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SUBGROUPS OF FINITE INDEX IN (2, 3, n)-TRIANGLE GROUPS

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Abstract. For an integer $n \geq 7$, let $\Delta(n)$ denote the (2, 3, n)-triangle group, and let $M(n)$ be the positive integer determined by the conditions that $\Delta(n)$ has a subgroup of index m for all $m \geq M(n)$, but no subgroup of index $M(n) - 1$. The main purpose of the paper is to obtain information (bounds, in some cases explicit values) concerning the function $M(n)$ (cf. Theorem 1). We also show that $\Delta(n)$ is replete (i.e., has a subgroup of index m for every integer $m \geq 1$) if, and only if, n is divisible by 20 or by 30 (see Theorem 2).

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1. Introduction and main results. For a positive integer n, let

$$\Delta(n) = \langle x, y \mid x^2 = y^3 = (xy)^n = 1 \rangle$$

be the (2, 3, n)-triangle group; that is, the quotient of the inhomogeneous modular group

$$\Gamma = \langle x, y \mid x^2 = y^3 = 1 \rangle$$

by the normal closure of the relator $(xy)^n$. For $n < 6$, the group $\Delta(n)$ is finite, whereas for $n = 6$ it is an infinite soluble group associated with symmetries of the Euclidean plane. For $n > 6$, $\Delta(n)$ is an infinite insoluble group associated with a hyperbolic triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{n}$. It is this last case that we are interested in.

It follows in particular from [**5**, Theorem 3] that, given $n \geq 7$, there exists a positive integer $M(n)$ such that $\Delta(n)$ has a subgroup of index m for all $m \geq M(n)$, but no subgroup of index $M(n) - 1$; this also follows from [**2**], where it is shown that all but finitely many alternating groups A_k occur as quotients of $\Delta(n)$ for each fixed $n \geq 7$. Determining the numbers $M(n)$ precisely appears, in general, very difficult; the main purpose of the present paper is to establish the following information on $M(n)$, where we include some previously know facts re-proven here for the sake of completeness.

THEOREM 1. *Let n be an integer with* $n \geq 7$.

(i) *For* $n \geq 53$, *we have* $M(n) \leq 6n$, *with equality holding if n is a prime.*
(ii) *For* $21 \leq n \leq 52$, *we have* $M(n) \leq 20n$.
(iii) *For* $17 \leq n \leq 20$, *we have* $M(n) \leq 19n$.

(iv) *For* $7 \leq n \leq 16$, *we have*

$$\begin{array}{ll} M(7) = 168, & M(12) \leq 240, \\ M(8) \leq 240, & M(13) \leq 143, \\ M(9) \leq 180, & M(14) \leq 154, \\ M(10) = 10, & M(15) \leq 210, \\ M(11) \leq 110, & M(16) \leq 128. \end{array}$$

Call a group *replete* if it contains a subgroup of index m for every integer $m \geq 1$. For instance, the modular group Γ is replete, whereas $\Delta(7)$ has no proper subgroup of index less than 7. Our second main result determines those n for which $\Delta(n)$ is replete.

THEOREM 2. *Let* $n \geq 7$ *be an integer. Then the hyperbolic triangle group* $\Delta(n)$ *is replete if and only if n is divisible by* 20 *or by* 30.

The paper is organised as follows. In the next section we recall those facts concerning coset diagrams needed in our present context, and we show by means of the genus formula for Γ that for $n \geq 7$ prime the triangle group $\Delta(n)$ does not have a subgroup of index $6n - 1$ (Proposition 3).

Section 3 describes two surgery processes (join and composition of diagrams) that produce new coset diagrams from given ones. These processes and their properties, explained in Lemmas 6 and 8, respectively, are basic for most of what follows: In Section 4, these processes are used to obtain a generic existence result (for $n \geq 53$); in Section 5, we apply them to establish a corresponding existence result in the range $17 \leq n \leq 52$ (see Parts (ii) and (iii) of Theorem 1); while Section 6 uses these processses to derive the estimates given in Part (iv) of Theorem 1 for $n = 8, 9$ and $11 \leq n \leq 16$. Moreover, Proposition 28 in Section 8 establishes the exact value of $M(10)$, again relying on the processes of join and composition.

Section 7 is of a somewhat different flavour; its main purpose is the computation of $M(7)$, which is accomplished by somewhat more general arguments of an arithmetic nature pertaining to the genus formula for $\Delta(n)$. The fact itself that $M(7) = 168$ is not new; it follows, for instance, from Conder's analysis of permutation representations of the (2, 3, 7)-triangle group in [**1**]. We include a somewhat different argument for the sake of completeness.

The paper concludes with the proof of Theorem 2 in Section 9.

2. Permutations and coset diagrams. Our main tool will be coset diagrams over Γ and $\Delta(n)$, and operations involving them. This technique was systematically developed by Graham Higman in the 1960s and 1970s, and is explained, for instance, in [**1**] and [**2**]; a thorough introduction to coset diagrams over the Hecke groups $C_2 * C_q$ with q prime can be found in [**3**, Section 3]. Here we recall only those facts needed in the present context.

Let G be a subgroup of the modular group Γ with index $(\Gamma : G) = m$. Then Γ acts on the m-set Γ/G of left cosets of G in Γ by left multiplication, giving rise to a transitive permutation representation $\varphi_G : \Gamma \to S(\Gamma/G)$ of Γ on Γ/G such that $\mathrm{stab}_{\varphi_G}(1 \cdot G) = G$. Identifying Γ/G with the standard m-set $[m] = \{1, 2, \ldots, m\}$ by means of a bijection $\psi : \Gamma/G \to [m]$ sending the coset $1 \cdot G$ to 1, we obtain a transitive permutation representation $\tilde{\varphi}_G : \Gamma \to S_m$ such that $\mathrm{stab}_{\tilde{\varphi}_G}(1) = G$. Since Γ is generated by two elements x and y, the representation φ_G is determined up to similarity once we specify permutations $\tilde{\varphi}_G(x) = \sigma$ and $\tilde{\varphi}_G(y) = \tau$. These permutation σ, τ satisfy the relations

$$\sigma^2 = \tau^3 = 1, \tag{2.1}$$

and generate a transitive subgroup of S_m. Conversely, given permutations $\sigma, \tau \in S_m$ satisfying relations (2.1) and generating a transitive subgroup $\langle \sigma, \tau \rangle$ of S_m, then mapping $x \mapsto \sigma$ and $y \mapsto \tau$ yields a permutation representation $\varphi : \Gamma \to S_m$ such that $(\Gamma : G) = m$, where $G = \mathrm{stab}_\varphi(1)$. Hence, existence of a subgroup of index m in Γ is equivalent to the existence of permutations $\sigma, \tau \in S_m$ satisfying (2.1) and generating a transitive subgroup in S_m. Similarly, existence of a subgroup of index m in the triangle group $\Delta(n)$ is equivalent to the existence of permutations $\sigma, \tau \in S_m$ satisfying relations

$$\sigma^2 = \tau^3 = (\tau\sigma)^n = 1, \tag{2.2}$$

and generating a transitive subgroup of S_m.

We shall find it convenient to translate the data (σ, τ) into a geometric language. More precisely, to a pair (σ, τ) of permutations specifying a subgroup of index m in Γ, there corresponds a diagram D consisting of m labelled vertices, red undirected loops, blue undirected loops, red undirected edges and blue directed edges, constructed as follows: The vertices of D are labelled with the elements of the standard set $[m]$; for $i, j \in [m]$ such that $\sigma(i) = j$, the vertices labelled i and j are joined by an undirected red edge (a loop if $i = j$); for $i, j \in [m]$ with $i \neq j$ and $\tau(i) = j$, we draw a directed blue edge from vertex i to vertex j, while for $i = j$ we attach an undirected blue loop to vertex i. Thus, by construction, such a diagram D satisfies the following:

- (D1) Each vertex of D either has a red loop, or is incident with exactly one red edge.
- (D2) Each vertex of D either has a blue loop, or is contained in precisely one oriented blue triangle.
- (D3) The red and blue edges together give a connected figure.

Conversely, a diagram D satisfying Conditions (D1)–(D3) specifies a pair of permutations $\sigma, \tau \in S_m$ satisfying relations (2.1) and generating a transitive subgroup, and hence a subgroup of index m in Γ. In the same vein we can speak of diagrams for a triangle group $\Delta(n)$. Clearly, every diagram for $\Delta(n)$ can be viewed as a diagram for Γ (specifying the preimage of the original subgroup of $\Delta(n)$ in Γ under the canonical map $\Gamma \to \Delta(n)$), while a diagram for Γ can be interpreted as a diagram for $\Delta(n)$ if and only if the cycle lengths of permutation $\tau\sigma$ divide n.

If necessary, we shall keep track of the cycles of the permutation $\tau\sigma$ by undirected green loops and directed green edges. Starting with the vertex a, say, we follow the red loop or edge at a, to vertex b, say, then follow the blue loop or edge (the latter according to its orientation) from b to vertex c, say. Then there is an undirected green loop at a if $a = c$, and a green edge from a to c otherwise. Each vertex has a green loop, or is a vertex of exactly one (oriented) green polygon. If $\tau\sigma$ has $f(r)$ cycles of length r, and a total of h cycles, then clearly $m = \sum_{r \geq 1} rf(r)$ and $h = \sum_{r \geq 1} f(r)$. If G is a subgroup of index m in the modular group Γ, then the partition $m = f(1) + 2f(2) + \cdots + mf(m)$ of m obtained in this way from the corresponding permutation representation φ_G (or a representation similar to φ_G, or a diagram for G) is called the *cusp-split* of G. Denoting by $e(2)$ and $e(3)$ the number of 1-cycles in $\sigma = \varphi_G(x)$ and $\tau = \varphi_G(y)$, respectively, and by p the genus of the Riemann surface associated with G, we have the *genus formula*

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(see [**8**, Formula (2)] or [**4**])

$$m = 3e(2) + 4e(3) + 12(p-1) + 6h. \tag{2.3}$$

As a first application of the genus formula, we show a result to the effect that $\Delta(n)$ does not have subgroups for certain indices.

PROPOSITION 3. *If $n \geq 7$ is prime, then $\Delta(n)$ does not have a subgroup of index $6n-1$.*

Proof. Suppose for a contradiction that G is a subgroup of index $6n-1$ in $\Delta(n)$, and let D be a coset diagram for G. Interpreting D as a diagram over Γ, the genus formula (2.3) implies that

$$6n - 1 \geq -12 + 6h, \tag{2.4}$$

since $e(2), e(3), p \geq 0$. However, as D is associated with $\Delta(n)$, and n is prime, the cusp-split of G (or its preimage $\tilde{G}$ in Γ) must consist of 1s and ns. Considering the resulting equation

$$6n - 1 = f(1) + nf(n)$$

modulo n, we find that $f(1) \equiv -1 \mod n$, so $f(1) \geq n-1$ and, consequently, $h \geq n+4$. Combining the last inequality with (2.4), we get

$$6n - 1 \geq -12 + 6(n+4) = 6n + 12,$$

a contradiction. □

REMARK. The results of application of Formula 2.3 may also be achieved by using necessary conditions for transitivity of a group generated by given permutations, as derived in [**6**] or [**7**].

COROLLARY 4. *For $n \geq 7$ prime, we have $M(n) \geq 6n$.*

3. Surgery on diagrams.

To simplify our illustrations throughout the paper, we shall adhere to the following conventions: (i) green edges are omitted; (ii) we assume that the blue triangles (indicated by bold lines) are oriented clockwise; (iii) red edges are indicated by light lines; (iv) red and blue loops are omitted (they may be inferred at vertices not on an edge of the appropriate colour).

In this section, we shall introduce and discuss two basic processes producing new diagrams from given ones.

DEFINITION 5. *The join of diagrams*. Let D_1 and D_2 be coset diagrams over Γ, and suppose that $D_1 and D_2$ have red loops at vertices x_1 and x_2, respectively. Then we combine the diagrams D_1 and D_2 into a new figure by replacing the red loops in question by an undirected red edge joining x_1 and x_2. The new (mixed) graph is called the *join* of D_1 and D_2, and is denoted as $D_1 * D_2$.

LEMMA 6 (The Joining Lemma). *Suppose that D_1 is a coset diagram over Γ with m_1 vertices and with a red loop at vertex x_1, with x_1 contained in a green polygon of size k_1, and that D_2 is a diagram having a total of m_2 vertices, with a red loop attached to vertex x_2, where x_2 is contained in a green polygon of size k_2. Then the join $D_1 * D_2$ is a*

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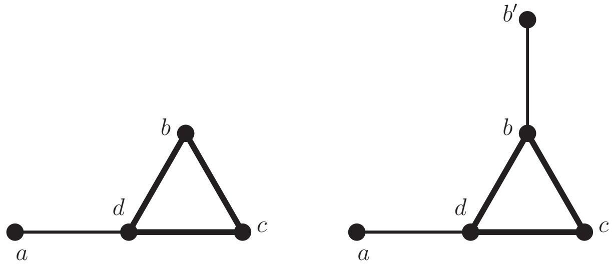

Figure 1. The join operation.

coset diagram over Γ *with* $m_1 + m_2$ *vertices, and the green polygon in* $D_1 * D_2$ *containing* x_1 *has size* $k_1 + k_2$*, and contains* x_2*. The sizes of all other green polygons are unchanged.*

Proof. The figure $D_1 * D_2$ involves $m_1 + m_2$ vertices, and is connected by construction plus the fact that D_1 and D_2 are connected. Also, the requirements (D1)–(D3) concerning red and blue loops, red undirected edges and blue directed edges are again met so that $D_1 * D_2$ is indeed a coset diagram over Γ. Moreover, the only green polygons affected by the process of joining are those through x_1 or x_2. If the green polygon n D_1 containing x_1 is $(y_1, \ldots, y_{k_1-1}, x_1)$ and that in D_2 containing x_2 is $(z_1, \ldots, z_{k_2-1}, x_2)$, then $D_1 * D_2$ has a green polygon $(y_1, \ldots, y_{k_1-1}, x_1, z_1, \ldots, z_{k_2-1}, x_2)$ involving x_1 and x_2, and of size $k_1 + k_2$, as claimed. □

Figure 1 shows the result of joining a 4-vertex diagram with one green polygon of size 4, and a 1-vertex diagram (which, of course, has a red, blue and green loops at its only vertex). The result is a 5-vertex diagram with one green polygon of size 5, in accordance with Lemma 6.

DEFINITION 7. (Free triangles).

(i) By a *free triangle* in a coset diagram over Γ we mean a blue triangle having red loops attached to (at least) two of its vertices.

(ii) A coset diagram over Γ is $F(r)$ if it has at least r free triangles.

Clearly, if a coset diagram has more than three vertices, then, by connectedness, a free triangle cannot have red loops at all three vertices; that is, in such a diagram, each free triangle has exactly two red loops.

Suppose that we have two coset diagrams over Γ, D_1 with m_1 vertices and D_2 with m_2 vertices, where $m_1, m_2 > 3$. We choose disjoint sets of labels $1, 2, \ldots, m_1, m_1 + 1, \ldots, m_1 + m_2$. Suppose further that (x_2, x_3, x_4) is a free triangle in D_1, and that (y_2, y_3, y_4) is a free triangle in D_2. Then D_1 has red loops at x_2 and x_3, say, and a red edge (x_1, x_4). Also, D_1 has a blue triangle (x_2, x_3, x_4), and a green polygon, including $\ldots x_1, x_2, x_3, x_4, \ldots$, and similarly for D_2. We form a new figure $D_1 + D_2$ by replacing the red loops at x_2, y_2, x_3 and y_3 with red edges (x_2, y_2) and $x_3, y_3)$. The new figure involves $m_1 + m_2$ vertices, is connected and satisfies Conditions (D1)–(D3). Moreover, the new diagram inherits most of the green structure from D_1 and D_2; only the green polygons involving x_2, y_2, x_3, and y_3 are affected. Indeed, $D_1 + D_2$ has green polygons $\ldots x_1, x_2, y_3, x_4, \ldots$ and $\ldots y_1, y_2, x_3, y_4, \ldots$. Note that these two green polygons have the same sizes as the corresponding ones in D_1 and D_2. Consequently, if all green polygons in D_1 and D_2 have sizes which divide n, then those of $D_1 + D_2$ also have sizes

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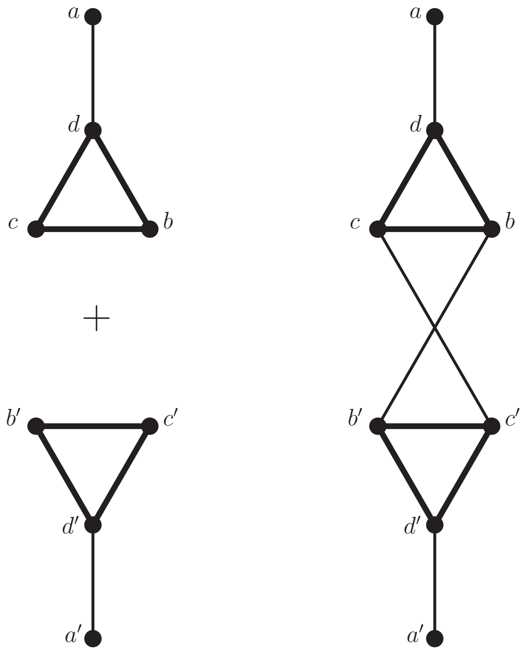

Figure 2. The composition operation.

dividing n; i.e. all are coset diagrams over $\Delta(n)$. The diagram just constructed is called the *composition* of D_1 and D_2. See Figure 2 for an illustration.

If one of the diagrams D_1, D_2 has only three vertices, or if both diagrams have only three vertices, we can still form the composition by choosing two of the three vertices in the corresponding free triangle, the third vertex keeping its red loop.

LEMMA 8 (The Composition Lemma). *Let r_1 and r_2 be positive integers, let D_1 be an m_1-vertex $F(r_1)$ diagram with cusp-split $\{f(i)\}_{i\geq 1}$, and let D_2 be an $F(r_2)$ diagram with m_2 vertices and cusp-split $\{g(i)\}_{i\geq 1}$. Then the composition $D_1 + D_2$ is an $(m_1 + m_2)$-vertex $F(r_1 + r_2 - 2)$ diagram with cusp-split $\{f(i) + g(i)\}_{i\geq 1}$. Moreover, if D_1 and D_2 are diagrams over $\Delta(n)$, so is $D_1 + D_2$.*

Proof. The observations preceding the lemma, together with the fact that the process of composition uses two free triangles, establish all claims in the case when $m_1, m_2 > 3$. The remaining cases require a separate analysis, which is however quite similar to the above, and is left to the reader. □

4. An existence theorem. Our aim here is to show the following generic existence result.

PROPOSITION 9. *If $n \geq 53$ and $m \geq 6n$, then $\Delta(n)$ has a subgroup of index m. Equivalently, for $n \geq 53$, we have $M(n) \leq 6n$.*

The proof of Proposition 9 requires some preparation.

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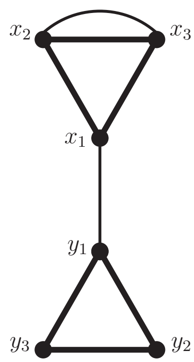

Figure 3. The 6-vertex diagram S.

LEMMA 10. *Let T be a 3-vertex diagram consisting of one blue triangle, with all three vertices carrying red loops, and let t be a positive integer. Then the join*

$$D = \underbrace{T * \cdots * T}_{t \text{ copies}}$$

of t disjoint copies of T gives a diagram with $3t$ vertices and one green cycle of size $3t$.

Proof. We use induction on t. If $t = 1$, i.e. $D = T$, one immediately checks that all three vertices of D are contained in the same green cycle. Suppose that the assertion of Lemma 10 holds for $t = t_0$ with some integer $t_0 \geq 1$, and let D be the join of t_0 copies of T. By the induction hypothesis, D has a green cycle of length $3t_0$; hence, by the joining lemma, $D * T$ has a green cycle of length $3t_0 + 3 = 3(t_0 + 1)$, as required. □

LEMMA 11. *Let S be the 6-vertex diagram over Γ consisting of two disjoint blue triangles $T_1 = (x_1, x_2, x_3)$ and $T_2 = (y_1, y_2, y_3)$, oriented clockwise, with red edges (x_2, x_3) and (x_1, y_1), and red loops at y_2 and y_3 (i.e. $S = T * T'$, where T' is a 3-vertex diagram with exactly one red loop); see Figure 3. Let r be a positive integer, and let $D = S * \cdots * S$ be the join of r disjoint copies of S. Then D is a diagram with a total of $6r$ vertices and cusp-split $(1^r, 5r)$, where the green loops sit at x_3 and the vertices corresponding to x_3.*

Proof. We use induction on r. If $r = 1$, that is, $D = S$, then one immediately checks that S has a green 5-cycle $(x_1, y_2, y_3, y_1, x_2)$, and a green loop at x_3. Suppose that the assertion of the lemma holds for $r = r_0$ with some integer $r_0 \geq 1$, and let D be the join of r_0 copies of S. By the induction hypothesis, D has a green cycle of size $5r_0$, and green loops at x_3, and the vertices corresponding to x_3. By the joining lemma, since the two vertices joined by a new red edge both sit in long cycles, $D * S$ has a green cycle of length $5r_0 + 5 = 5(r_0 + 1)$, and we have green loops at x_3, and at all vertices corresponding to x_3, as claimed. □

LEMMA 12. *If $n \geq 5r + 12$, or if $n = 5r + 6, 5r + 9, 5r + 10$, then there exists an $F(2)$ diagram with $n + r$ vertices and cusp-split $(1^r, n^1)$.*

Proof. Construct a diagram D by first joining r copies of the 6-vertex diagram S to obtain a diagram S_r, then join a copy of the 3-vertex diagram T to S_r from the left, and three copies of T from the right. By Lemmas 10 and 11 plus the joining lemma, the

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resulting diagram D has r green loops and a green cycle of size $5r + 12$. Moreover, the first and the last triangle are free, so D is $F(2)$, and the first two triangles on the right of S_r each carry a red loop. If we have $n \geq 5r + 12$, set $n = 5r + 12 + 3s + t$, where s is a non-negative integer, and $t \in \{0, 1, 2\}$. Then we join s copies of T at the right-hand side of D, and t copies of the 1-vertex diagram at the first two triangles to the right of S_r. By Lemma 10 plus the joining lemma, the result is an $F(2)$ diagram with $n + r$ vertices and the desired cusp-split.

If $n = 5r + 6$, we again construct the diagram S_r (i.e. the join of r copies of the 6-vertex diagram S), and then join one copy of T each from the left and the right to S_r to get the desired diagram.

Finally, if $n = 5r + 9$ or $5r + 10$, then we join one copy of T to S_r from the left, and two copies of T from the right, and, in the latter case, also join a 1-vertex diagram to the first triangle on the right of S_r. □

LEMMA 13. *If $n \geq 5r + 6$, or if $n = 5r$ with some $r > 0$, or if $n = 5r + 3, 5r + 4$, then there exists an $F(1)$ diagram with $n + r$ vertices and cusp-split* $(1^r, n^1)$.

Proof. For $n = 5r + 3, 5r + 4$, we begin by forming the diagram S_r, and then join a copy of T to S_r from the left. This gives the desired diagram for $n = 5r + 3$; while for $n = 5r + 4$, we also join a 1-vertex diagram to $T * S_r$ from the right.

For $n \geq 5r + 6$, we set $n = 5r + 6 + 3s + t$ with $s \geq 0$ and $t \in \{0, 1, 2\}$. We form the diagram $D = T * S_r * T$, and join s copies of T to D from the right. By Lemma 10 plus the joining lemma, this yields a diagram with $6r + 6 + 3s$ vertices, r green loops and a green cycle of length $5r + 6 + 3s$. Moreover, the first triangle is free. If $s = 0$, the triangle on the right of S_r at this stage is also free, so we can join t copies of the 1-vertex diagram on the right to obtain the desired $F(1)$ diagram. For $s = 1$, and $t \in \{0, 1\}$, we join t 1-vertex diagrams to the first triangle on the right of S_r, to obtain an $F(2)$ diagram with the desired cusp-split, while for $s = 1$ and $t = 2$ we get an $F(1)$ diagram of the required type. Finally, if $s \geq 2$, we again obtain in this way an $F(2)$ diagram with cusp-split $(1^r, n^1)$, as required.

Finally, for $n = 5r$ with $r \geq 1$, we first form $D = S_{r-1}$, a diagram with $6(r - 1)$ vertices, $r - 1$ green loops and a green cycle of size $5(r - 1)$. We then join a copy of T to D from the left (thus, in particular, obtaining a free triangle), and a copy of T' from the right, where, as before, T' is a 3-vertex diagram with exactly one red loop. The result is an $F(1)$ diagram with $6r$ vertices, r green loops and a green cycle of size $5(r - 1) + 3 + 2 = 5r$, as desired. □

Proof of Proposition 9. We distinguish cases according to the residue of n modulo 5.

CASE 1. $n = 5k$. By Lemma 12, there exists an $F(2)$ diagram with cusp-split $(1^r, n^1)$ for every r with $0 \leq r \leq k - 2$. More precisely, we use here the special case of Lemma 12 where $n = 5r + 10$ with $r = k - 2$, and the case where $n \geq 5r + 12$ for $0 \leq r \leq k - 3$. Composing K such diagrams for various suitably chosen r, we can get a diagram over $\Delta(n)$ with cusp-split $(1^S, n^K)$ for every integer S with $0 \leq S \leq K(k - 2)$. Setting $m = Kn + R$ with $0 \leq R < n$, we will be able to produce a diagram over $\Delta(n)$ with m vertices for all m satisfying $Kn \leq m \leq Kn + n - 1$, provided that

$$K(k - 2) \geq n - 1 = 5k - 1,$$

or, equivalently, whenever

$$k \geq \frac{2K-1}{K-5}. \tag{4.1}$$

Note that the function $g : (5, \infty) \to (0, \infty)$ given by $g(x) = \frac{2x-1}{x-5}$ is strictly decreasing for $x > 5$. Hence, Condition (4.1) on k is the strongest for $K = 6$, and we have shown existence, in Case 1, of an m-vertex diagram over $\Delta(n)$ for every $m \geq 6n$, whenever $k \geq 11$, that is, whenever $n \geq 55$.

CASE 2. $n = 5k + 1$. Using Lemma 12 in the case where $n = 5r + 6$ with $r = k - 1$, we obtain an $F(2)$ diagram with cusp-split $(1^{k-1}, n^1)$. Similarly, using the case where $n \geq 5r + 12$ and $0 \leq r \leq k - 3$, we find $F(2)$ diagrams with cusp-split $(1^r, n^1)$ for all integers r such that $0 \leq r \leq k - 3$. Moreover, by Lemma 13 in the special case where $n \geq 5r + 6$ and $r = k - 2$, there also exists an $F(1)$ diagram with cusp-split $(1^{k-2}, n^1)$.

We claim that by composing K such diagrams we can obtain diagrams with cusp-split $(1^S, n^K)$ for every integer S with $0 \leq S \leq K(k - 1)$. Indeed, composing K $F(2)$ diagrams with suitably chosen rs in the range $0 \leq r \leq k - 3$, we can reach every S with $0 \leq S \leq K(k - 3)$. Now suppose that we have chosen K $F(2)$ diagrams with $r = k - 3$. Replacing j of these by an $F(2)$ diagram with $r = k - 1$, for $j = 1, 2, \ldots, K$, we reach the values

$$S = (K - j)(k - 3) + j(k - 1) = K(k - 3) + 2j, \quad j = 1, 2, \ldots, K.$$

On the other hand, replacing $j - 1$ of $F(2)$ diagrams (from the second one onwards) by an $F(2)$ diagram with $r = k - 1$, for $j = 1, 2, \ldots, K$, and replacing the first diagram by an $F(1)$ diagram with $r = k - 2$, we reach every S-value of the form

$$S = (K - j)(k - 3) + (j - 1)(k - 1) + (k - 2) = K(k - 3) + 2j - 1, \quad j = 1, 2, \ldots, K.$$

This proves our claim.

Consequently, we are able to produce an m-vertex diagram over $\Delta(n)$ for all m with $Kn \leq m \leq Kn + (n - 1)$, provided that

$$K(k - 1) \geq n - 1 = 5k;$$

or, equivalently, whenever

$$k \geq \frac{K}{K-5}.$$

Since the function $g(x) = \frac{x}{x-5}$ is decreasing for $x > 5$, we can thus, in Case 2, find a diagram over $\Delta(n)$ with m vertices for every $m \geq 6n$, provided that $k \geq 6$ or $n \geq 31$.

CASE 3. $n = 5k + 2$. Using Lemma 12 in the case where $n \geq 5r + 12$ and with r in the range $0 \leq r \leq k - 2$, we find an $F(2)$ diagram with cusp-split $(1^r, n^1)$ for each r with $0 \leq r \leq k - 2$. Moreover, from Lemma 13 in the case where $n \geq 5r + 6$ and $r = k - 1$, we obtain an $F(1)$ diagram with cusp-split $(1^{k-1}, n^1)$. Composing K $F(2)$ diagrams with suitably chosen rs in the rangle $0 \leq r \leq k - 2$, we obtain an $F(2)$ diagram with cusp-split $(1^S, n^K)$ for all S such that $0 \leq S \leq K(k - 2)$. Further, replacing one or two of these diagrams (at the beginning or end of the chain) with an $F(1)$ diagram for $r = k - 1$, we can also reach the values $S = K(k - 2) + 1, K(k - 2) + 2$. Arguing as before, we thus find an m-vertex diagram over $\Delta(n)$ for all $m \geq 6n$, provided that $k \geq 11$ or $n \geq 57$.

CASE 4. $n = 5k + 3$. Using Lemma 12 in the case where $n \geq 5r + 12$ and with $r = 0, 1, \ldots, k - 2$, we find $F(2)$ diagrams with cusp-split $(1^r, n^1)$ for all integers r such that $0 \leq r \leq k - 2$. Also, from Lemma 13 in the special cases where $n = 5r + 3$ (with $r = k$), and where $n \geq 5r + 6$ with $0 \leq r \leq k - 1$, we obtain $F(1)$ diagrams with cusp-split $(1^r, n^1)$ for all r such that $0 \leq r \leq k$. Composing K $F(2)$ diagrams with various suitably chosen rs in the range $0 \leq r \leq k - 2$, we get $F(2)$ diagrams with cusp-split $(1^S, n^K)$ for all S in the range $0 \leq S \leq K(k - 2)$. Furthermore, replacing one or both of the free traingles (at the beginning and end of the chain) by an $F(1)$ diagram with cusp-split $(1^{k-1}, n^1)$ or $(1^k, n^1)$, we can also cover the values $S = K(k - 2) + j$ for $j = 1, 2, 3, 4$. This yields the sufficient condition

$$K(k - 2) + 4 \geq n - 1 = 5k + 2,$$

or, equivalently,

$$k \geq \frac{2K - 2}{K - 5},$$

for the existence of m-vertex diagrams over $\Delta(n)$ for all m in the range $Kn \leq m \leq Kn + (n - 1)$; and arguing as before we find that an m-vertex diagram over $\Delta(n)$ exists in Case 4 for all $m \geq 6n$, provided that $k \geq 10$ or $n \geq 53$.

CASE 5. $n = 5k + 4$. From Lemma 12, in the cases where $n \geq 5r + 12$ (with $0 \leq r \leq k - 2$) and $n = 5r + 9$ (with $r = k - 1$), we obtain $F(2)$ diagrams with cusp-split $(1^r, n^1)$ for all r such that $0 \leq r \leq k - 1$. Similarly, from Lemma 13 in the cases where $n = 5r + 4$ (with $r = k$) and $n \geq 5r + 6$ with $0 \leq r \leq k - 1$, we find $F(1)$ diagrams with cusp-split $(1^r, n^1)$ for all r in the range $0 \leq r \leq k$. Composing K $F(2)$ diagrams with various suitably chosen values of r, we get $F(2)$ diagrams with cusp-split $(1^S, n^K)$ for all integers S such that $0 \leq S \leq K(k - 1)$. Replacing one or two of these $F(2)$ diagrams with an $F(1)$ diagram for $r = k$, we see that we can also cover the values $S = K(k - 1) + j$ for $j = 1, 2$. Thus, our sufficient condition for existence of an m-vertex diagram over $\Delta(n)$ with m covering the range $Kn \leq m \leq Kn + (n - 1)$ becomes

$$K(k - 1) + 2 \geq n - 1 = 5k + 3,$$

and we find that an m-vertex diagram over $\Delta(n)$ exists for all $m \geq 6n$, provided that $k \geq 7$ or $n \geq 39$.

Combining our findings of Cases 1–5, we see that an m-vertex diagram over $\Delta(n)$ (and hence a subgroup of index m in $\Delta(n)$) exists for all $m \geq 6n$, provided that $n \geq 53$, which is the assertion of Proposition 9. □

Proposition 9 just established in conjunction with Corollary 4 now yields the following.

COROLLARY 14. *For prime* $n \geq 53$, *we have* $M(n) = 6n$.

Proposition 9 and Corollary 14, when taken together, establish Part (i) of Theorem 1.

5. A further existence theorem for $n \geq 17$.

In order to cover the range where $17 \leq n \leq 56$, we show the following result, thus establishing Parts (ii) and (iii) of Theorem 1.

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PROPOSITION 15.
(a) *If* $n \geq 21$ *and* $m \geq 20n$, *then* $\Delta(n)$ *has a subgroup of index* m.
(b) *For* $17 \leq n \leq 20$, *and every* m *with* $m \geq 19n$, *the triangle group* $\Delta(n)$ *has a subgroup of index* m.

Proof. (a) Let $m = Kn + R$, where R is such that $0 \leq R < n$. We show how to build a diagram with cusp-split $(1^R, n^K)$. Suppose that

$$r \leq N := \left[\frac{n-12}{5}\right],$$

where, for a real number x, $[x]$ denotes the largest integer less than or equal to x. By Lemma 12, there exists an $F(2)$ diagram with cusp-split $(1^r, n^1)$. If we compose K such diagrams, for various suitably chosen values of r, then, according to the composition lemma (Lemma 8), we obtain diagrams with cusp-split $(1^S, n^K)$ for every integer S such that $0 \leq S \leq KN$. Hence, in order to ensure existence of an m-vertex diagram over $\Delta(n)$ for all m in the range $Kn \leq m \leq Kn + (n-1)$, it thus suffices to require that

$$KN \geq n - 1. \tag{5.1}$$

Since $N \geq \frac{n-16}{5}$, we get the sufficient condition

$$K(n-16) \geq 5(n-1),$$

or, equivalently,

$$n \geq \frac{16K - 5}{K - 5}. \tag{5.2}$$

For $K = 20$, Inequality (5.2) gives $n \geq 21$, and, since the function $\frac{16x-5}{x-5}$ is decreasing for $x > 5$, the condition that $n \geq 21$ is enough to ensure the conclusion of Part (a).

(b) For $n = 17, 18, 19, 20$, we go back to the original condition (5.1). Observing that, for $n \geq 17$, we have $N \geq 1$, we see that (5.1) is satisfied for $17 \leq n \leq 20$ and every K satisfying $K \geq 19$. This establishes the assertion of Part (b). □

6. The cases where $8 \leq n \leq 16$.

In order to complete the proof of Theorem 1, it remains to estimate the function $M(n)$ in the range where $7 \leq n \leq 16$. For $8 \leq n \leq 16$, this is done in a series of lemmas in the present section. The case of the (2, 3, 7)-triangle group is considered from a somewhat different, more arithmetic, perspective in Section 7. Here we shall give complete details for Lemmas 16–18, while for the remaining results of this section we confine ourselves to listing relevant diagrams, leaving details of the construction to the interested reader.

LEMMA 16. *The triangle group* $\Delta(8)$ *has a subgroup of index* m *for every* $m \geq 240$.

Proof. The argument depends on three specific diagrams:
(i) $D(8)$: An 8-vertex $F(1)$ diagram with cusp-split (8^1), obtained by joining two copies of T, and then joining two 1-vertex diagrams to the right-hand blue triangle.
(ii) $D(9) = T * T * T'$: A 9-vertex $F(1)$ diagram with cusp-split $(1^1, 8^1)$.

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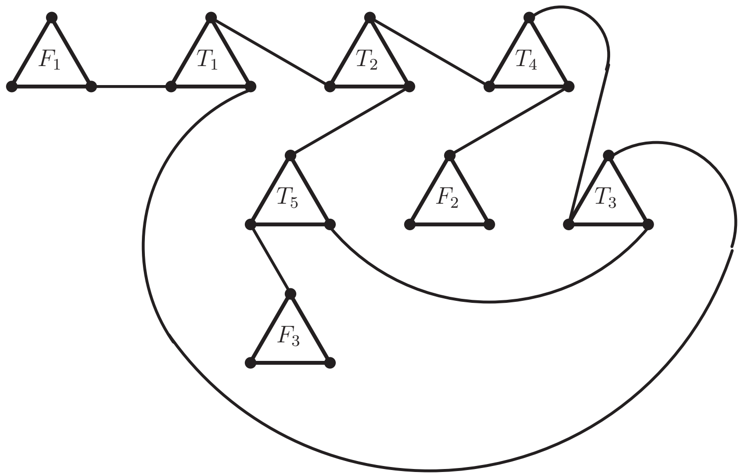

Figure 4. The diagram $D(24)$ for $\Delta(8)$.

(iii) $D(24)$: A 24-vertex $F(3)$ diagram with cusp-split (8^3), obtained as follows. Imagine five blue triangles T_1–T_5 connected by red edges in the following way: T_1 is connected to T_2 and T_3; T_2 is connected to T_1, T_4 and T_5; T_3 is connected to T_1, T_4 and T_5. This implies that T_4 is connected to T_2 and T_3, that T_5 is connected to T_2 and T_3, and leaves three free vertices (i.e. vertices not involved in a red edge), belonging to triangles T_1, T_4 and T_5, where we attach a free triangle each (denoted F_1–F_3); see Figure 4.

Now suppose that $m \geq 240$, and set $m = 8k + r$ with $0 \leq r \leq 7$. Then $k \geq 30$. Let $k - r = 3s + t$ with $0 \leq t \leq 2$. As $r \leq 7$, we have $k - r \geq 23$, so $s \geq 7$. Take s copies of $D(24)$, and combine them into a single diagram D by applying composition. By the composition lemma, D is a $24s$-vertex $F(s+2)$ diagram with cusp-split (8^{3s}). Again using composition, we can add t copies of $D(8)$ and r copies of $D(9)$, since $t + r \leq 9$ and D is $F(9)$. The result is a single diagram with m vertices and cusp-split

$$(1^r, 8^{3s+r+t}) = (1^r, 8^{(m-r)/8}),$$

which shows that $\Delta(8)$ has a subgroup of index m. □

Lemma 17. *The triangle group* $\Delta(9)$ *has a subgroup of index* m *for every* $m \geq 180$.

Proof. Here the argument depends on the following diagrams:[1]

(i) $D(9) = T * T * T$: A 9-vertex $F(2)$ diagram with cusp-split (9^1) (cf. Lemma 10).

(ii) $D(10)$: A 10-vertex $F(1)$ diagram with cusp-split $(1^1, 9^1)$ obtained from the join $T * T * T'$ by joining a 1-vertex diagram to the triangle in the middle.

(iii) $D(36)$: A 36-vertex $F(4)$ diagram with cusp-split (9^4), obtained in the following way. Let T_1–T_8 be eight blue triangles connected by red edges as follows: T_1 is connected to T_2 and T_3; T_2 is connected to T_1, T_4 and T_5; T_3 is connected to T_1, T_6 and T_8; T_4 is connected to T_2 and T_8; T_5 is connected to T_2, T_6 and T_7;

[1]The reader will observe that, in the interest of flexibility of notation, $D(9)$ here denotes a 9-vertex diagram different from the one used in the proof of Lemma 16; a similar abuse of notation occurring in other places without explicit mention.

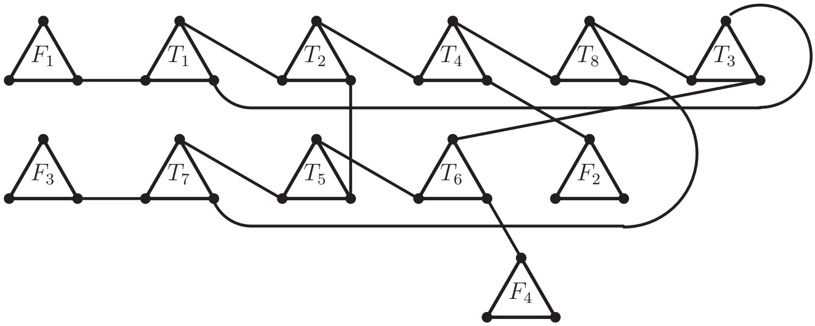

Figure 5. The diagram $D(36)$ for $\Delta(9)$.

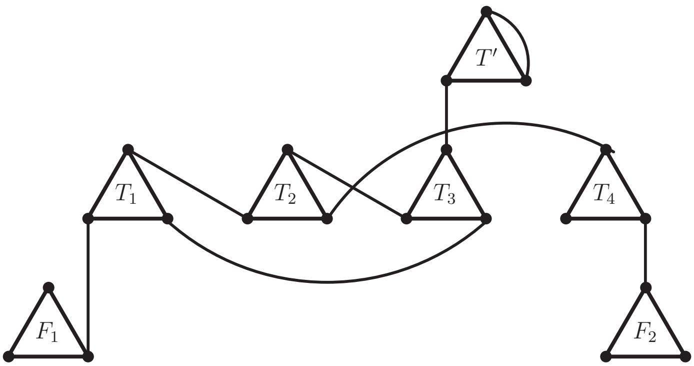

Figure 6. The diagram $D(21)$ for $\Delta(10)$.

and T_8 is connected to T_3, T_4 and T_7. This implies that T_6 is connected to T_3 and T_5; and that T_7 is connected to T_5 and T_8; and leaves four free vertices, one each at triangles T_1, T_4, T_6 and T_7, at each of which we attach a free triangle (denoted F_1–F_4) to obtain the required $F(4)$ diagram with cusp-split (9^4); see Figure 5.

Now suppose that $m \geq 180$, and let $m = 9k + r$ with $0 \leq r \leq 8$. Then $k \geq 20$. Composing three copies of $D(36)$, we obtain an $F(8)$ diagram D with cusp-split (9^{12}). Next, we compose D with $(k - r - 12)$ copies of $D(9)$ to get an $F(8)$ diagram D' with cusp-split (9^{k-r}); and finally, since $r \leq 8$, we can compose D' with r copies of $D(10)$ to arrive at a diagram with m vertices and cusp-split $(1^r, 9^k)$, which shows that $\Delta(9)$ has a subgroup of index m. □

LEMMA 18. *If* $m \geq 180$, *then* $\Delta(10)$ *has a subgroup of index* m.

Proof. The diagrams we use are as follows:

(i) $D(10)$: A 10-vertex $F(2)$ diagram with cusp-split (10^1), obtained from $T * T * T$ by joining a 1-vertex diagram to the middle triangle;

(ii) $D(21)$: A 21-vertex $F(2)$ diagram with cusp-split $(1^1, 10^2)$, obtained in the following way. Consider four blue triangles T_1–T_4 connected by red edges as follows: T_1 is connected to T_2 and T_3; T_2 is connected to T_1, T_3 and T_4. This leaves four free vertices, one each at triangles T_1 and T_3, and two at T_4, which we use to attach two free triangles, a copy of T', and a loop (at T_4); see Figure 6.

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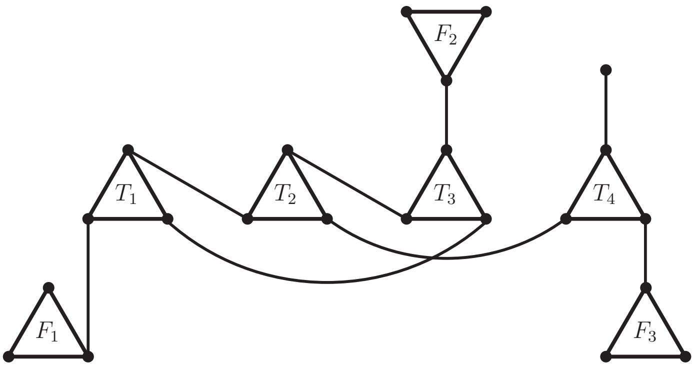

Figure 7. The diagram $D(22)$ for $\Delta(11)$.

Suppose that $m \geq 180$, and let $m = 10k + r$ with $0 \leq r \leq 9$. Then $k \geq 18$. For $r > 0$, we first compose r copies of $D(21)$ to obtain an $F(2)$ diagram D with $21r$ vertices and cusp-split $(1^r, 10^{2r})$. If we now compose D with $k - 2r$ copies of $D(10)$, we get a diagram with m vertices and cusp-split

$$(1^r, 10^{2r+(k-2r)}) = (1^r, 10^k),$$

which gives the required subgroup of index m in the triangle group $\Delta(10)$. □

LEMMA 19. *The triangle group $\Delta(11)$ has a subgroup of index m for every $m \geq 110$.*

Proof. The required diagrams are as follows:

(i) $D(11)$: An 11-vertex $F(1)$ diagram with cusp-split (11^1), obtained from $T * T * T$ by joining two 1-vertex diagrams to the second and third blue triangle.

(ii) $D(12)$: A 12-vertex $F(2)$ diagram with cusp-split $(1^1, 11^1)$, obtained from $T * T * T$ by joining a copy of T' to the triangle in the middle.

(iii) $D(22)$: A 22-vertex $F(3)$ diagram with cusp-split (11^2), obtained as described next. Take four blue triangles T_1–T_4 and connect them by red edges as follows: T_1 is connected to T_2 and T_3; T_2 is connected to T_1, T_3 and T_4. This leaves four free vertices, one each at triangles T_1 and T_3, and two at triangle T_4; and we attach a free triangle each at T_1 and T_3, and another free triangle plus a 1-vertex diagram at T_4; see Figure 7.

□

LEMMA 20. *If $m \geq 240$, then $\Delta(12)$ has a subgroup of index m.*

Proof. Here the required diagrams are as follows:

(i) $D(12)$: A 12-vertex $F(3)$ diagram with cusp-split (12^1), obtained by joining three free triangles to a blue triangle in the middle.

(ii) $D(13)$: A 13-vertex $F(1)$ diagram with cusp-split $(1^1, 12^1)$, obtained by joining a 1-vertex diagram to one of the two middle triangles in the diagram $T * T * T * T'$.

□

LEMMA 21. *The triangle group $\Delta(13)$ has a subgroup of index m for every $m \geq 143$.*

Proof. The required diagrams are as follows:

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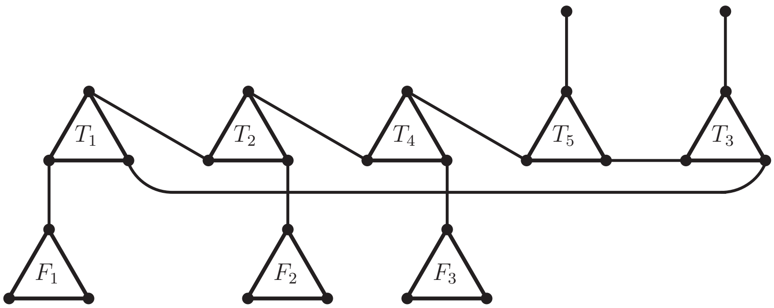

Figure 8. The diagram $D(26)$ for $\Delta(13)$.

(i) $D(13)$: A 13-vertex $F(2)$ diagram with cusp-split (13^1), obtained by joining a 1-vertex diagram to one of the two middle triangles in the diagram $T * T * T * T$.

(ii) $D(14)$: A 14-vertex $F(1)$ diagram with cusp-split $(1^1, 13^1)$, obtained by joining two 1-vertex diagrams to the middle triangles in the diagram $T * T * T * T'$.

(iii) $D(15)$: A 15-vertex $F(1)$ diagram with cusp-split $(1^2, 13^1)$, obtained by joining two copies of T' to the second and third triangle of a diagram of the form $T * T * T$.

(iv) $D(26)$: A 26-vertex $F(3)$ diagram with cusp-split (13^2), obtained in the following way. Take five blue triangles T_1–T_5, and connect them by red edges as follows: T_1 is connected to T_2 and T_3; T_2 is connected to T_1 and T_4; T_3 is connected to T_1 and T_5; T_4 is connected to T_2 and T_5. This leaves five free vertices, one on each triangle, where we attach two copies of a 1-vertex diagram (at T_3 and T_5), and three free triangles (at T_1, T_2, and T_4); cf. Figure 8.

(v) $D(27)$: A 27-vertex $F(3)$ diagram with cusp-split $(1^1, 13^2)$, obtained as described next. Take five blue triangles T_1–T_5, and connect them by red edges as follows: T_1 is connected by a red edge to T_2 and T_3; T_2 is connected to T_1 (as mentioned) and T_4; T_3 is connected to T_1 and T_5; T_4 is connected to T_2 and T_5; finally, T_5 is connected to T_3 and T_4 (as stated before). This leaves five free vertices, one for each triangle T_1–T_5. We attach free triangles at T_1, T_3 and T_4, a red loop at T_2 and a blue triangle with an internal edge at T_5 to obtain the desired diagram; see Figure 9.

□

Lemma 22. *If* $m \geq 154$, *then* $\Delta(14)$ *has a subgroup of index* m.

Proof. The required diagrams are as follows:

(i) $D(14)$: A 14-vertex $F(2)$ diagram with cusp-split (14^1), obtained by joining two 1-vertex diagrams to the two middle triangles of a diagram of the form $T * T * T * T$.

(ii) $D(15)$: A 15-vertex $F(2)$ diagram with cusp-split $(1^1, 14^1)$, obtained from a diagram of the form $T * T * T * T$ by joining a copy of T' at one of the two middle triangles.

(iii) $D(16)$: A 16-vertex $F(1)$ diagram with cusp-split $(1^2, 14^1)$, obtained as follows. Form a 9-vertex diagram D of the form $D = T * T * T'$, as well as a 6-vertex diagram $D' = T * T'$, join D and D', and join a 1-vertex diagram to the first triangle of D'.

□

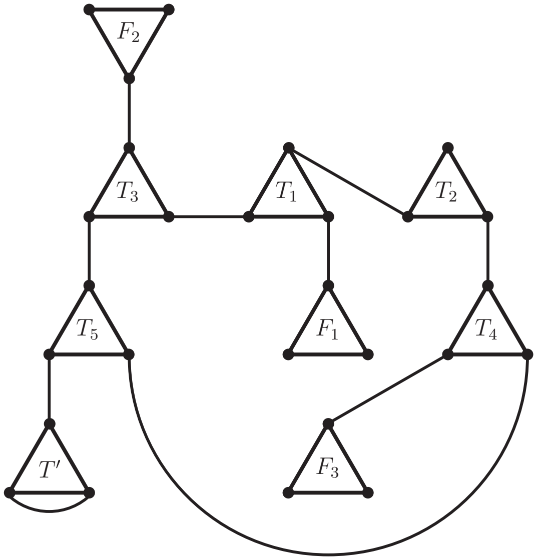

Figure 9. The diagram $D(27)$ for $\Delta(13)$.

LEMMA 23. *The triangle group* $\Delta(15)$ *has a subgroup of index m for each* $m \geq 210$.

Proof. The required diagrams are as follows:

(i) $D(15)$: A 15-vertex $F(3)$ diagram with cusp split (15^1), obtained by first building diagrams D, D' of the form $D = T * T * T$ and $D' = T * T$, and then joining D via its middle triangle to D'.

(ii) $D(16)$: A 16-vertex $F(2)$ diagram with cusp-split $(1^1, 15^1)$, obtained as follows. First produce a diagram D of the form $T * T * T$, and a diagram D' built by joining a 1-vertex diagram to a diagram of the form $T * T'$, then join D via the middle triangle to D', thus getting the desired 16-vertex diagram.

□

LEMMA 24. *If* $m \geq 128$, *then* $\Delta(16)$ *has a subgroup of index m.*

Proof. Suitable diagrams are as follows:

(i) $D(16)$: A 16-vertex $F(3)$ diagram with cusp-split (16^1), obtained by building an $F(2)$ diagram D of the form $T * T * T$, and an $F(1)$ diagram D' by starting from a diagram of the form $T * T$ and joining a 1-vertex diagram to the left triangle, then joining D via the middle triangle to the left triangle of D'.

(ii) $D(17)$: A 17-vertex $F(1)$ diagram with cusp-split $(1^1, 16^1)$, obtained by first building a diagram D of the form $T * T * T * T * T'$, and then joining 1-vertex diagrams to two of the three interior triangles of D.

(iii) $D(18)$: An 18-vertex $F(2)$ diagram with cusp-split $(1^2, 16^1)$, obtained by first building diagrams D, D' of the form $T * T * T$ and $T' * T * T'$, respectively, and then joining D and D' via their middle triangles. □

7. Some remarks on the genus formula for $\Delta(n)$**.** Let G be a subgroup of index m in $\Delta(n)$, and let D be a corresponding diagram over $\Delta(n)$. If D has $e(2)$ red loops, $e(3)$ blue loops and cusp-split $\{f(d)\}_{d|n}$, and if p denotes the genus of the Riemann surface associated with G, then these data are related by the formula

$$(n-6)m = 3ne(2) + 4ne(3) + 6\sum_{\substack{d|n \\ d<n}}(n-d)f(d) + 12n(p-1). \tag{7.1}$$

This is the genus formula for $\Delta(n)$; cf. [**4**].

A subgroup of finite index in $\Delta(n)$ is called *full* if it has a cusp-split consisting only of 1s and ns. Of course, if n is prime, then every finite-index subgroup is full. Our next result records some observations concerning the non-negative integers m, $e(2)$, $e(3)$, $f(1)$ and p associated with a full subgroup in $\Delta(n)$.

LEMMA 25. *For a full subgroup of index m in $\Delta(n)$, we have*

$$(n-6)m = 3ne(2) + 4ne(3) + 6(n-1)f(1) + 12n(p-1). \tag{7.2}$$

Moreover, we have

$$m \equiv e(2) \pmod 2, \tag{7.3}$$
$$m \equiv e(3) \pmod 3, \tag{7.4}$$
$$m \equiv f(1) \pmod n, \tag{7.5}$$

as well as

$$r + (n-1)t \equiv 0 \pmod 2, \tag{7.6}$$

where

$$m = e(2) + 2r = e(3) + 3s = f(1) + nt. \tag{7.7}$$

Furthermore, for n odd, we have $m \equiv e(2) \pmod 4$. Also, for n even, r and t have the same parity so that $e(2)$ is determined modulo 4 from the knowledge of m and $f(1)$ (as integers).

Proof. Equation (7.2) is just the genus formula (7.1) in the case when all green polygons have size 1 or n.

Consider a coset diagram D for the subgroup in question. Then the congruences (7.3)–(7.5) follow by looking at the partition of the m vertices of D affected by the red, blue and green structures, respectively.

If we use equation (7.7) to substitute for $e(2)$, $e(3)$ and $f(1)$ in (7.2), then we find, after division by $6n$, that

$$-2m + r + 2s + (n-1)t = 2(p-1),$$

from which we deduce the congruence (7.6). The remaining assertions are now immediate. □

REMARK. In certain cases, for instance if n is prime to 6, the congruences in Lemma 25 follow from Formula (7.2), so we can simply study the equation in these cases. In general, we have to consider the system of equation and congruences.

Our next result provides sufficient conditions for the existence of (non-negative) solutions to this system of equation and congruences.

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LEMMA 26.

(a) *If* $m > 6n - 1 + 18n/(n-6)$, *then the system of equation and congruences in Lemma* 25 *has a solution in non-negative integers* $e(2)$, $e(3)$, $f(1)$, p.

(b) *If* $n \geq 25$, *then there are non-negative solutions for all* $m \geq 6n$.

Proof. (a) Suppose that we have a (non-negative) solution $(e(2), e(3), f(1), p)$ to the system under consideration. If $e(2) \geq 4$, then we can decrease it by 4 and increase p by 1 to get another solution. In this way we can reduce $e(2)$ modulo 4, and ensure that $e(2) \leq 3$. Likewise, we may assume that $e(3) \leq 2$. Now suppose that we have a solution with $e(2) \leq 3$, $e(3) \leq 2$ and $f(1) \geq n$. If n is odd, we can decrease $f(1)$ by n, and increase p by $\frac{n-1}{2}$; if n is even and $e(2) \leq 1$, then we decrease $f(1)$ by n, increase $e(2)$ by 2 and increase p by $\frac{n-2}{2}$; if n is even and $e(2) \geq 2$, then we decrease $f(1)$ by n, decrease $e(2)$ by 2 and increase p by $\frac{n}{2}$. On readily checks, in each case we obtain a new solution meeting the bounds $e(2) \leq 3$ and $e(3) \leq 2$. Thus, we may further assume that $f(1) \leq n-1$.

Now choose non-negative integers $e(2)$, $e(3)$, $f(1)$, r and t such that $e(2) \leq 3$, $e(3) \leq 2$, $f(1) \leq n-1$ such that Congruence (7.6) and equation (7.7) are satisfied. Substituting for $e(2)$, $e(3)$ and $f(1)$ by means of equation (7.7), we find that expression

$$N := (n-6)m - 3ne(2) - 4ne(3) - 6(n-1)f(1)$$

is a multiple of $12n$, so we only have to ensure that p is non-negative, i.e. $N > -24n$, in order to get a solution to the system. However, by our assumptions on $e(2)$, $e(3)$, $f(1)$ and m, we have

$$\begin{aligned} N &\geq (n-6)m - 9n - 8n - 6(n-1)^2 \\ &> (n-6)\big(6n - 1 + 18n/(n-6)\big) - 17n - 6(n-1)^2 = -24n, \end{aligned}$$

establishing Part (a).

(b) Suppose that $n \geq 25$. Then Part (a) gives solutions for all $m \geq 7n-1$. Now let $m = 6n + k$ with $0 \leq k \leq n-2$. Since $m \equiv f(1) \pmod{n}$, we may take $f(1) = k$. Defining N as before, we now have

$$\begin{aligned} N &\geq (n-6)m - 9n - 8n - 6k(n-1) \\ &= (n-6)(6n+k) - 17n - 6k(n-1). \end{aligned}$$

As before, a sufficient condition for the existence of a solution for such m is that $N > -24n$, from which we obtain the sufficient condition

$$(n-6)(6n+k) - 17n - 6k(n-1) > -24n,$$

or, equivalently,

$$6n - 29 > 5k.$$

However, since $k \leq n-2$, the last condition is satisfied, provided that

$$5(n-2) < 6n - 29,$$

or $n > 19$. □

COROLLARY 27. *We have* $M(7) = 168$.

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Proof. By Part (a) of Lemma 26, we have non-negative solutions to equation (7.2) for all $m > 167$. By [**9**, Theorem 4.1], this implies existence of a subgroup of index m in $\Delta(7)$ for all $m \geq 168$. On the other hand, it is easy to check that equation (7.2) does not have a solution for $m = 167$; thus, a fortiori, there is no subgroup of that index.[2] □

REMARK. The genus formula (7.1) can also be used to obtain non-trivial lower bounds for $M(n)$ in certain cases. For instance, one shows by an argument similar to the proof of Proposition 3 that, for n a prime, $\Delta(n)$ has no subgroup of index $Kn - 1$ for all integers K with $1 \leq K < \frac{6n-23}{p-6}$. For large prime n, this only yields the bound $M(n) \geq 6n$, which was already observed in Corollary 4; but for $p = 7$, 11, 13 and 17, we get better results: $M(7) \geq 126$, $M(11) \geq 88$, $M(13) \geq 91$ and $M(17) \geq 119$. One also finds in this way that $M(9) \geq 18$, $M(25) \geq 25$ and $M(49) \geq 49$. For $n = q^r$ a prime power with $r = 2$ and $q > 7$, or for $q \geq 2$ and exponents $r > 2$, this method fails however.

8. Determination of $M(10)$.

We begin by observing that the restrictions on images σ, τ of generators x, y of $\Delta(n)$ coming from the modular relations $x^2 = y^3 = 1$ plus the requirement that $\langle \sigma, \tau \rangle$ be transitive are rather severe for diagrams with few vertices. For instance, for two vertices, we must have $\tau = 1$ and $\sigma = (1, 2)$ (two vertices with blue loops attached to them, connected by a red edge). For three, four and five vertices, τ must have a single 3-cycle, with any extra vertices linked in via σ. A little analysis shows that, for $m \leq 5$, the only possibilities are as follows, up to the labelling of vertices:

m = 1: $\sigma = \tau = \tau\sigma = 1$;

m = 2: $\sigma = \tau\sigma = (1, 2)$, $\tau = 1$;

m = 3: $\sigma = 1$, $\tau = \tau\sigma = (1, 2, 3)$ or
$\sigma = (1, 2)$, $\tau = (1, 2, 3)$, $\tau\sigma = (1, 3)$;

m = 4: $\sigma = (3, 4)$, $\tau = (1, 2, 3)$, $\tau\sigma = (1, 2, 3, 4)$ or
$\sigma = (1, 2)(3, 4)$, $\tau = (1, 2, 3)$, $\tau\sigma = (1, 3, 4)$;

m = 5: $\sigma = (1, 4)(3, 5)$, $\tau = (1, 2, 3)$, $\tau\sigma = (1, 4, 2, 3, 5)$.

We are now ready to establish the following refinement of Lemma 18.

PROPOSITION 28. *The triangle group $\Delta(10)$ has a subgroup of index m for each $m \geq 1$, except for $m = 4, 8$ and 9; in particular, we have $M(10) = 10$.*

Proof. We begin by listing some diagrams, which will be used to build families of subgroups:

(i) $D(6)$: A 6-vertex $F(1)$ diagram with cusp-split $(1^1, 5^1)$, obtained by joining a copy of T to a copy of T'.

(ii) $D(10)$: A 10-vertex $F(2)$ diagram with cusp-split (10^1), obtained by joining a 1-vertex diagram to the middle triangle in a copy of $T * T * T$.

(iii) $D(12)$: A 12-vertex $F(2)$ diagram with cusp-split $(2^1, 10^1)$, obtained from a 6-vertex diagram consisting of two blue triangles with a red double bond by joining two copies of T.

(iv) $D(15)$: A 15-vertex $F(1)$ diagram with cusp-split $(5^1, 10^1)$, obtained by joining a copy of $T * T$ to a 9-vertex diagram built from a copy of $T * T * T$ by replacing

[2] We note that Corollary 27 also follows immediately from results in [**1**].

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the top red loops of the first and the right-hand loop of the last triangle by a red edge connecting these triangles.

(v) $D(21)$: A 21-vertex $F(2)$ diagram with cusp-split $(1^1, 10^2)$; see Figure 6.

Now let $m \geq 1$, and write $m = 10k + r$ with $0 \leq r \leq 9$. In each case, we shall give a collection of diagrams that can be combined by composition to obtain the required m-vertex diagram, leaving out only certain small numbers m, which need to be handled separately.

$r = 0$. Here we have $k \geq 1$, so we can compose k copies of $D(10)$ to obtain the desired diagram.

$r = 1$. Suppose that $k \geq 2$. Then we compose one copy of $D(21)$ and $(k - 2)$ copies of $D(10)$; this leaves out the cases where $m = 1$ or $m = 11$.

$r = 2$. For $k \geq 1$, we may compose $(k - 1)$ copies of $D(10)$ and one copy of $D(12)$; this leaves out the case where $m = 2$.

$r = 3$. For $k \geq 3$, we may compose one copy each of $D(12)$ and $D(21)$, and $(k - 3)$ copies of $D(10)$; this leaves out the cases where $m = 3, 13, 23$.

$r = 4$. Suppose that $k \geq 2$. Then we compose two copies of $D(12)$ and $(k - 2)$ copies of $D(10)$ to get the desired $(10k + 4)$-vertex diagram; this leaves out the cases where $m = 4, 14$.

$r = 5$. For $k \geq 1$, we can compose one copy of $D(15)$ with $(k - 1)$ copies of $D(10)$ to get the required diagram; this leaves out the case where $m = 5$.

$r = 6$. Here we compose one copy of $D(6)$ with k copies of $D(10)$.

$r = 7$. Let $k \geq 2$. Then we may compose one copy each of $D(12)$ and $D(15)$, and $(k - 2)$ copies of $D(10)$; this leaves out the cases where $m = 7, 17$.

$r = 8$. For $k \geq 1$, we can compose one copy each of $D(6)$ and $D(12)$, and $(k - 1)$ copies of $D(10)$; this leaves out the case where $m = 8$.

$r = 9$. Let $k \geq 3$. Then we may compose two copies of $D(12)$, one copy of $D(15)$ and $(k - 3)$ copies of $D(10)$ to get the desired $(10k + 9)$-vertex diagram; this leaves out the cases where $m = 9, 19, 29$.

To complete the proof, it remains to produce diagrams having cusp-splits compatible with $\Delta(10)$ for $m = 1, 2, 3, 5, 7, 11, 13, 14, 17, 19, 23, 29$, and to show that there are no such diagrams with $m = 4, 8, 9$. The first task is routine (though fairly tedious for the later values), and is left as an exercise to the reader.

From our survey of subgroups of small index, we already know that the 4-vertex diagrams over Γ have cusp-splits (4^1) and $(1^1, 3^1)$, neither of which is compatible with $\Delta(10)$. This shows that $\Delta(10)$ does not have a subgroup of index 4.

Next, the genus formula for Γ shows that an 8-vertex diagram can have at most two green cycles. However, since $8 \nmid 10$, and as 8 cannot be expressed as the sum of two divisors of 10, no 8-vertex diagram exists over $\Delta(10)$; thus, there is no subgroup of index 8 in $\Delta(10)$.

The case where $m = 9$ is somewhat more difficult. By the genus formula, a 9-vertex diagram can have at most three green cycles, and the only cusp-split compatible with $\Delta(10)$ and this restriction is $(2^2, 5^1)$. Moreover, again by the genus formula, such a diagram should have three blue triangles, four red edges and a red loop, and it is not hard to convince oneself that such a diagram does not exist. □

REMARK. By arguments similar to those in the proof of Proposition 28, one can also show that $\Delta(12)$ has a subgroup of index m for each $m \geq 1$, except for $m = 5$ and

11; in particular, $M(12) = 12$. The main ingredients of a proof of this are a 6-vertex $F(2)$ diagram, and $F(1)$ diagrams with 3 and 4 vertices.

9. Proof of Theorem 2. Combining Lemmas 16 and 17 and 19–24 with Corollaries 4 and 27, and Propositions 9–28, we obtain the assertion of Theorem 1.

A first useful step in proving Theorem 2 consists in the following observation.

LEMMA 29. *Let n and n' be integers such that $n, n' \geq 7$ and $n \mid n'$. In this situation, if $\Delta(n)$ has a subgroup of index m for some positive integer m, then so has $\Delta(n')$.*

Proof. If $\Delta(n)$ has a subgroup of index m, then there exists an m-vertex diagram over $\Delta(n)$, whose all green cycles have lengths dividing n. Since n divides n', the same diagram may also be viewed as a diagram over $\Delta(n')$, showing that $\Delta(n')$ also has a subgroup of index m. Alternatively, the assertion of the lemma also follows from the fact that $\Delta(n)$ is a homomorphic image of $\Delta(n')$. □

With Proposition 28 and Lemma 29 in hand, we can now proceed to the proof of Theorem 2.

First suppose that $\Delta(n)$ is replete. Then, in particular, it has a subgroup of index 2. From the list before Proposition 28, the corresponding diagram must have a green cycle of length 2, so $2 \mid n$. Also, $\Delta(n)$ has a subgroup of index 5, and our list shows that the corresponding diagram must have a green 5-cycle, implying that $5 \mid n$. Combining these results, we see that $10 \mid n$. Moreover, $\Delta(n)$ must have a subgroup of index 4. From the list, we see that in this case the corresponding diagram must have either a green 3-cycle or a green 4-cycle so that n is either divisible by 4 or 3, and hence by 20 or 30.

Suppose conversely that n is divisible by 20 or 30. We want to show that in this case $\Delta(n)$ is replete. By Lemma 29, it suffices to show that $\Delta(20)$ and $\Delta(30)$ are replete. Proposition 28 in conjunction with Lemma 29 shows that there are subgroups of index m in $\Delta(20)$ and $\Delta(30)$ for every index m, except possibly for $m = 4$, 8 or 9. We already know that there exists a 4-vertex diagram with cusp-split (4^1), which is compatible with $\Delta(20)$, and a 4-vertex diagram with cusp-split $(1^1, 3^1)$, which is compatible with $\Delta(30)$. Hence, both $\Delta(20)$ and $\Delta(30)$ contain a subgroup of index 4. Moreover, it is easy to find 8-vertex diagrams with cusp-splits (4^2) and $(2^1, 6^1)$, and 9-vertex diagrams with cusp-splits $(4^1, 5^1)$ and $(3^1, 6^1)$. Thus, $\Delta(20)$ and $\Delta(30)$ are replete, finishing the proof of Theorem 2.

We conclude this paper with the following.

PROBLEM 1. Determine the exact value of $M(n)$ for all n in the range $8 \leq n \leq 52$.

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