MODULI OF ENDOMORPHISMS OF SEMISTABLE VECTOR BUNDLES OVER A COMPACT RIEMANN SURFACE

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Introduction. Mumford and Suominen in **[8]** and Newstead in **[11]** have considered the moduli problem of classifying the endomorphisms of finite-dimensional vector spaces. Using similar ideas we consider the moduli problem for endomorphisms of indecomposable semistable vector bundles over a compact connected Riemann surface of genus $g \geq 2$.

In this paper we develop the 3-dimensional case, which gives an idea of how to solve the moduli problem in general. First we give the algebras which can occur as algebras of endomorphisms of vector bundles of rank 3. Then we give necessary and sufficient conditions for a vector bundle to have a particular algebra of endomorphisms. Such conditions show that for any non-zero nilpotent endomorphism, there are extensions of vector bundles from which it can be reconstructed. Thus the problem of parametrizing endomorphisms is largely reduced to one of parametrizing extensions. We construct the corresponding universal families of extensions; unfortunately, this is not quite sufficient for us to obtain moduli spaces for endomorphisms themselves. However, in some cases we can obtain local universal families of endomorphisms. The algebra of endomorphisms depends on how the extensions are related.

In Section 1 we state the moduli problem for endomorphisms of vector bundles. In Section 2 we recall from **[3]** the relations between the extensions and the algebras of endomorphisms. Section 3 contains the constructions of the universal families of extensions which partially solve the moduli problem.

1. Moduli of endomorphisms. Throughout this paper X will denote a compact connected Riemann surface of genus greater than 1, and $S(n, d)$ the set of (isomorphism classes of) indecomposable semistable non-simple vector bundles of rank n and slope d over X.

Let $P(n, d)$ be the set of pairs (E, ϕ) where E is in $S(n, d)$ and $\phi: E \to E$ is an endomorphism of vector bundles. We say that two pairs (E, ϕ) and (F, ψ) are equivalent, written $(E, \phi) \sim (F, \psi)$, if there exists an isomorphism $\alpha: E \to F$ such that $\psi \circ \alpha = \alpha \circ \phi$.

By a family of endomorphisms parametrized by a variety S we shall mean a pair (E, Φ), where E is a vector bundle over $X \times S$ and Φ an endomorphism of E, such that for each $s \in S$ the restriction $(E, \Phi)_{X \times s}$ is in $P(n, d)$. Two families (E, Φ) and (F, Ψ) parametrized by S are equivalent if and only if the restrictions $(E, \Phi)_{X \times s}$ and $(F, \Psi)_{X \times s}$ are equivalent for each $s \in S$. Given a family of endomorphisms (E, Φ) parametrized by S and a morphism $h': T \to S$, the family $h^*(E, \Phi) = (h^*E, h^*\Phi)$, where $h = id_X \times h'$, is called the induced family of endomorphisms.

Definition 1. A *fine moduli space* for $P(n, d)$ is a variety M and a family of endomorphisms (U, Φ) parametrized by M, such that for any family of endomorphisms

(W, Ψ) parametrized by a variety S there exists a unique morphism $h': S \to M$ such that the induced family is equivalent to the family (W, Ψ).

For any indecomposable vector bundle E over X the algebra of endomorphisms is a special algebra i.e. $\mathrm{END}(E) \cong (1) \oplus \mathrm{Nil}(E)$ (see [**1**]). Since any endomorphism is the sum of a scalar multiple of the identity and a nilpotent endomorphism we shall concentrate on the study of non-zero nilpotent endomorphisms.

2. Algebras of endomorphisms. Let E be a vector bundle over X of rank n. If E is semistable then for any $x \in X$ the map $e_x: \mathrm{End}(E) \to \mathrm{End}(E_x)$ defined as $\phi \mapsto \phi_x$ is injective.

If E is also indecomposable the image $e_x(\mathrm{END}(E)) = G(E)$ is a special algebra in $M_{n\times n}(\mathbb{C})$. Denote by $N(E)$ the subalgebra $e_x(\mathrm{Nil}(E))$. Let H be the set of all non-zero subspaces $W \subset E_x$ which are invariant under $G(E)$, i.e. $\phi(W) \subset W$ for all ϕ in $G(E)$. Let V be a minimal subspace of H. For any $v \in V$, define $N(v) \subset V$ as the vector space $\{w \in E_x \mid \phi(v) = w$ for some ϕ in $N(E)\}$. From the minimal property of V we see that $N(v) = 0$ or $N(v) = V$. If $N(v) = V$ then there is a nilpotent element $\psi \in N(E)$ such that $\psi(v) = v$, which is a contradiction. Hence $N(v) = 0$ and so v is a common eigenvector for $G(E)$. By inductive procedure we can see that E_x has a flag invariant under $G(E)$. The existence of such flag implies that one can choose a basis of E_x such that, for all ϕ_x in $G(E)$, ϕ_x is an upper triangular matrix with all its diagonal entries equal.

Hence we have the following proposition.

PROPOSITION 1. *If E is in $S(n, d)$ then* $\dim \mathrm{END}(E) \leq 1 + \frac{1}{2}n(n-1)$.

One question is: which special algebras, i.e. local rings, with fixed dimension can occur as algebras of endomorphisms of vector bundles in $S(n, d)$? In [**3**] we prove that for E in $S(3, d)$, $\dim \mathrm{END}(E) \neq 4$. Actually, $\mathrm{END}(E)$ is isomorphic to one of the algebras $\mathbb{C}[t]/(t^2)$, $\mathbb{C}[t]/(t^3)$ or $\mathbb{C}[r, s]/(r, s)^2$.

If E is in $S(3, d)$ and non-simple then there is a nilpotent endomorphism $\phi: E \to E$ such that $\phi^2 = 0$ and $\phi \neq 0$. Denote by E_2 the kernel of ϕ and by L the image. Since E is semistable and $\phi^2 = 0$, E_2 and L are vector bundles over X and define the exact sequences

$$\xi: 0 \to E_2 \xrightarrow{j} E \xrightarrow{\pi} L \to 0 \quad \text{and} \quad \rho: 0 \to L \xrightarrow{i} E_2 \xrightarrow{p} L' \to 0$$

where $j \circ i \circ \pi \sim \phi$.

The type of algebra of endomorphisms depends on the relation between the extensions ξ and ρ and on whether L' is isomorphic to L or not.

REMARK 1. In [**3**] we proved the following results.

(i) If $L \ncong L'$ then $\mathrm{END}(E) \cong \mathbb{C}[t]/(t^2)$.
(ii) If $L \cong L'$ and $\rho = 0$ then $\mathrm{END}(E) \cong \mathbb{C}[r, s]/(r, s)^2$.
(iii) If $L \cong L'$ and $\rho \neq 0$ then from the surjective homomorphism $p_*: \mathrm{Ext}(L, E_2) \to \mathrm{Ext}(L, L)$ we see that:
(1) if $p_*(\xi) = 0$ then $\mathrm{END}(E) \cong \mathbb{C}[r, s]/(r, s)^2$;
(2) if $p_*(\xi) = \lambda\rho$ for some $\lambda \in \mathbb{C}^*$ then $\mathrm{END}(E) \cong \mathbb{C}[t]/(t^3)$;
(3) if $p_*(\xi) \neq \lambda\rho$ for $\lambda \in \mathbb{C}$ then $\mathrm{END}(E) \cong \mathbb{C}[t]/(t^2)$.

Denote by Ω^1, Ω^2 and Ω^3 the subsets of $S(3, 0)$ of those vector bundles satisfying conditions (1), (2) and (3) respectively.

If $p_*(\xi) = \lambda\rho$ for some $\lambda \in \mathbb{C}^*$ then we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & E_2 \arrow[r, "j"] \arrow[d, "\rho"'] & E \arrow[r, "\pi"] \arrow[d, "\alpha"'] & L \arrow[r] \arrow[d, equal] & 0, \\
0 \arrow[r] & L \arrow[r, "i"] \arrow[d] & E_2 \arrow[r, "\lambda p"] \arrow[d] & L \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

which can be completed as follows.

(A)
$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& L \arrow[r, equal] \arrow[d, "i"'] & L \arrow[d, "\beta"'] & & \\
0 \arrow[r] & E_2 \arrow[r, "j"] \arrow[d, "\rho"'] & E \arrow[r] \arrow[d, "\alpha"'] & L \arrow[r] \arrow[d, equal] & 0. \\
0 \arrow[r] & L \arrow[r, "i"'] \arrow[d] & E_2 \arrow[r, "\lambda p"'] \arrow[d] & L \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

In this case, the composition $j \circ \alpha$ is a nilpotent endomorphism ψ such that $\psi^3 = 0$ and ψ^2 is equivalent to $\lambda^{-1}\phi$.

Now let L, L' be two line bundles with the same slope, and $\rho: 0 \to L \xrightarrow{i} F \xrightarrow{p} L' \to 0$ an extension of L' by L. If $\xi: 0 \to F \xrightarrow{j} E \xrightarrow{\pi} L \to 0$ is an extension of L by F then

(i) E is semistable;

(ii) the pair of extensions (ξ, ρ) defines a nilpotent endomorphism $(j \circ i \circ \pi) = \phi$ of index 2, i.e. $\phi \neq 0$ but $\phi^2 = 0$;

(iii) if ξ and ρ are non-trivial then E is indecomposable;

(iv) if ρ is trivial then E is indecomposable if and only if either

  (a) $L \not\cong L'$ and ξ is not in either of the subspaces $\mathrm{Ext}(L, L)$ or $\mathrm{Ext}(L, L')$ of $\mathrm{Ext}(L, F) = \mathrm{Ext}(L, L \oplus L')$, or

  (b) $L \cong L'$ and ξ is not in the image of $i_*: \mathrm{Ext}(L, L) \to \mathrm{Ext}(L, F) = \mathrm{Ext}(L, L \oplus L)$ for any of the inclusions $i: L \to L \oplus L$.

If $L \cong L'$ denote by $S(L)$ the image of such inclusions.

With a pair (ξ, ρ) of extensions as above we obtain a pair (E, ϕ) in $P(3, d)$.

Denote by $E(3, d)$ the set of pairs of extensions (ξ, ρ) as above.

Remark 2. (i) Note the pairs (ξ, ρ) and $(\xi, \lambda\rho)$ define the same nilpotent endomorphism of index 2. Moreover, if $L = L'$ and $\alpha = 1 + \mu i p$ is an automorphism of F then, in general, the extension $\alpha\xi: 0 \to F \xrightarrow{j\alpha} E \xrightarrow{\pi} L \to 0$ is not equivalent to ξ, but the corresponding endomorphisms $ji\pi$ and $j\alpha i\pi$ are identical.

(ii) If α extends to an automorphism of E, the extension $\alpha\xi$ is equivalent to ξ, so this problem does not arise. In particular, there is no problem when all automorphisms of F extend to E, which happens

(a) when F is simple, i.e. $L \not\cong L'$, and ρ non-trivial,

(b) in the case $L \cong L'$, ρ non-trivial, precisely when E belongs to Ω^1 or Ω^2.

To study the moduli problem for endomorphisms we split $P(3, d)$ as follows:

$$\begin{aligned}
P^0 &= \{(E, \phi) \in P(3, d) : \phi^3 = 0 \text{ but } \phi^2 \neq 0\},\\
P^1 &= \{(E, \phi) \in P(3, d) : L \cong L' \text{ and } E_2 \not\cong L \oplus L'\},\\
P^2 &= \{(E, \phi) \in P(3, d) : L \not\cong L' \text{ and } E_2 \not\cong L \oplus L'\},\\
P^3 &= \{(E, \phi) \in P(3, d) : L \not\cong L' \text{ and } E_2 \cong L \oplus L'\},\\
P^4 &= \{(E, \phi) \in P(3, d) : L \cong L' \text{ and } E_2 \cong L \oplus L'\},
\end{aligned}$$

where as before, E_2 denotes the kernel of ϕ, L is the image of ϕ and $L' = E_2/L$.

Without loss of generality we assume that $d = 0$.

For each set P^i we assign a set P_i of equivalence classes of pairs of extensions in $E(3, d)$. The equivalence relation on the pairs of extensions depends on each of the sets P^i, so we shall treat them differently.

3. Moduli spaces. We split this section into five parts. In each one we construct a universal family of extensions and show how these provide a partial solution to the moduli problem for endomorphisms.

Let E and F be two families of vector bundles over X parametrized by a variety S such that $\dim H^i(X, \mathrm{Hom}(E_s, F_s))$ is independent of $s \in S$ for $i = 0, 1$. Let $p : X \times S \to S$ be the projection and let us denote the vector bundle $R^1_p(\mathrm{Hom}(E, F))$ by V. Hence, we have the commutative diagram

$$\begin{array}{ccc}
X \times V & \xrightarrow{\pi} & V\\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f}\\
X \times S & \xrightarrow[p]{} & S
\end{array}$$

Lange in [7], using Grothendieck's universal properties of vector bundles, proved that if

$$H^i(S, R^0_p \mathrm{Hom}(E, F) \otimes R^1_p \mathrm{Hom}(E, F)^*) = 0$$

for $i = 1, 2$ then $H^1(X \times S, \mathrm{Hom}(E, F) \otimes p^* R^1_p(\mathrm{Hom}(E, F))^*) \cong \mathrm{END}(V)$ and the identity in $\mathrm{END}(V)$ induces a universal family of extensions

$$\Omega : 0 \to g^*F \to Z \to g^*E \to 0$$

of E by F parametrized by V such that Ω_v is the extension represented by v, for each $v \in V$. Moreover, if $\mathbb{P}(V)$ is the projective bundle associated to V then (see Corollary 4.5 in [7]) there exists a universal family

$$P\Omega : 0 \to g^*F \otimes p^*\mathcal{O}_{\mathbb{P}(V)}(1) \to \tilde{Z} \to g^*E \to 0.$$

of extensions over $X \times \mathbb{P}(V)$ which parametrizes all the classes of non-splitting extensions of E_s by F_s over X modulo the equivalence relation of identifying extensions which differ by a non-zero constant. From the universal properties of Ω and $P\Omega$ and the canonical map $\pi: V-\{0\} \to \mathbb{P}(V)$ we see that $(id_x \times \pi)^*\mathbb{P}(V)$ is equivalent to $\Omega_{|V-\{0\}}$.

We shall prove that for some cases such universal extensions exist.

I. Let P_1 be the set of equivalence classes of pairs (ξ, ρ), where $\rho \neq 0$ and $L \cong L'$. Two pairs (ξ, ρ) and (ξ', ρ') are equivalent if and only if $\xi = \xi'$ and $\rho = \lambda\rho'$ for some λ in $\mathbb{C}^*$. We shall construct a moduli space for P_1.

Let T be the vector space $\mathrm{Ext}_X(\mathbf{1}, \mathbf{1}) \cong H^1(X, \mathbf{1})$ and denote by $\mathbb{P}(T)$ the projective space of T. If $\mathbb{H}$ is the hyperplane bundle over $\mathbb{P}(T)$ then there exists a universal extension

$$P\beta: 0 \to p^*\mathbb{H} \to W \to \mathbf{1} \to 0 \tag{1}$$

over $X \times \mathbb{P}(T)$ that parametrizes all classes of non-splitting extensions of $\mathbf{1}$ by $\mathbf{1}$, modulo the equivalence relation of identifying extensions which differ by a non-zero constant, (see [**9**, Lemma 2.3]).

Let us consider the families of vector bundles $p^*\mathbb{H}$ and W over $X \times \mathbb{P}(T)$. We recall from [**4**] the proof that

$$H^i(\mathbb{P}(T), R_p^0 \operatorname{Hom}(p^*\mathbb{H}, W) \otimes R_p^1 \operatorname{Hom}(p^*\mathbb{H}, W)^*) = 0$$

for $i = 1, 2$. Basically we need the following lemmas.

Lemma 1. $R_p^0(W) \cong \mathbb{H}$.

Proof. From the exact sequence (1) we have the exact sequence

$$0 \to \mathbf{1} \to W \otimes p^*\mathbb{H}^* \to p^*\mathbb{H}^* \to 0$$

of vector bundles over $X \times \mathbb{P}(T)$, which induces the following exact sequence

$$\begin{aligned} 0 \to \mathbf{1} \xrightarrow{f} p_*W \otimes \mathbb{H}^* \to \mathbb{H}^* \xrightarrow{\partial} H^1(X, \mathbf{1}) \otimes_{\mathbb{P}(T)} \mathcal{O} \xrightarrow{g} R_p^1(W) \otimes \mathbb{H}^* \\ \to H^1(X, \mathbf{1}) \otimes_{\mathbb{P}(T)} \mathcal{O} \otimes \mathbb{H}^* \to 0 \end{aligned} \tag{2}$$

over $\mathbb{P}(T)$. Since $\dim H^0(X, W_t) = 1$ for all $t \in \mathbb{P}(T)$, we see that $p_*W \otimes \mathbb{H}^*$ is a line bundle and hence the inclusion $f: \mathbf{1} \to p_*W \otimes \mathbb{H}^*$ is an isomorphism, so that p_*W is isomorphic to $\mathbb{H}$, which proves the Lemma.

Since the map $f: \mathbf{1} \to p_*W \otimes H^*$ is an isomorphism we have from the exact sequence (2) the following exact sequence

$$0 \to \mathbb{H}^* \xrightarrow{\partial} H^1(X, \mathbf{1}) \otimes_{\mathbb{P}(T)} \mathcal{O} \xrightarrow{g} R_p^1(W) \otimes \mathbb{H}^* \to H^1(X, \mathbf{1}) \otimes_{\mathbb{P}(T)} \mathcal{O} \otimes \mathbb{H}^* \to 0,$$

which we split in two, namely

$$0 \to \mathbb{H}^* \xrightarrow{\partial} H^1(X, \mathbf{1}) \otimes_{\mathbb{P}(T)} \mathcal{O} \to I_g \to 0$$

and

$$0 \to I_g \to R_p^1(W) \otimes \mathbb{H}^* \to H^1(X, \mathbf{1}) \otimes_{\mathbb{P}(T)} \mathcal{O} \otimes \mathbb{H}^* \to 0,$$

where I_g is the image vector bundle. Since g is a homomorphism of vector bundles of constant rank, I_g^* is a vector bundle over $\mathbb{P}(T)$. We take the dual sequences

$$0 \to I_g^* \to H^1(X, \mathbf{1})^* \otimes_{\mathbb{P}(T)} \mathcal{O} \xrightarrow{\partial} \mathbb{H} \to 0 \tag{3}$$

and

$$0 \to H^1(X, \mathbf{1})^* \otimes_{\mathbb{P}(T)} \mathcal{O} \otimes \mathbb{H} \to R_p^1(W)^* \otimes \mathbb{H} \to I_g^* \to 0. \tag{4}$$

From the exact sequence (4) we see that:

LEMMA 2. *$H^i(\mathbb{P}(T), R_p^1(W)^* \otimes \mathbb{H}) \cong H^i(\mathbb{P}(T), I_g^*)$ for $i \geq 1$.*

Proof. The lemma follows from the equality $H^i(\mathbb{P}(T), \mathbb{H}) \otimes H^1(X, \mathbf{1})^* = 0$ for all $i \geq 1$ and the cohomology sequence associated to the exact sequence (4).

LEMMA 3. *If $\partial : H^1(X, \mathbf{1})^* \otimes_{\mathbb{P}(T)} \mathcal{O} \to \mathbb{H}$ is the surjective homomorphism in (3) then the induced map $\partial^* : H^0(\mathbb{P}(T), \mathcal{O}) \otimes H^1(X, \mathbf{1})^* \to H^1(\mathbb{P}(T), \mathbb{H})$ is an isomorphism.*

Proof. The map ∂ coincides with the tautological surjection $T^* \times \mathbb{P}(T) \to \mathbb{H}$. It is a standard fact (following easily from the definition of $\mathbb{H}$) that this induces an isomorphism of spaces of sections.

LEMMA 4. *$H^i(\mathbb{P}(T), I_g^*) = 0$ for $i \geq 0$.*

Proof. Part of the cohomology sequence of (3) is

$$\to H^i(\mathbb{P}(T), \mathbb{H}) \to H^{i+1}(\mathbb{P}(T), I_g^*) \to H^{i+1}(\mathbb{P}(T), \mathcal{O}) \otimes H^1(X, \mathbf{1})^* \to .$$

Now $H^i(\mathbb{P}(T), \mathbb{H}) = 0 = H^i(\mathbb{P}(T), \mathcal{O})$ for $i \geq 1$, so $H^i(\mathbb{P}(T), I_g^*) = 0$ for $i \geq 2$. Thus we have the exact sequence

$$0 \to H^0(\mathbb{P}(T), I_g^*) \to H^0(\mathbb{P}(T), \mathcal{O}) \otimes H^1(X, \mathbf{1})^* \xrightarrow{\partial} H^0(\mathbb{P}(T), \mathbb{H}) \to H^1(\mathbb{P}(T), I_g^*) \to 0.$$

But from Lemma 3 we know that ∂ is an isomorphism, hence $H^i(\mathbb{P}(T), I_g^*) = 0$ for $i = 0, 1$.

PROPOSITION 2. *$H^i(\mathbb{P}(T), R_p^0 \operatorname{Hom}(p^*\mathbb{H}, W) \otimes R_p^1 \operatorname{Hom}(p^*\mathbb{H}, W)^*) = 0$ for $i = 1, 2$.*

Proof. From Lemma 1, $R_p^0(W) \cong \mathbb{H}$; hence the Proposition follows from Lemmas 2 and 4.

THEOREM 1. *There exists a fine moduli space for P_1.*

Proof. From Lange's results there is a universal extension

$$\Omega : 0 \to g^*W \xrightarrow{j} Z \to g^*p^*(\mathbb{H}) \to 0 \tag{5}$$

of vector bundles over $X \times V$, where V is $R_p^1 \operatorname{Hom}(p^*\mathbb{H}, W)$, which parametrizes all extensions of $\mathbf{1}$ by W_t, for all $t \in \mathbb{P}(T)$. Let $\tilde{V}$ be the complement, in V, of the zero section $s_0 : \mathbb{P}(T) \to R_p^1 \operatorname{Hom}(p^*\mathbb{H}, W)$.

From the restrictions of the exact sequence Ω and the induced extension $P\beta$, we have over $X \times \tilde{V}$, the following exact sequences

$$\Omega : 0 \to g^*W \xrightarrow{j} Z \xrightarrow{\pi} g^*p^*(\mathbb{H}) \to 0$$

and (6)

$$g^*P\beta : 0 \to g^*p^*\mathbb{H} \xrightarrow{i} g^*W \xrightarrow{\pi} \mathbf{1} \to 0.$$

Let M_1 be $\tilde{V} \times \operatorname{Pic}_0(X)$. From the universal properties of the extensions $P\beta$ and Ω we see that the pair of extensions $(\Omega, g^*P\beta)$ induces on $X \times M_1$ a pair of extensions which, after tensoring by the pull-back of the Poincaré bundle L, define the moduli space for P_1.

Let (Z, Φ) be the family of endomorphisms over $X \times \tilde{V}$ given by $\Phi = j \circ i \circ \pi$. The restriction $Z_{X \times v}$ is in $S(3, 0)$, for each $v \in \tilde{V}$. However we know, from Section 2, that not all the vector bundles $Z_{X \times v}$ have the same algebra of endomorphisms.

If V and $\mathbb{P}(T)$ are as above then the monomorphism $j:\mathbb{H}^{*2}\to H^1(X,\mathbf{1})\otimes_{\mathbb{P}(T)}\mathcal{O}\otimes\mathbb{H}^*$ induces the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_\Pi \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d, "i"] & \mathbb{H}^{*2} \arrow[r] \arrow[d, "j"] & 0 \\ 0 \arrow[r] & K_\Pi \arrow[r] & V \arrow[r, "\Pi"] & H^1(X,\mathbf{1})\otimes_{P(T)}\mathcal{O}\otimes\mathbb{H}^* \arrow[r] & 0 \end{tikzcd}$$

where i is an inclusion. Hence K_Π is a subbundle of S and $\Pi(S)=\mathbb{H}^{*2}$. This means that for each $s\in(S)_t$, $\Pi_t(s)=\lambda\beta_t$, where $\lambda\in\mathbb{C}^*$ and $\beta_t\in[t]$. Denote by S^1 the kernel K_Π, by S^2 the space $S-S^1$ and by S^3 the space $V-S$.

THEOREM 2. *Let Z be the family of vector bundles parametrized by $\tilde{V}$ given in Theorem* 1. *Then* (i) $Z_m\in\Omega^1$ *iff* $m\in S^1$, (ii) $Z_m\in\Omega^2$ *iff* $m\in S^2$, (iii) $Z_m\in\Omega^3$ *iff* $m\in S^3$.

Proof. The Theorem follows from the definition of Ω^i and S^i.

Now let (Z_1,Φ) denote the family of endomorphisms over $X\times M_1$ induced by (Z,Φ). For $i=1,2,3$ let P^{1i} denote the set of the pairs $(E,\Phi)\in P^1$ such that $E\in\Omega^i$, let $M_{1i}=(\tilde{V}\cap S^i)\times\mathrm{Pic}_0(X)$ and let (Z_{1i},Φ) denote the restriction of (Z_1,Φ) to $X\times M_{1i}$. We deduce at once from Theorem 2 and Remark 2(b) the following result.

COROLLARY. *For $i=1,2$, the family of endomorphisms (Z_{1i},Φ) is a universal family for P^{1i}.*

When $i=3$, the situation is more complicated (see Remark 2 again).

II. Denote by P^0 the set of (equivalence classes of) pairs (E,ϕ) in $P(3,d)$ such that $\phi^3=0$ but $\phi^2\neq 0$. For each pair (E,ϕ) in P^0 we have the following commutative diagram of vector bundles

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & & \\ \Omega: & 0 \arrow[r] & F \arrow[r, "i"] \arrow[d] & E \arrow[r] \arrow[d, "\alpha"] & L \arrow[r] & 0 \\ \beta: & 0 \arrow[r] & L \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & L \arrow[r] & 0 \\ & & 0 & 0 & & \end{tikzcd}$$

where F and L are the image and kernel of ϕ respectively. We consider the problem of parametrizing such diagrams up to equivalence of the extensions β and Ω. Denote by P_0 the set of equivalence classes of the diagrams.

A diagram as above defines a pair $(E,i\circ\alpha)$ in P^0. Two pairs (E,ϕ) and (E,ϕ') in P^0 define the same element in P_0 iff $\phi=(1+\psi)\phi'$ with ψ nilpotent. Again we solve the moduli problem for P_0.

Let T_0 be the space $T-\{0\}$, where $T\cong H^1(X,\mathbf{1})$ and let

$$\beta:0\to\mathbf{1}\to W_0\xrightarrow{p}\mathbf{1}\to 0$$

be the extension over $X\times T_0$ that parametrizes all the non-trivial extensions of $\mathbf{1}$ by $\mathbf{1}$ (see [**10**]). Actually, if $h':T_0\to\mathbb{P}(T)$ is the natural map then $h^*=(id_X\times h')^*(P\beta)=\beta$, where $P\beta$ is the extension (1). Moreover, $h^*(W)=W_0$.

Let $\pi : X \times T_0 \to T_0$ be the projection and denote by V_0 the vector bundle associated to $R^1_\pi(W_0)$. From the following commutative diagram

$$\begin{array}{ccc} H^1(X \times \mathbb{P}(T), W \otimes p^*R^1_p(W)^*) & \xrightarrow{\cong} & \mathrm{END}(V) \\ \downarrow & & \downarrow \\ H^1(X \times T_0, W_0 \otimes \pi^* R^1_\pi(W_0^*)) & \xrightarrow{\delta} & \mathrm{END}(V_0) \end{array}$$

we have that the identity in $\mathrm{END}(V)$ defines an element β_0 in $H^1(X \times T_0, W_0 \otimes \pi^* R^1_\pi(W_0)^*)$ such that $\delta(\beta_0) = id_{V_0}$. If $g : X \times V_0 \to X \times T_0$ is the induced homomorphism then the element β_0 defines under the following natural maps

$$\begin{aligned} H^1(X \times T_0, W_0 \otimes \pi^* R^1_\pi(W)^*) &\xrightarrow{\subset} H^1(X \times T_0, W_0 \otimes g_* \mathcal{O}_{X \times V_0}) \\ &\cong H^1(X \times T_0, g_* g^* W_0) \\ &\cong H^1(X \times V_0, g^* W_0), \end{aligned}$$

a universal extension

$$\Omega : 0 \to g^* W_0 \to Z_0 \to \mathbf{1} \to 0$$

over $X \times V_0$ which parametrizes all the extensions of the trivial line bundle $\mathbf{1}$ by W_t, where W_t is a non-trivial extension of $\mathbf{1}$ by $\mathbf{1}$. We shall prove that there is a variety $S_0 \subset V_0$ which parametrizes the diagrams for which $\beta \otimes L^*$ corresponds to a point of T_0.

The image of the extension β under the homomorphism $H^1(X \times T_0, \mathbf{1}) \xrightarrow{\partial} H^0(T_0, R^1_p \operatorname{Hom}(\mathbf{1}, \mathbf{1}))$ given in the Leray spectral sequence defines a nowhere-vanishing section $s = \partial(\beta)$. This has the property that for each $t \in T_0$, $s(t) \in H^1(X, \mathbf{1})$ is precisely the extension represented by t. The surjective homomorphism $p : W_0 \to \mathbf{1}$ over $X \times T_0$ induces a surjective homomorphism $R^1_p \operatorname{Hom}(\mathbf{1}, W_0) \to R^1_p \operatorname{Hom}(\mathbf{1}, \mathbf{1})$ of vector bundles over T_0. Let us denote by S_0 the subspace $p^{-1}(s(T_0)) \subset V_0$. We now have two extensions $\beta_| : 0 \to \mathbf{1} \xrightarrow{i} g^* W_0 \xrightarrow{p} \mathbf{1} \to 0$ and $\Omega_| : 0 \to g^* W_0 \to Z \to \mathbf{1} \to 0$ which are the restrictions of β and Ω to $X \times S_0$.

Let s_1 and s_2 be the images of $\Omega_|$ and $\beta_|$ respectively under the homomorphisms $H^1(X \times S_0, g^* W_0) \xrightarrow{f} H^0(S_0, R^1_p(g^* W_0))$ and $H^1(X \times S_0, \mathbf{1}) \xrightarrow{h} H^0(S_0, R^1_p(\mathbf{1}))$ given in the Leray spectral sequences. The surjective homomorphism $p : g^* W_0 \to \mathbf{1}$ induces homorphisms p_1^* and p_2^* such that the diagram

$$\begin{array}{ccc} H^1(X \times S_0, g^* W_0) & \xrightarrow{f} & H^0(S_0, R^1_p(g^* W_0)) \\ \downarrow{\scriptstyle p_{1*}} & & \downarrow{\scriptstyle p_{2*}} \\ H^1(X \times S_0, \mathbf{1}) & \xrightarrow[h]{} & H^0(S_0, R^1_p(\mathbf{1})) \end{array}$$

commutes.

From the definition of S_0, we see that $p_{2*}(s_1) = s_2$, and so $p_{2*}(f(\Omega_|)) = p_{2*}(s_1) = s_2 = h(\beta_|)$. Hence, $h(p_{1*}(\Omega_|)) = s_1 = h(\beta_|)$. If $g \geq 3$, $H^1(T_0, \mathbf{1}) = 0$, so $H^1(S_0, \mathbf{1}) = 0$, the map h is injective and hence $p_{1*}(\Omega_|)$ and $\beta_|$ define the same family of extensions. The equality $p_{1*}(\Omega_|) = \beta_|$ implies that there is a unique homomorphism $\alpha : Z \to g^*(W)$ such that the diagram

$$\begin{tikzcd}
\Omega_l : 0 \arrow[r] & g^*W_0 \arrow[r, "j"] \arrow[d, "p_{l^*}"'] & Z \arrow[r] \arrow[d, "\alpha"'] & \mathbf{1} \arrow[r] \arrow[d, equal] & 0 \\
\beta_l : 0 \arrow[r] & \mathbf{1} \arrow[r, "i"] & g^*W_0 \arrow[r, "p"] & \mathbf{1} \arrow[r] & 0
\end{tikzcd}$$

of vector bundles over $X \times S_0$ commutes. Such diagram can be completed as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathbf{1} \arrow[r, equal] \arrow[d] & \mathbf{1} \arrow[d] & & \\
0 \arrow[r] & g^*W_0 \arrow[r, "j"] \arrow[d] & Z \arrow[r] \arrow[d, "\alpha"'] & \mathbf{1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{1} \arrow[r] \arrow[d] & g^*W_0 \arrow[r] \arrow[d] & \mathbf{1} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Let M^0 be $S_0 \times \mathrm{Pic}_0(X)$ and consider the family of diagrams given by $(\pi_{12}^*\Omega_l \otimes \pi_{13}^*L, \pi_{12}^*\beta_l \otimes \pi_{13}^*L)$ over $X \times M_0$. This proves:

THEOREM 3. *If $g \geq 3$, there exists a fine moduli space for the diagrams in P_0 which lie over $S_0 \times \mathrm{Pic}_0(X)$.*

III. Let P_2 be the set of equivalence classes of pairs (ξ, ρ) where $\rho \neq 0$ and $L \not\cong L'$. Two pairs are equivalent iff $\rho = \lambda\rho'$ for some $\lambda \in \mathbb{C}^*$ and $\xi = \xi'$.

Denote by Δ^c the variety $\mathrm{Pic}_0(X) \times \mathrm{Pic}_0(X) - \Delta$, where Δ is the diagonal subvariety. If $p_{1k} : X \times \Delta^c \to X \times \mathrm{Pic}_0(X)$ are the projections for $k = 2, 3$ then let L_{k-1} be p_{1k}^*L. Take the vector bundle $\mathrm{Hom}(L_2, L_1)$ over $X \times \Delta^c$. Since $R^0_{\pi_2}(\mathrm{Hom}(L_2, L_1)) = 0$ we see from Lange's results that there is an extension

$$\Omega : 0 \to \pi_2^*\mathbb{H} \otimes g^*L_1 \to W \to g^*L_2 \to 0 \tag{7}$$

over $X \times \mathbb{P}(V)$, where V is $R^1_{\pi_2}\mathrm{Hom}(L_2, L_1)$, $g : X \times V \to X \times \Delta^c$, $\mathbb{P}(V)$ is the projective bundle of V and $\mathbb{H}$ the hyperplane bundle over $\mathbb{P}(V)$. Such an extension parametrizes all the classes of non-splitting extensions of two non-isomorphic line bundles with zero degree.

Let us take the vector bundle $\mathrm{Hom}(\pi_2^*\mathbb{H} \otimes g^*L_1, W)$ over $X \times \mathbb{P}(V)$ and let Z be the vector bundle $R^1_{\pi_2}\mathrm{Hom}(g^*L_1, W) \otimes \mathbb{H}^*$ over $\mathbb{P}(V)$. To use Lange's result we need to prove that

$$H^i(\mathbb{P}(V), R^0_{\pi_2}(g^*L_1^* \otimes W) \otimes R^1_{\pi_2}(g^*L_1^* \otimes W)^*) = 0 \tag{8}$$

for $i = 1, 2$ or that there exists a unique element in

$$H^1(X \times \mathbb{P}(V), g^*L_1^* \otimes W \otimes \pi_2^*R^1_{\pi_2}(g^*L_1^* \otimes W)^*)$$

which maps to the identity in $\mathrm{END}(Z)$.

To compute the cohomology groups we see that the exact sequence (7) induces the following exact sequence

$$0 \to \mathbb{H} \to R^0_{\pi_2}(W \otimes g^*L_1^*) \to R^0_{\pi_2}(g^*L_2 \otimes g^*L_1^*) \to R^1_{\pi_2}(\mathbf{1}) \otimes \mathbb{H}$$
$$\to R^1_{\pi_2}(W \otimes g^*L_1^*) \to R^1_{\pi_2}(g^*L_2 \otimes g^*L_1^*) \to 0. \tag{9}$$

Since $R^0_{\pi_2}(g^*L_2 \otimes g^*L_1^*) = 0$, we have that $R^0_{\pi_2}(W \otimes g^*L_1^*) \cong \mathbb{H}$. Hence $R^0_{\pi_2}(g^*L_1^* \otimes W) \otimes R^1_{\pi_2}(g^*L_1^* \otimes W)^* \cong \mathbb{H} \otimes R^1_{\pi_2}(W \otimes g^*L^*)^*$. From the exact sequence (9) we obtain the exact sequence

$$0 \to R^1_{\pi_2}(g^*L_2 \otimes g^*L_1^*)^* \otimes \mathbb{H} \to R^1_{\pi_2}(W \otimes g^*L_1^*)^* \otimes \mathbb{H} \to R^1_{\pi_2}(\mathbf{1})^* \to 0. \tag{10}$$

Part of the cohomology sequence of (10) is

$$H^i(\mathbb{P}(V), R^1_{\pi_2}(g^*L_2 \otimes g^*L_1^*)^* \otimes \mathbb{H}) \to H^i(\mathbb{P}(V), R^1_{\pi_2}(W \otimes g^*L_1^*)^* \otimes \mathbb{H})$$
$$\to H^i(\mathbb{P}(V), \mathcal{O}) \otimes H^1(X, \mathcal{O})^* \to .$$

Since $\mathbb{P}(V)$ is a projective bundle over Δ^c we have that

$$H^i(\mathbb{P}(V), \mathcal{O}) \cong H^i(\Delta^c, \mathcal{O}) \tag{11}$$

and from the commutative diagram

$$\begin{tikzcd} X \times \mathbb{P}(V) \arrow[r, "g"'] \arrow[d, "\pi_2"'] & X \times \Delta^c \arrow[d, "p"] \\ \mathbb{P}(V) \arrow[r, "f"'] & \Delta^c \end{tikzcd}$$

we see that

$$H^i(\mathbb{P}(V), R^1_{\pi_2}(g^*L_2 \otimes g^*L_1^*)^* \otimes \mathbb{H})^* \cong H^i(\Delta^c, \sigma^*V^* \otimes V^*), \tag{12}$$

where σ is the canonical involution on Δ^c.

In this case we arrive at the problem that we do not know if the cohomology groups (11) and (12) are zero or not, nor even if there exists a unique element which maps to the identity in END(Z).

Lange in [7] distinguishes between global families of extensions (the concept that we have been using) and families of extensions over a variety M (see [7, page 105]). He defines "family of extensions" using an open cover of M. Over each open set there is a collection of extensions "glued" together to define an extension over the open set. If the covering may be taken to be M itself, then the family is said to be "globally defined".

Using similar ideas we could introduce a more general definition of families of endomorphisms.

Definition. A family of endomorphisms parametrized by M is a collection of pairs $\{(E, \phi)_m\}$ such that there exist

(i) an open cover $\{U_\alpha\}$ of M,

(ii) a vector bundle W_α and

(iii) an endomorphism ϕ_α of W_α in each open set U_α,

such that for each pair $(E, \phi)_m$ with $m \in U_\alpha$, $(W_\alpha, \phi_\alpha)_m$ is equivalent to $(E, \phi)_m$. The family is said to be global if the open cover is M itself.

The global families of endomorphisms correspond to the concept of families of endomorphisms that we have been using (see Definition 1). If we use this new concept of family of endomorphisms (not global) then we can solve the moduli problem for P^2 as follows:

Recall from above, that Z is a vector bundle over $P(V)$ and V is a vector bundle over Δ^c. Hence we can cover $P(V)$ by affine open sets $\{U_\alpha\}$ which induce a cover $\{V_\alpha\}$ over Z. The cohomology groups

$$H^1(U_\alpha, R_p^0(\mathrm{Hom}(g^*L_1, W)) \otimes R_p^1(\mathrm{Hom}(g^*L_1, W)^*))$$

are zero for $i = 1, 2$, since U_α is affine.

Now we can apply Lange's results to prove that there is a universal extension of (the restrictions of) $g^*(L_1)$ by W over $X \times V_\alpha$. Such an extension parametrizes all the extensions of a line bundle L_1 by W_t, where W_t is in U_α. Hence we have the following Theorem.

Theorem 4. *There exists a (local) universal extension for P_2.*

In this case W_t is simple, so by Remark 2(a) the extensions are determined by the endomorphisms. Hence

Corollary. *There exists a (local) universal family of endomorphisms for P^2.*

IV. Let P_3 be the set of all extensions $\xi: 0 \to L \oplus L' \to E \to L \to 0$ with $L \not\cong L'$ and $\mu(L) = \mu(L') = 0$. As in the previous case take Δ^c and the line bundles L_1 and L_2 over $X \times \Delta^c$. If $p: X \times \Delta^c \to \Delta^c$ is the projection then the points in

$$R_p^1(\mathrm{Hom}(g^*(L_1), g^*(L_1 \oplus L_2))) = V'$$

represent extensions in P_3.

To construct the (global) universal extension we must prove that the cohomology group $H^i(\Delta^c, V')$ is zero, for $i = 1, 2$.

As in the previous case, we only obtain a local universal extension, using a Stein cover of Δ^c.

Theorem 5. *There exists a (local) universal family of extensions for P_3.*

V. Let P_4 be the set of extensions $\xi: 0 \to L \oplus L \to E \to L \to 0$ such that E is indecomposable.

If R is the space $\mathrm{Ext}_x(\mathbf{1}, \mathbf{1} \oplus \mathbf{1})$ then there exists a universal extension

$$0 \to \mathbf{1} \oplus \mathbf{1} \to Y \to \mathbf{1} \to 0 \tag{13}$$

over $X \times R$ that parametrizes all the extensions of $\mathbf{1}$ by $\mathbf{1} \oplus \mathbf{1}$ (see [10]). Take $S(\mathbf{1}) \subset R$ as in Section 2 and denote $R - S(\mathbf{1})$ by S_0. If M_4 is $S_0 \times \mathrm{Pic}_0(X)$ then the extension (13) induces over $X \times M_4$ the universal extension

$$\xi: 0 \to L_0 \oplus L_0 \to Y_0 \otimes L_0 \to L_0 \to 0, \tag{14}$$

where L_0 and Y_0 are the corresponding pull back of the Poincaré bundle and the restriction of the vector bundle Y to $X \times S_0$.

Thus, from the universal properties of the exact sequence (14) we have the following Theorem.

THEOREM 6. *There is a fine moduli space for P_4.*

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