mathbf{T})} \\ &= \Big(\frac{rN}{rN+\log\theta}\Big)^{1/2} \quad a.s.\end{aligned} \tag{1.14}$$

where $1<\theta<e$ in the definition of $\beta_2(\mathbf{T})$. The reason why θ cannot be equal to e is shown following (2.20) below. When $0\le r<\infty$, it is clear that lim sup results in Corollary 1.4 are different from lim inf results (1.14) if $\|\mathbf{a}_\mathbf{T}\|\to\infty$.

In order to obtain a limit result, we consider the following condition (1.15) of Theorem 1.8 when $r=\infty$ in (1.12). In condition (1.15), note in particular that there is a case that $\|\mathbf{b}_\mathbf{T}\|$ converges to a positive constant as well as $\|\mathbf{b}_\mathbf{T}\|\to\infty$ (or 0) in (1.12), as $\|\mathbf{T}\|\to\infty$.

Theorem 1.8 *Let* $\mathbf{X}(\mathbf{t})$ *and* $\sigma_*(\cdot)$ *be as in Theorem* 1.2. *For each* $i=1,2,\dots,N$, *let* $a_i(\mathbf{T})$ *and* $b_i(\mathbf{T})$ *be positive continuous functions on* $(0,\infty)^N$ *such that*

$$\lim_{\|\mathbf{T}\|\to\infty} \frac{\log\big(\|\mathbf{b}_\mathbf{T}\|/\|\mathbf{a}_\mathbf{T}\|\big)}{\log_\theta|\log\|\mathbf{b}_\mathbf{T}\||} = \infty. \tag{1.15}$$

If condition (1.11) *is satisfied, then*

$$(1.16)\qquad \liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a_T})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_3(\mathbf{T})} \ge 1 \quad a.s.$$

Combining Theorems 1.2 and 1.8, we arrive at the following limit result.

Corollary 1.9 *Under the assumptions of Theorem* 1.8 *we have*

$$(1.17)\qquad \begin{aligned} &\lim_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a_T})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_i(\mathbf{T})} \\ &\qquad = \lim_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_i(\mathbf{T})} \\ &\qquad = 1, \quad i=1,2,3 \quad a.s. \end{aligned}$$

For example, in the case of $N=1$, if we put $\mathbf{b_T}=b_1(T_1)=1$ and $\mathbf{a_T}=a_1(T_1)=1/T_1=h$ $(0<h<1)$ in (1.17), then we obtain the modulus of continuity (1.7) of Theorem 1.1; if $\mathbf{b_T}=T_1$ and $\mathbf{a_T}=a_1(T_1)$ in (1.17), then we obtain the large incremental result [26, (2.9)].

For a one-parameter Wiener process with $\sigma_*(h)=\sqrt{h}$, initial results that are preliminaries to Corollaries 1.4, 1.7, and 1.9 can be found in [10].

The structures of the main theorems above and the techniques for their proofs can be applied to develop a similar limit theory for increments of l^∞-valued, l^p-valued, $1\le p<\infty$, or finite dimensional space-valued multiparameter random fields, extensions of stochastic processes that are dealt with in several papers: Ornstein–Uhlenbeck processes in [7], Gaussian processes in [4, 5, 25], and Lévy Brownian motion in [23, 35].

Returning to our present exposition, we present two examples.

Example 1.10 (Large incremental result) Let $\mathbf{X}(\mathbf{t})$ and $\sigma_*(\,\cdot\,)$ be as in Theorem 1.2. For $\mathbf{T}=(T_1,T_2,T_3,T_4)>\mathbf{0}$ with $T_1>T_2>T_3>T_4$, let

$$\mathbf{b_T}=\big(\sqrt{(T_1-T_3)^2-(T_1-T_2)^2},\sqrt{(T_2-T_4)^2-(T_3-T_4)^2},T_2-T_3,T_1-T_4\big),$$
$$\mathbf{a_T}=(T_2-T_3)\mathbf{b_T}/\|\mathbf{b_T}\|.$$

For convenience, take $T_1=T^2e^T$, $T_2=e^T$, $T_3=T$, $T_4=1$ for $T>1$. Then $\mathbf{a_T}$ and $\mathbf{b_T}$ satisfy all the conditions of Theorems 1.2, 1.3, 1.5 and 1.6 with

$$\begin{aligned} \|\mathbf{a_T}\| &= T_2-T_3=e^T-T, \\ \|\mathbf{b_T}\| &= T_1+T_2-T_3-T_4=(T^2+1)e^T-(T+1), \\ \beta_1(T) &= \{8\log[((T^2+1)e^T-(T+1))/(e^T-T)] \\ &\quad +2\log\log[(T^2+1)e^T-(T+1)]\}^{1/2}, \\ \mathbf{a_T}<\mathbf{b_T} &\quad\text{and}\quad r=2\log\theta \text{ in (1.12)}. \end{aligned}$$

Thus, by Corollary 1.4, we have

$$\begin{aligned}\limsup_{T\to\infty}\sup_{\|\mathbf{t}\|\le(T^2+1)e^T-T-1}\sup_{\|\mathbf{s}\|\le e^T-T}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(e^T-T)\,\beta_1(T)}&=\limsup_{T\to\infty}\frac{\|\mathbf{X}(\mathbf{b_T}+\mathbf{a_T})-\mathbf{X}(\mathbf{b_T})\|_\infty}{\sigma_*(e^T-T)\beta_1(T)}\\&=1\quad\text{a.s.}\end{aligned}$$

and by (1.14) of Corollary 1.7, we have

$$\begin{aligned}\liminf_{T\to\infty}\sup_{\|\mathbf{t}\|\le(T^2+1)e^T-T-1}\sup_{\|\mathbf{s}\|\le e^T-T}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(e^T-T)\,\beta_1(T)}&=\liminf_{T\to\infty}\sup_{\|\mathbf{t}\|\le(T^2+1)e^T-T-1}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a_T})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(e^T-T)\,\beta_1(T)}\\&=\sqrt{8/9}\qquad\text{a.s.}\end{aligned}$$

On the other hand, from Corollary 1.9, we can obtain a modulus of continuity as follows.

Example 1.11 (Modulus of continuity) Let $\mathbf{X}(\mathbf{t})$ and $\sigma_*(\,\cdot\,)$ be as in Theorem 1.2. For convenience, let $\mathbf{T}=(T_1,T_2,T_3)=(1,2,3)/h^2$ for $0<|h|<6/7$, then $\|\mathbf{T}\|\to\infty$ if and only if $h\to 0$. Put $\mathbf{a_T}=(1/T_1,1/T_2,1/T_3)$ and $\mathbf{b_T}=|h|\mathbf{a_T}/\|\mathbf{a_T}\|$. Then $\mathbf{a_T}$ and $\mathbf{b_T}$ satisfy conditions of Corollary 1.9 with

$$\mathbf{a_T}=(1,1/2,1/3)h^2=:\mathbf{a}_h,\quad \|\mathbf{a_T}\|=7h^2/6,\quad \|\mathbf{b_T}\|=|h|,$$

$$\beta_1(\mathbf{T})=\Big\{2\Big(3\log\Big(\frac{6}{7|h|}\Big)+\log\log\frac{1}{|h|}\Big)\Big\}^{1/2}=:\beta_h.$$

Thus, by (1.17), we arrive at

$$\begin{aligned}\lim_{h\to0}\sup_{\|\mathbf{t}\|\le|h|}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a}_h)-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(7h^2/6)\beta_h}&=\lim_{h\to0}\sup_{\|\mathbf{t}\|\le|h|}\sup_{\|\mathbf{s}\|\le 7h^2/6}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(7h^2/6)\beta_h}\\&=1\quad\text{a.s.}\end{aligned}$$

2 Proofs

We shall accomplish the proofs of Theorems 1.2, 1.3, 1.5, 1.6, and 1.8 via several lemmas. The following lemma is another version of Fernique's lemma [15] for l^∞-valued Gaussian random fields which is proved in a way similar to that of [24, Lemma 2.1] by using condition (1.8). Theorem 1.2 is verified by Lemma 2.2, which follows from Lemma 2.1.

Lemma 2.1 *Let $\mathbb{D}$ be a compact subset of $\mathbb{R}^N$ with Euclidean norm $\|\cdot\|$ and let $\{Z_k(\mathbf{t}), \mathbf{t} \in \mathbb{D}\}_{k=1}^\infty$ be a sequence of real-valued separable and centered Gaussian random fields. Assume that $\{\mathbf{U}(\mathbf{t}) = (Z_1(\mathbf{t}), Z_2(\mathbf{t}), \cdots), \mathbf{t} \in \mathbb{D}\}$ is an l^∞-valued Gaussian random field with l^∞-norm $\|\cdot\|_\infty$. Suppose that*

$$0 < \Gamma_k := \sup_{\mathbf{t}\in\mathbb{D}} \{E(Z_k(\mathbf{t}))^2\}^{1/2} < \infty, \quad \Gamma := \sup_{k\geq 1} \Gamma_k,$$
$$\sigma_k^2(\|\mathbf{t}-\mathbf{s}\|) := E\{Z_k(\mathbf{t}) - Z_k(\mathbf{s})\}^2 \leq \varphi_k^2(\|\mathbf{t}-\mathbf{s}\|),$$
$$\varphi_*(h) = \sup_{k\geq 1} \varphi_k(h),$$

where $\sigma_k(h)$ and $\varphi_k(h)$ are positive nondecreasing and continuous functions of $h > 0$. If condition (1.8) *is satisfied, then for $\lambda > 0$, $x > 0$ and $\mathcal{B} > (4\sqrt{2}+4)\sqrt{N}$, there exists a constant $c > 0$ such that*

$$P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} \|\mathbf{U}(\mathbf{t})\|_\infty \geq x\Big(\Gamma + \mathcal{B}\int_0^\infty \varphi_*(\lambda 2^{-y^2})\,dy\Big)\Big\} \leq c\,\frac{m(\mathbb{D})}{\lambda^N}\exp(-x^2/2),$$

where $m(\mathbb{D})$ is the Lebesgue measure of $\mathbb{D}$.

From Lemma 2.1, we obtain the following large deviation probability for the l^∞-valued Gaussian random field $\mathbf{X}(\,\cdot\,)$ of our investigation.

Lemma 2.2 *Let $\mathbf{X}(\mathbf{t})$ and $\sigma_*(\,\cdot\,)$ be as in Theorem* 1.2. *For each $i = 1, 2, \ldots, N$, let $a_i(\mathbf{T})$ and $b_i(\mathbf{T})$ be positive continuous functions on $(0,\infty)^N$. Then, for any $\varepsilon > 0$, there exists a positive constant C_ε depending only on ε such that*

$$P\Big\{\sup_{\|\mathbf{t}\|\leq\|\mathbf{b}_\mathbf{T}\|}\ \sup_{\|\mathbf{s}\|\leq\|\mathbf{a}_\mathbf{T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s}) - \mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)} \geq x\Big\} \leq C_\varepsilon\Big(\frac{\|\mathbf{b}_\mathbf{T}\|}{\|\mathbf{a}_\mathbf{T}\|}\Big)^N \exp\Big(-\frac{x^2}{2+\varepsilon}\Big)$$

for all $x > 0$.

We omit the proof, which is similar to that of [24, Lemma 2.2] and does not make use of equation (1.2) in Theorem 1.1 or of the classical method of positive dyadic rational numbers that is used in the proof of Theorem 1.1. Lemma 2.2 plays a key role in obtaining inequality (2.3).

Proof of Theorem 1.2 Let $\theta = \sqrt{1+\varepsilon}$ for any given $\varepsilon > 0$. Define

$$E_k = \{\mathbf{T} : \theta^k \leq \sigma_*(\|\mathbf{a}_\mathbf{T}\|) \leq \theta^{k+1}\}, \quad -\infty < k < \infty,$$
$$E_{k,j} = \{\mathbf{T} : 2^j \leq \frac{\|\mathbf{b}_\mathbf{T}\|}{\|\mathbf{a}_\mathbf{T}\|} \leq 2^{j+1}, \mathbf{T} \in E_k\}, \quad 0 < j < \infty,$$
$$\|\mathbf{a}_{\mathbf{T}_{k,j}}\| = \sup\{\|\mathbf{a}_\mathbf{T}\| : \mathbf{T} \in E_{k,j}\},$$
$$\|\mathbf{b}_{\mathbf{T}_{k,j}}\| = \sup\{\|\mathbf{b}_\mathbf{T}\| : \mathbf{T} \in E_{k,j}\}.$$

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By condition (1.9), we have

$$(2.1)\qquad \limsup_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_\mathbf{T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\,\beta_1(\mathbf{T})}$$
$$\le \limsup_{|k|+l\to\infty} \sup_{j\ge l>0} \sup_{\mathbf{T}\in E_{k,j}} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_\mathbf{T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\,\beta_1(\mathbf{T})}$$
$$\le \limsup_{|k|+l\to\infty} \sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{\mathbf{T}_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{\mathbf{T}_{k,j}}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\theta^k\, D(k,j)},$$

where $D(k,j)=\left\{2(\log 2^{Nj}+\log\log\theta^{|k|+j\log_\theta 2})\right\}^{1/2}$. We are to show that

$$(2.2)\qquad \limsup_{|k|+l\to\infty} \sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{\mathbf{T}_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{\mathbf{T}_{k,j}}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\theta^k\, D(k,j)}$$
$$\le \theta \limsup_{|k|+l\to\infty} \sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{\mathbf{T}_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{\mathbf{T}_{k,j}}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_{\mathbf{T}_{k,j}}\|)\, D(k,j)}$$
$$\le \theta^2 \quad \text{a.s.}$$

for any θ as defined above. By Lemma 2.2, there exists $C_\varepsilon > 0$, depending only on $\varepsilon > 0$, such that

$$(2.3)\qquad P\Big\{ \sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{\mathbf{T}_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{\mathbf{T}_{k,j}}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_{\mathbf{T}_{k,j}}\|)D(k,j)} \ge \theta\Big\}$$
$$\le C_\varepsilon \sum_{j\ge l} \Big(\frac{\|\mathbf{b}_{\mathbf{T}_{k,j}}\|}{\|\mathbf{a}_{\mathbf{T}_{k,j}}\|}\Big)^N \exp\Big(-\frac{2(1+\varepsilon)}{2+\varepsilon}(\log 2^{Nj}+\log\log\theta^{|k|+j\log_\theta 2})\Big)$$
$$\le C_\varepsilon \sum_{j\ge l} 2^{-\varepsilon' Nj}|k\vee 1|^{-1-\varepsilon'}$$
$$\le C_\varepsilon\, |k\vee 1|^{-1-\varepsilon'}\, 2^{-\varepsilon' Nl}$$

for $|k|+l$ large enough, where $\varepsilon' = \varepsilon/(2+\varepsilon)$ and $k\vee 1 = \max\{k,1\}$. Hence we have

$$\sum_{l=1}^{\infty}\sum_{|k|=1}^{\infty} P\Big\{ \sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{\mathbf{T}_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{\mathbf{T}_{k,j}}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_{\mathbf{T}_{k,j}}\|)D(k,j)} \ge \theta\Big\} < \infty,$$

and (2.2) follows from the Borel–Cantelli lemma. Combining (2.2) with (2.1) yields (1.10) by the arbitrariness of θ. ■

The following Lemmas 2.3–2.5 are essential for the proof of Thoerem 1.3, and Lemma 2.3 is a well-known version of the second Borel–Cantelli lemma.

Lemma 2.3 *Let $\{A_k, k \geq 1\}$ be a sequence of events. If*

(i) $\displaystyle\sum_{k=1}^{\infty} P(A_k) = \infty,$

(ii) $\displaystyle\liminf_{n\to\infty} \sum_{1\leq j<k\leq n} \frac{P(A_j \cap A_k) - P(A_j)P(A_k)}{\big(\sum_{j=1}^n P(A_j)\big)^2} \leq 0,$

then $P(A_n, \text{i.o.}) = 1$.

Lemma 2.4 ([31]) *Let $\{X_j, j = 1, 2, \dots, n\}$ be centered and stationary normal random variables with $E(X_iX_j) = r_{ij}$ and $r_{ii} = 1$. Let $I_c^{+1} = [c, \infty)$ and $I_c^{-1} = (-\infty, c)$. Denote by F_j the event $\{X_j \in I_{c_j}^{\varepsilon_j}\}$ for $c_j \in (-\infty, \infty)$, $j = 1, 2, \dots, n$, where ε_j is either $+1$ or -1. Let $K \subset \{1, 2, \dots, n\}$, then $P\{\bigcap_{j\in K} F_j\}$ is an increasing function of r_{ij} if $\varepsilon_i\varepsilon_j = +1$; otherwise, it is decreasing.*

The proof of Lemma 2.5 is similar to that of [6, Lemma 5].

Lemma 2.5 *Assume condition (1.11) of Theorem 1.3 is satisfied. For $i = 0, 1, 2, 3$, let $\mathbf{a}^{(i)} = (a_1^{(i)}, \dots, a_N^{(i)})$ be positive N-dimensional vectors such that $a_j^{(3)} - a_j^{(2)} > a_j^{(2)} - a_j^{(1)} > 0$ for each $j = 1, 2, \dots, N$. then there exists a positive constant c such that*

$$\int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|} d\sigma_*^2(x) - \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} d\sigma_*^2(x) \leq c\,\frac{\sigma_*^2(\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|)\|\mathbf{a}^{(3)}-\mathbf{a}^{(2)}\|\|\mathbf{a}^{(2)}-\mathbf{a}^{(1)}\|}{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|^2}.$$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 Let $j_{\mathbf{T}} \geq 1$ be an integer such that $\sigma_{j_{\mathbf{T}}}(\|\mathbf{a}_{\mathbf{T}}\|) = \sigma_*(\|\mathbf{a}_{\mathbf{T}}\|)$, where $j_{\mathbf{T}}$ depends on $\|\mathbf{a}_{\mathbf{T}}\|$. Then

$$\limsup_{\|\mathbf{T}\|\to\infty} \frac{\|\mathbf{X}(\mathbf{b}_{\mathbf{T}}) - \mathbf{X}(\mathbf{b}_{\mathbf{T}} - \mathbf{a}_{\mathbf{T}})\|_\infty}{\sigma_*(\|\mathbf{a}_{\mathbf{T}}\|)\beta_1(\mathbf{T})} \geq \limsup_{\|\mathbf{T}\|\to\infty} \frac{X_{j_{\mathbf{T}}}(\mathbf{b}_{\mathbf{T}}) - X_{j_{\mathbf{T}}}(\mathbf{b}_{\mathbf{T}} - \mathbf{a}_{\mathbf{T}})}{\sigma_{j_{\mathbf{T}}}(\|\mathbf{a}_{\mathbf{T}}\|)\beta_1(\mathbf{T})}. \tag{2.4}$$

Let $\{\mathbf{T}_i = (T_{1i}, \dots, T_{Ni}) > \mathbf{0}\}_{i=1}^\infty$ be an increasing sequence with $\mathbf{T}_0 = \mathbf{1}$ whose points $\mathbf{T}_i$ are determined by the relation

$$b_l(\mathbf{T}_i) - a_l(\mathbf{T}_i) = b_l(\mathbf{T}_{i-1}), \quad l = 1, \dots, N$$

with $\mathbf{T}_m$ $(m = 1, 2, \cdots, i-1)$ defined by induction, where $\mathbf{1} < \mathbf{T}_i \leq \mathbf{T} \leq \mathbf{T}_{i+1}$. This can be done by noting that $b_l(\mathbf{T}) - a_l(\mathbf{T})$ are increasing on $(0, \infty)^N$, because $b_l(\mathbf{T})/a_l(\mathbf{T})$ (> 1) are increasing. For estimating a lower bound of the right-hand side of (2.4), we simply let $j_i = j_{\mathbf{T}_i}$, $\mathbf{a}_i = \mathbf{a}_{\mathbf{T}_i} = (a_1(\mathbf{T}_i), \dots, a_N(\mathbf{T}_i))$ and $\mathbf{b}_i = \mathbf{b}_{\mathbf{T}_i} = (b_1(\mathbf{T}_i), \dots, b_N(\mathbf{T}_i))$, $i \geq 1$, and set

$$Z_i := \frac{X_{j_i}(\mathbf{b}_i) - X_{j_i}(\mathbf{b}_i - \mathbf{a}_i)}{\sigma_{j_i}(\|\mathbf{a}_i\|)}.$$

The proof of Theorem 1.3 will be completed by showing that

$$\limsup_{i\to\infty} \frac{Z_i}{\beta_1(\mathbf{T}_i)} \ge 1 \quad \text{a.s.}$$

For any given $0 < \varepsilon < 1$, let $B_i = \{Z_i > x_i\}$, $i \ge 1$, where $x_i = (1-\varepsilon)\beta_1(\mathbf{T}_i)$. First we show that $\sum_{i=1}^{\infty} P(B_i) = \infty$. For a large i, we have

$$\begin{aligned} P(B_i) &\ge \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x_i} - \frac{1}{x_i^3}\Big)\exp\Big(-\frac{1}{2}x_i^2\Big) \\ &\ge \exp\Big(-\frac{1}{2-\varepsilon}x_i^2\Big) \ge \left(\frac{\|\mathbf{a}_i\|^N}{\|\mathbf{b}_i\|^N |\log\|\mathbf{b}_i\||}\right)^{1-\varepsilon}, \\ \sum_{i=i_0}^{m} P(B_i) &\ge \frac{1}{|\log\|\mathbf{b}_m\||^{1-\varepsilon}} \sum_{i=i_0}^{m} \frac{\|\mathbf{a}_i\|^N}{\|\mathbf{b}_i\|^N}, \end{aligned}$$

for some i_0 with $m \ge i \ge i_0$. Further, we have

$$\begin{aligned} |\log\|\mathbf{b}_m\|| &\le c\sum_{i=i_0}^{m} \log\frac{\|\mathbf{b}_i\|}{\|\mathbf{b}_{i-1}\|} = c\sum_{i=i_0}^{m}\log\left(1 + \frac{\|\mathbf{a}_i\|}{\|\mathbf{b}_{i-1}\|}\right) \\ &\le c\sum_{i=i_0}^{m}\log\left(1+\frac{c_0\|\mathbf{a}_{i-1}\|}{\|\mathbf{b}_{i-1}\|}\right) \end{aligned} \tag{2.5}$$

for sufficiently large $c_0 > 1$, where $c > 1$ is a constant. The last inequality of (2.5) follows from the fact that there is $c_0 > 1$ big enough such that

$$\frac{\|\mathbf{a}_i\|}{\|\mathbf{a}_{i-1}\|} \le \frac{\|\mathbf{b}_i\|}{\|\mathbf{b}_{i-1}\|} \le \frac{\|\mathbf{b}_i\|}{\|\mathbf{b}_i\| - \|\mathbf{a}_i\|} = \frac{1}{1-(\|\mathbf{a}_i\|/\|\mathbf{b}_i\|)} < c_0$$

for all $i_0 \le i \le m$. It follows from (2.5) that there exists a constant $K > 0$ such that

$$N|\log\|\mathbf{b}_m\|| \le cN\sum_{i=i_0}^{m}\log\Big(1+\frac{c_0\|\mathbf{a}_{i-1}\|}{\|\mathbf{b}_{i-1}\|}\Big) \le K\sum_{i=i_0}^{m}\Big(\frac{{c_0}^2\|\mathbf{a}_i\|}{\|\mathbf{b}_i\|}\Big)^N.$$

Therefore, we have

$$\sum_{i=1}^{m} P(B_i) \ge \frac{N}{K{c_0}^{2N}}|\log\|\mathbf{b}_m\||^{\varepsilon} \to \infty \quad \text{as } m\to\infty,$$

and hence condition (i) of Lemma 2.3 is satisfied.

Next, it suffices to show that condition (ii) of Lemma 2.3 holds. Let $i < k$. If $j_i \ne j_k$, then

$$E\{(X_{j_i}(\mathbf{b}_i) - X_{j_i}(\mathbf{b}_i - \mathbf{a}_i))(X_{j_k}(\mathbf{b}_k) - X_{j_k}(\mathbf{b}_k - \mathbf{a}_k))\} = 0$$

by independence of $X_k(\cdot)$. But, if $j_i = j_k$, then

$$(2.6)\quad E(Z_iZ_k) = \frac{1}{\sigma_{j_i}(\|\mathbf{a}_i\|)\sigma_{j_i}(\|\mathbf{a}_k\|)} E\big\{ (X_{j_i}(\mathbf{b}_i) - X_{j_i}(\mathbf{b}_i - \mathbf{a}_i)) \times (X_{j_i}(\mathbf{b}_k) - X_{j_i}(\mathbf{b}_k - \mathbf{a}_k))\big\}$$
$$= -\frac{1}{2\sigma_{j_i}(\|\mathbf{a}_i\|)\sigma_{j_i}(\|\mathbf{a}_k\|)} \big\{ \sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i\|) - \sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i - \mathbf{a}_k\|) - \sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i + \mathbf{a}_i\|) + \sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i - \mathbf{a}_k + \mathbf{a}_i\|)\big\}$$
$$= \frac{1}{2\sigma_{j_i}(\|\mathbf{a}_i\|)\sigma_{j_i}(\|\mathbf{a}_k\|)} \big\{ \sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i + \mathbf{a}_i\|) - \sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i\|) - (\sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i - \mathbf{a}_k + \mathbf{a}_i\|) - \sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i - \mathbf{a}_k\|))\big\}.$$

If the right-hand side of (2.6) is less than or equal to zero, that is, if $\sigma^2_{j_i}(\cdot)$ is a nearly concave function with $0 < \alpha \le 1/2$, then $P(B_i \cap B_j) \le P(B_i)P(B_j)$ by Lemma 2.4, and hence (ii) of Lemma 2.3 holds true.

On the contrary, if the right-hand side of (2.6) is larger than zero, that is, if $\sigma^2_{j_i}(\cdot)$ is a nearly convex function with $1/2 < \alpha < 1$, then

$$(2.7)\quad E(Z_iZ_k) \le \frac{1}{\sigma_{j_i}(\|\mathbf{a}_i\|)\sigma_{j_i}(\|\mathbf{a}_k\|)} \times \Big\{ \int_{\|\mathbf{b}_k-\mathbf{b}_i-\mathbf{a}_k+\mathbf{a}_i\|}^{\|\mathbf{b}_k-\mathbf{b}_i+\mathbf{a}_i\|} d\sigma^2_{j_i}(x) - \int_{\|\mathbf{b}_k-\mathbf{b}_i-\mathbf{a}_k\|}^{\|\mathbf{b}_k-\mathbf{b}_i-\mathbf{a}_k+\mathbf{a}_i\|} d\sigma^2_{j_i}(x)\Big\}.$$

Applying Lemma 2.5 with $\mathbf{a}^{(0)} = \mathbf{b}_k - \mathbf{b}_i - \mathbf{a}_k$, $\mathbf{a}^{(1)} = \mathbf{0}$, $\mathbf{a}^{(2)} = \mathbf{a}_i$, and $\mathbf{a}^{(3)} = \mathbf{a}_k + \mathbf{a}_i$, the right-hand side of (2.7) is less than or equal to

$$(2.8)\quad \frac{c\sigma^2_{j_i}(\|\mathbf{b}_k - \mathbf{b}_i + \mathbf{a}_i\|)\|\mathbf{a}_i\|\,\|\mathbf{a}_k\|}{\sigma_{j_i}(\|\mathbf{a}_i\|)\sigma_{j_i}(\|\mathbf{a}_k\|)\|\mathbf{b}_k - \mathbf{b}_i - \mathbf{a}_k\|^2}.$$

By the definition of $\{\mathbf{T}_i\}_{i=1}^\infty$, we have

$$(2.9)\quad \mathbf{b}_k - \mathbf{b}_i = \sum_{l=i+1}^{k} \mathbf{a}_l, \quad \|\mathbf{b}_k - \mathbf{b}_i + \mathbf{a}_i\| = \Big\|\sum_{l=i}^{k}\mathbf{a}_l\Big\| \le (k-i+1)\|\mathbf{a}_k\|,$$

$$(2.10)\quad \|\mathbf{b}_k - \mathbf{b}_i - \mathbf{a}_k\| = \Big\|\sum_{l=i+1}^{k-1}\mathbf{a}_l\Big\| \ge (k-i-1)\|\mathbf{a}_i\|, \quad k \ge i+2.$$

Noting that $\|\mathbf{b}_i\|/\|\mathbf{a}_i\|$ is increasing and $\big\|\sum_{l=i+1}^{k}\mathbf{a}_l\big\| = \|\mathbf{b}_k - \mathbf{b}_i\| < \|\mathbf{b}_k\|$, it follows that

$$(2.11)\quad \frac{\|\mathbf{a}_i\|}{\|\mathbf{a}_k\|} \ge \frac{\|\mathbf{b}_i\|}{\|\mathbf{b}_k\|} \ge 1 - \frac{\|\mathbf{b}_k - \mathbf{b}_i\|}{\|\mathbf{b}_k\|} = 1 - \frac{\|\sum_{l=i+1}^{k}\mathbf{a}_l\|}{\|\mathbf{b}_k\|} \ge 1 - \rho$$

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for some $0 < \rho < 1$. From (2.7), (2.8), (2.9), (2.10), and (2.11) and the property of slowly varying function $L(\cdot)$, we arrive at

$$
\begin{aligned}
E(Z_iZ_k) &\le \frac{c\,\sigma_{j_i}^2((k-i+1)\|\mathbf{a}_k\|)\|\mathbf{a}_i\|\|\mathbf{a}_k\|}{\sigma_{j_i}(\|\mathbf{a}_i\|)\sigma_{j_i}(\|\mathbf{a}_k\|)(k-i-1)^2\|\mathbf{a}_i\|^2} \\
&\le c\,\frac{\sigma_{j_i}^2((k-i+1)\|\mathbf{a}_k\|)\|\mathbf{a}_k\|}{\sigma_{j_i}((1-\rho)\|\mathbf{a}_k\|)\sigma_{j_i}(\|\mathbf{a}_k\|)(k-i-1)^2(1-\rho)^2\|\mathbf{a}_k\|} \\
&= c\,\frac{(k-i+1)^{2\alpha}\|\mathbf{a}_k\|^{2\alpha}L^2((k-i+1)\|\mathbf{a}_k\|)}{(1-\rho)^{\alpha+2}\|\mathbf{a}_k\|^{2\alpha}L((1-\rho)\|\mathbf{a}_k\|)L(\|\mathbf{a}_k\|)(k-i-1)^2} \\
&\le c\,(k-i)^{\alpha'-1}, \quad k \ge i+2,
\end{aligned}
$$

where $\alpha < \alpha' < 1$. The remainder of the proof is exactly the same as the corresponding proof of [27, Theorem 2]. The details are omitted. ■

Proof of Theorem 1.5 First, consider the case $0 < r \le \infty$. Given condition (1.12), there exists a positive number γ such that

$$
\frac{\|\mathbf{b}_\mathbf{T}\|}{\|\mathbf{a}_\mathbf{T}\|} \ge \big|\log\|\mathbf{b}_\mathbf{T}\|\big|^{\gamma/\log\theta},
$$

provided $\|T\|$ is large enough. Thus, it follows from (1.12) and Lemma 2.2 that, for any $\varepsilon > 0$,

$$
\begin{aligned}
(2.12)\qquad & P\Big\{\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|}\sup_{\|\mathbf{s}\|\le\|\mathbf{a}_\mathbf{T}\|}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\,\beta_3(\mathbf{T})} > \sqrt{1+\varepsilon}\Big\} \\
&\le C_\varepsilon\Big(\frac{\|\mathbf{b}_\mathbf{T}\|}{\|\mathbf{a}_\mathbf{T}\|}\Big)^N\exp\Big(-\frac{2+2\varepsilon}{(2+\varepsilon)^2}\log\Big(\frac{\|\mathbf{b}_\mathbf{T}\|}{\|\mathbf{a}_\mathbf{T}\|}\Big)^N\Big) \\
&\le C_\varepsilon\,|\log\|\mathbf{b}_\mathbf{T}\||^{-N\gamma\varepsilon/((2+\varepsilon)\log\theta)} \to 0 \quad \text{as } |T\| \to \infty,
\end{aligned}
$$

and [25, Lemma 1.1.5] implies

$$
\liminf_{\|\mathbf{T}\|\to\infty}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|}\sup_{\|\mathbf{s}\|\le\|\mathbf{a}_\mathbf{T}\|}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\,\beta_3(\mathbf{T})} \le 1 \quad \text{a.s.}
$$

Hence, by (1.12), we get

$$
\begin{aligned}
(2.13)\qquad & \liminf_{\|\mathbf{T}\|\to\infty}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|}\sup_{\|\mathbf{s}\|\le\|\mathbf{a}_\mathbf{T}\|}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\,\beta_2(\mathbf{T})} \\
&= \liminf_{\|\mathbf{T}\|\to\infty}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_\mathbf{T}\|}\sup_{\|\mathbf{s}\|\le\|\mathbf{a}_\mathbf{T}\|}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a}_\mathbf{T}\|)\,\beta_3(\mathbf{T})}\,\frac{\beta_3(\mathbf{T})}{\beta_2(\mathbf{T})} \\
&\le \Big(\frac{rN}{1+rN}\Big)^{1/2} \quad \text{a.s.}
\end{aligned}
$$

Next, consider the case $r = 0$. It follows from (1.12) that for any $\varepsilon > 0$,

$$\frac{\|\mathbf{b_T}\|}{\|\mathbf{a_T}\|} < |\log \|\mathbf{b_T}\||^{\varepsilon/((2+\varepsilon)N\log\theta)}$$

for $\|\mathbf{T}\|$ large enough. Similarly to (2.12), we have

$$\begin{aligned} P\Big\{ \sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_2(\mathbf{T})} > \sqrt{\varepsilon/2}\Big\} &\le c\Big(\frac{\|\mathbf{b_T}\|}{\|\mathbf{a_T}\|}\Big)^N \exp\Big(-\frac{\varepsilon}{2(2+\varepsilon)}\beta_2^2(\mathbf{T})\Big) \\ &\le c\,|\log\|\mathbf{b_T}\||^{-\varepsilon^2/((2+\varepsilon)^2\log\theta)} \to 0 \quad \text{as } \|\mathbf{T}\|\to\infty, \end{aligned}$$

which implies that

$$\liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{s})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_2(\mathbf{T})} \le 0 \quad \text{a.s.}$$

Combining this inequality with (2.13) completes the proof of Theorem 1.5. ∎

To prove Theorem 1.6, we need the following two lemmas. The proof of Lemma 2.6 is similar to that of [6, Lemma 5].

Lemma 2.6 *Assume that condition* (1.11) *of Theorem* 1.3 *is satisfied. Let* $\mathbf{a} > \mathbf{0}$ *and* $\mathbf{b} > \mathbf{1}$ *be N-dimensional vectors. Then there exists a positive constant C such that*

$$\Big|\int_{\|\mathbf{a}\|\,\|\mathbf{b}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|} d\sigma_*^2(x) - \int_{\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}\|} d\sigma_*^2(x)\Big| \le C\frac{\sigma_*^2(\|\mathbf{a}\|\|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{b}-\mathbf{1}\|^2}.$$

Lemma 2.7 ([4,21,22]) *Let $\mathbb{N} = (n_1,\dots,n_N)$ be an N-dimensional vector, where $n_1,\dots,n_N = 1,2,\dots,L$, and let $\{l_\mathbb{N} := (l_{n_1},\dots,l_{n_N})\}$ be a subsequence of $\{\mathbb{N}\}$. Suppose that $\{Y(\mathbb{N})\}$ is a sequence of N-parameter standard normal random variables with $\Lambda(\mathbb{N},\mathbb{N}') := \mathrm{Cov}\,(Y(\mathbb{N}),Y(\mathbb{N}'))$ for $\mathbb{N}\ne\mathbb{N}'$ such that $\delta := \max_{\mathbb{N}\ne\mathbb{N}'}|\Lambda(\mathbb{N},\mathbb{N}')| < 1$ and*

$$|\lambda(\mathbb{N},\mathbb{N}')| := |\Lambda(l_\mathbb{N},l_{\mathbb{N}'})| < \|\mathbb{N}-\mathbb{N}'\|^{-\nu}$$

for some $\nu > 0$. Denote $\mathbf{m} = (m_1,\dots,m_N)$ with $m_i \le L$, $1\le i\le N$. Set $u = \big\{(2-\eta)\log\big(\prod_{i=1}^N m_i\big)\big\}^{1/2}$, where $0<\eta<(1-\delta)\nu/(1+\nu+\delta)$. Then we have

$$P\Big\{\max_{\mathbf{1}\le\mathbb{N}\le\mathbf{m}} Y(l_\mathbb{N}) \le u\Big\} \le \big\{\Phi(u)\big\}^{\prod_{i=1}^N m_i} + c\Big(\prod_{i=1}^N m_i\Big)^{-\delta_0},$$

where $\Phi(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,dy$, $\delta_0 = \{\nu(1-\delta)-\eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\} > 0$ and $c > 0$ is a constant independent of $\mathbb{N}$ and u.

Proof of Theorem 1.6 The inequality (1.13) is obvious when $r = 0$. In what follows, we assume that $0 < r \leq \infty$. Let $\theta = 1 + \varepsilon$ for $0 < \varepsilon < 1$. Define

$$B_{k,\mathbf{j}} = \{\mathbf{T} : \theta^{k-1} \leq \|\mathbf{b_T}\| \leq \theta^k, \theta^{j_i - 1} \leq a_i(\mathbf{T}) \leq \theta^{j_i},\ 1 \leq i \leq N\},$$

where k and j_i are integers. Denote $\mathbf{j} = (j_1, \dots, j_N)$, $\boldsymbol{\theta}^{a\mathbf{j}} = (\theta^{a j_1}, \dots, \theta^{a j_N})$ for $-\infty < a < \infty$ and $\underline{j} = \frac{1}{N}\sum_{i=1}^N j_i$. In the sequel, we always consider k and $\mathbf{j}$ such that $B_{k,\mathbf{j}} \neq \varnothing$. Note that $\|\mathbf{a_T}\| \geq \theta^{\underline{j}-1}$ for $\mathbf{T} \in B_{k,\mathbf{j}}$. By condition (1.12), there exists $\gamma > 0$ such that $\underline{j} \leq k + 1 - \gamma(\log\log\theta^{|k|})/(\log\theta)^2 =: K$ for $|k|$ sufficently large. Noting that

$$\lim_{\|\mathbf{T}\|\to\infty} \frac{\beta_2(\mathbf{T})}{\beta_3(\mathbf{T})} = \Big(\frac{rN+1}{rN}\Big)^{1/2},$$

the inequality (1.13) is proved if we show that

$$\liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\leq\|\mathbf{b_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a_T}) - \mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_3(\mathbf{T})} \geq 1 \quad \text{a.s.} \tag{2.14}$$

By (1.12), we can write

$$\begin{aligned}
&\liminf_{\|\mathbf{T}\|\to\infty} \sup_{\|\mathbf{t}\|\leq\|\mathbf{b_T}\|} \frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a_T}) - \mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\,\beta_3(\mathbf{T})} \\
&\geq \liminf_{|k|\to\infty} \inf_{\underline{j}\leq K} \sup_{\|\mathbf{t}\|\leq\theta^{k-1}} \frac{\|\mathbf{X}(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}}) - \mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}}\|)\sqrt{2N\log\theta^{k-\underline{j}+1}}} \\
&\qquad - \limsup_{|k|\to\infty} \sup_{\underline{j}\leq K} \sup_{\|\mathbf{t}\|\leq\theta^k} \sup_{\boldsymbol{\theta}^{\mathbf{j}-1}\leq\mathbf{s}\leq\boldsymbol{\theta}^{\mathbf{j}}} \\
&\qquad\qquad \frac{\|\mathbf{X}(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}}) - \mathbf{X}(\mathbf{t}+\mathbf{s})\|_\infty}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}} - \boldsymbol{\theta}^{\mathbf{j-1}}\|)\sqrt{2N\log\theta^{k-\underline{j}+1}}} \frac{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}} - \boldsymbol{\theta}^{\mathbf{j-1}}\|)}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j-1}}\|)} \\
&=: J_1 - J_2.
\end{aligned} \tag{2.15}$$

First, we claim that

$$J_1 \geq 1 \qquad \text{a.s.} \tag{2.16}$$

By the definition of $\sigma_*(h)$, there exists an integer $\zeta \geq 1$ such that $\sigma_\zeta(\|\boldsymbol{\theta}^{\mathbf{j}}\|) = \sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}}\|)$. Put

$$\beta(\mathbf{k},\mathbf{j}) = (\beta(k, j_1), \dots, \beta(k, j_N)) = \frac{1}{\sqrt{N}M}(\theta^{k-1-j_1}, \dots, \theta^{k-1-j_N})$$

for sufficiently large $M > 0$. Then

$$J_1 \geq \liminf_{|k|\to\infty} \inf_{\underline{j}\leq K} \max_{\mathbf{1}\leq\mathbf{l}\leq\beta(\mathbf{k},\mathbf{j})} \frac{X_\zeta(M\mathbf{l}\boldsymbol{\theta}^{\mathbf{j}} + \boldsymbol{\theta}^{\mathbf{j}}) - X_\zeta(M\mathbf{l}\boldsymbol{\theta}^{\mathbf{j}})}{\sigma_\zeta(\|\boldsymbol{\theta}^{\mathbf{j}}\|)\sqrt{2\log(\Pi_{i=1}^N\beta(k, j_i))}}. \tag{2.17}$$

Let

$$Z_{\mathbf{j}}(\mathbf{l}) = \frac{X_\zeta(M\mathbf{l}\boldsymbol{\theta}^{\mathbf{j}} + \boldsymbol{\theta}^{\mathbf{j}}) - X_\zeta(M\mathbf{l}\boldsymbol{\theta}^{\mathbf{j}})}{\sigma_\zeta(\|\boldsymbol{\theta}^{\mathbf{j}}\|)}, \quad \mathbf{1} \le \mathbf{l} \le \beta(\mathbf{k}, \mathbf{j}).$$

Similarly to (2.6), we have, for all $\mathbf{l}$ and $\mathbf{l}'$ with $\mathbf{l} > \mathbf{l}'$,

$$\begin{aligned}(2.18)\quad \lambda_{\mathbf{j}}(\mathbf{l}, \mathbf{l}') &:= Cov(Z_{\mathbf{j}}(\mathbf{l}), Z_{\mathbf{j}}(\mathbf{l}')) \\ &= \frac{1}{2\sigma_\zeta^2(\|\boldsymbol{\theta}^{\mathbf{j}}\|)}\Big\{\sigma_\zeta^2(\|M(\mathbf{l}-\mathbf{l}')\boldsymbol{\theta}^{\mathbf{j}} + \boldsymbol{\theta}^{\mathbf{j}}\|) - \sigma_\zeta^2(\|M(\mathbf{l}-\mathbf{l}')\boldsymbol{\theta}^{\mathbf{j}}\|) \\ &\qquad - \big(\sigma_\zeta^2(\|M(\mathbf{l}-\mathbf{l}')\boldsymbol{\theta}^{\mathbf{j}}\|) - \sigma_\zeta^2(\|M(\mathbf{l}-\mathbf{l}')\boldsymbol{\theta}^{\mathbf{j}} - \boldsymbol{\theta}^{\mathbf{j}}\|)\big)\Big\}.\end{aligned}$$

If the right-hand side of (2.18) is less than or equal to zero, then it follows from Lemma 2.4 that for any $0 < \varepsilon < 1$,

$$\begin{aligned}(2.19)\quad P\Big\{\inf_{\underline{j} \le K} \max_{\mathbf{1} \le \mathbf{l} \le \beta(\mathbf{k}, \mathbf{j})} \frac{Z_{\mathbf{j}}(\mathbf{l})}{\sqrt{2\log\big(\Pi_{i=1}^N \beta(k, j_i)\big)}} \le \sqrt{1-\varepsilon}\Big\} \\ \le \sum_{\underline{j} \le K}\Big\{\Phi\Big(\sqrt{(2-2\varepsilon)\log\big(\Pi_{i=1}^N \beta(k, j_i)\big)}\Big)\Big\}^{\Pi_{i=1}^N \beta(k, j_i)}.\end{aligned}$$

On the other hand, if the right-hand side of (2.18) is positive, that is, σ_ζ^2 is a nearly convex function, then it follows from the regular variation of σ_ζ^2 and Lemma 2.6 with $\mathbf{a} = \boldsymbol{\theta}^{\mathbf{j}}$ and $\mathbf{b} = M(\mathbf{l} - \mathbf{l}')$ that

$$\begin{aligned}|\lambda_{\mathbf{j}}(\mathbf{l}, \mathbf{l}')| &\le \frac{1}{\sigma_\zeta^2(\|\boldsymbol{\theta}^{\mathbf{j}}\|)}\Big|\int_{\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\mathbf{l}-\mathbf{l}')\|}^{\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\mathbf{l}-\mathbf{l}')+\mathbf{1}\|} d\sigma_\zeta^2(x) - \int_{\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\mathbf{l}-\mathbf{l}')-\mathbf{1}\|}^{\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\mathbf{l}-\mathbf{l}')\|} d\sigma_\zeta^2(x)\Big| \\ &\le C\,\frac{\sigma_\zeta^2(\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\mathbf{l}-\mathbf{l}')+\mathbf{1}\|)}{\sigma_\zeta^2(\|\boldsymbol{\theta}^{\mathbf{j}}\|)\,\|M(\mathbf{l}-\mathbf{l}')-\mathbf{1}\|^2} \\ &\le C\,\frac{\|M(\mathbf{l}-\mathbf{l}')+\mathbf{1}\|^2}{\|M(\mathbf{l}-\mathbf{l}')-\mathbf{1}\|^2}\,\|M(\mathbf{l}-\mathbf{l}')+\mathbf{1}\|^{2\alpha-2} \\ &< \xi\,\|\mathbf{l}-\mathbf{l}'\|^{-\nu}\end{aligned}$$

for sufficiently small $\xi > 0$, where $\nu = 1 - \alpha > 0$. We now apply Lemma 2.7 for

$$\begin{gathered}Y(l_{\mathbf{l}}) = Z_{\mathbf{j}}(\mathbf{l}), \quad \mathbf{1} \le \mathbf{l} \le \beta(\mathbf{k}.\mathbf{j}), \quad \mathbf{m} = \beta(\mathbf{k}, \mathbf{j}), \\ |\lambda(\mathbf{l}, \mathbf{l}')| = |\lambda_{\mathbf{j}}(\mathbf{l}, \mathbf{l}')| < \xi\,\|\mathbf{l}-\mathbf{l}'\|^{-\nu}, \quad \nu = 1 - \alpha > 0, \\ u = \{(2-\eta)\log(\Pi_{i=1}^N \beta(k, j_i))\}^{1/2}, \quad \eta = 2\varepsilon.\end{gathered}$$

Then we have

$$
\begin{aligned}
(2.20)\qquad P\Big\{\inf_{\underline{j}\le K}\max_{\mathbf{1}\le\mathbf{l}\le\beta(\mathbf{k},\mathbf{j})}\frac{Z_{\mathbf{j}}(\mathbf{l})}{\sqrt{2\log\big(\Pi_{i=1}^N\beta(k,j_i)\big)}}\le\sqrt{1-\varepsilon}\Big\}
&\le\sum_{\underline{j}\le K}\Big\{(\Phi(u))^{\Pi_{i=1}^N\beta(k,j_i)}+c\Big(\prod_{i=1}^N\beta(k,j_i)\Big)^{-\delta_0}\Big\}\\
&\le\sum_{\underline{j}\le K}\{\exp\big(-c\,\theta^{\varepsilon N(k-\underline{j})}\big)+c\,(\theta^{N(k-\underline{j})})^{-\delta_0}\}\\
&\le c\sum_{\underline{j}\le K}\theta^{-N\delta_0(k-\underline{j})}\le c\theta^{-N\delta_0\gamma(\log_\theta\log\theta^{|k|})/\log\theta}\\
&\le c\,|k|^{-N\delta_0\gamma/\log\theta}
\end{aligned}
$$

for sufficiently large $|k|$. Note that the right-hand side of (2.19) is less than or equal to that of (2.20). Taking $\theta>1$ such that $\log\theta<N\delta_0\gamma$ in (2.20), the Borel–Cantelli lemma implies (2.16) via (2.17).

Next, we show that

$$(2.21)\qquad J_2\le 2c\,\varepsilon^{\alpha/2}\quad\text{a.s.}$$

for any small $\varepsilon>0$, where $c>0$ is a constant. Since $\sigma_*(h)$ is regularly varying, we have

$$\frac{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j-1}}\|)}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j-1}}\|)}\le c\,\varepsilon^{\alpha/2}.$$

Therefore, (2.21) is proved if we show that

$$(2.22)\qquad \limsup_{|k|\to\infty}\sup_{\underline{j}\le K}\sup_{\|\mathbf{t}\|\le\theta^k}\sup_{\boldsymbol{\theta}^{\mathbf{j}-1}\le\mathbf{s}\le\boldsymbol{\theta}^{\mathbf{j}}}\frac{\|\mathbf{X}(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}})-\mathbf{X}(\mathbf{t}+\mathbf{s})\|_\infty}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j-1}}\|)\sqrt{2N\log\theta^{k-\underline{j}+1}}}\le 2\quad\text{a.s.}$$

Similarly to the proof of Lemma 2.2, it follows that for sufficiently large $|k|$,

$$
\begin{aligned}
P\Big\{\sup_{\|\mathbf{t}\|\le\theta^k}\sup_{\boldsymbol{\theta}^{\mathbf{j}-1}\le\mathbf{s}\le\boldsymbol{\theta}^{\mathbf{j}}}\frac{\|\mathbf{X}(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}})-\mathbf{X}(\mathbf{t}+\mathbf{s})\|_\infty}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j-1}}\|)\sqrt{2N\log\theta^{k-\underline{j}+1}}}\ge 2+\varepsilon\Big\}
&\le c\frac{\theta^{Nk}}{\|\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j-1}}\|^N}\exp\Big(-\frac{2(2+\varepsilon)^2}{2+\varepsilon}N\log\theta^{k-\underline{j}+1}\Big)\\
&\le c(\theta^{3N})^{-(k-\underline{j})}.
\end{aligned}
$$

Since

$$\sum_{|k|=1}^\infty\sum_{\underline{j}\le K}(\theta^{3N})^{-(k-\underline{j})}\le c\sum_{|k|=1}^\infty|k|^{-\gamma/\log\theta}<\infty,$$

we obtain (2.22) and hence (2.14) holds true via (2.21), (2.16) and (2.15). This proves Theorem 1.6. ∎

Proof of Theorem 1.8 In case $r = \infty$ in the condition (1.12) of Theorem 1.5, by considering

$$\lim_{r\to\infty}\Big(\frac{rN}{1+rN}\Big)^{1/2} = 1$$

in (1.13) of Theorem 1.6, it follows from (1.13) and condition (1.15) that the inequality (1.16) is immediate as long as $\lim_{\|\mathbf{T}\|\to\infty}\|\mathbf{b_T}\| = \infty$ (or 0) in (1.12). Thus it remains to prove (1.16) under the construction $B_{k,\mathbf{j}}$ in the proof of Theorem 1.6 when

$$0 < \liminf_{\|\mathbf{T}\|\to\infty}\|\mathbf{b_T}\| \le \limsup_{\|\mathbf{T}\|\to\infty}\|\mathbf{b_T}\| < \infty \tag{2.23}$$

in (1.15). The latter (2.23) in (1.15) implies that, as $\|\mathbf{T}\|\to\infty$,

$$\frac{\|\mathbf{b_T}\|}{\|\mathbf{a_T}\|}\to\infty \quad \text{or } \|\mathbf{a_T}\|\to 0. \tag{2.24}$$

By (2.23) and (2.24), it is clear that

$$\lim_{\|\mathbf{T}\|\to\infty}\frac{\log(\|\mathbf{b_T}\|/\|\mathbf{a_T}\|)}{\log\log(\|\mathbf{b_T}\|/\|\mathbf{a_T}\|)} = \infty.$$

Hence, for sufficiently small $\underline{j} < 0$, it follows from the construction $B_{k,\mathbf{j}}$ that

$$\underline{j} < k+1-\frac{1}{(\log\theta)^2}\log\log\theta^{k-\underline{j}} =: J,$$

and there exists $M > 0$ large enough such that

$$\beta(\mathbf{k},\mathbf{j}) = (\beta(k,j_1),\dots,\beta(k,j_N)) = \frac{1}{\sqrt{NM}}(\theta^{k-1-j_1},\dots,\theta^{k-1-j_N}) > \mathbf{1}$$

by (2.23) and (2.24). Thus, we have

$$\begin{aligned}
&\liminf_{\|\mathbf{T}\|\to\infty}\sup_{\|\mathbf{t}\|\le\|\mathbf{b_T}\|}\frac{\|\mathbf{X}(\mathbf{t}+\mathbf{a_T})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\mathbf{a_T}\|)\beta_3(\mathbf{T})} \\
&\qquad \ge \liminf_{\underline{j}\to-\infty}\inf_{\underline{j}<J}\sup_{\|\mathbf{t}\|\le\theta^{k-1}}\frac{\|\mathbf{X}(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}})-\mathbf{X}(\mathbf{t})\|_\infty}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}}\|)\sqrt{2\log\big(\Pi_{i=1}^N\beta(k,j_i)\big)}} \\
&\qquad\quad -\limsup_{\underline{j}\to-\infty}\sup_{\underline{j}<J}\sup_{\|\mathbf{t}\|\le\theta^k}\sup_{\boldsymbol{\theta}^{\mathbf{j-1}}\le\mathbf{s}\le\boldsymbol{\theta}^{\mathbf{j}}} \\
&\qquad\qquad \frac{\|\mathbf{X}(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}})-\mathbf{X}(\mathbf{t}+\mathbf{s})\|_\infty}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j-1}}\|)\sqrt{2N\log\theta^{k-\underline{j}+1}}}\frac{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j-1}}\|)}{\sigma_*(\|\boldsymbol{\theta}^{\mathbf{j-1}}\|)} \\
&\qquad =: J_1'-J_2'.
\end{aligned} \tag{2.25}$$

In order to prove

$$J_1' \geq 1 \quad \text{a.s.,} \tag{2.26}$$

we proceed along the lines of the proof of (2.16). Then we arrive at

$$P\left\{ \inf_{\underline{j}<J} \max_{\mathbf{1}\leq\mathbf{l}\leq\beta(\mathbf{k},\mathbf{j})} \frac{Z_{\mathbf{j}}(\mathbf{l})}{\sqrt{2\,\log\left(\Pi_{i=1}^N \beta(k,j_i)\right)}} \leq \sqrt{1-\varepsilon} \right\} \leq c\,(k-\underline{j})^{-N\delta_0/\log\theta}$$

for sufficiently large $k-\underline{j}$. Setting $\underline{j}=-\underline{j}'$ ($\underline{j}'\geq 1$), we obtain (2.26) by the Borel–Cantelli lemma. It is easy to show that

$$J_2' = 0 \qquad \text{a.s.} \tag{2.27}$$

along the lines of the proof of (2.21). Combining (2.25), (2.26), and (2.27) yields (1.16) as well under (2.23). This completes the proof of Theorem 1.8. ■

Acknowledgement The authors wish to thank the referee for careful reading of their manuscript, and for a number of inspiring questions and helpful suggestions that have led to improving the presentation of our results.

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