8

Supersymmetric strings and T-duality

8.1 T-duality of supersymmetric strings

We noticed in section 7.5, when considering the low energy spectrum of the type II superstrings compactified on tori, that there is an equivalence between them. We saw much the same things happen for the heterotic strings in section 7.4 too. This is of course T-duality, as we should examine it further here and check that it is the familiar exact equivalence. Just as in the case of bosonic strings, doing this when there are open string sectors present will uncover D-branes of various dimensions.

8.1.1 T-duality of type II superstrings

T-duality on the closed oriented Type II theories has a somewhat more interesting effect than in the bosonic case[12, 8]. Consider compactifying a single coordinate X^9, of radius R. In the $R \to \infty$ limit the momenta are $p^9_{\mathrm{R}} = p^9_{\mathrm{L}}$, while in the $R \to 0$ limit $p^9_{\mathrm{R}} = -p^9_{\mathrm{L}}$. Both theories are $SO(9,1)$ invariant but under *different* $SO(9,1)$s. T-duality, as a right-handed parity transformation (see (4.18)), reverses the sign of the right-moving $X^9(\bar{z})$; therefore by superconformal invariance it does so on $\widetilde{\psi}^9(\bar{z})$. Separate the Lorentz generators into their left- and right-moving parts $M^{\mu\nu} + \widetilde{M}^{\mu\nu}$. Duality reverses all terms in $\widetilde{M}^{\mu 9}$, so the $\mu 9$ Lorentz generators of the T-dual theory are $M^{\mu 9} - \widetilde{M}^{\mu 9}$. In particular this reverses the sign of the helicity $\tilde{s}_4$ and so switches the chirality on the right-moving side. If one starts in the IIA theory, with opposite chiralities, the $R \to 0$ theory has the same chirality on both sides and is the $R \to \infty$ limit of the IIB theory, and vice versa. In short, T-duality, as a one-sided spacetime parity operation, reverses the relative chiralities of the right- and left-moving ground states. The same is true if one dualises on any odd number of dimensions, whilst dualising on an even number returns the original type II theory.

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Since the IIA and IIB theories have different R–R fields, T_9 duality must transform one set into the other. The action of duality on the spin fields is of the form

$$S_\alpha(z) \to S_\alpha(z), \qquad \tilde{S}_\alpha(\bar{z}) \to P_9 \tilde{S}_\alpha(\bar{z}) \tag{8.1}$$

for some matrix P_9, the parity transformation (nine-reflection) on the spinors. In order for this to be consistent with the action $\tilde{\psi}^9 \to -\tilde{\psi}^9$, P_9 must anticommute with Γ^9 and commute with the remaining Γ^μ. Thus $P_9 = \Gamma^9\Gamma^{11}$ (the phase of P_9 is determined, up to sign, by hermiticity of the spin field). Now consider the effect on the R–R vertex operators (7.27). The Γ^{11} just contributes a sign, because the spin fields have definite chirality. Then by the Γ-matrix identity (7.28), the effect is to add a 9-index to G if none is present, or to remove one if it is. The effect on the potential C ($G = dC$) is the same. Take as an example the type IIA vector C_μ. The component C_9 maps to the IIB scalar C, while the $\mu \neq 9$ components map to $C_{\mu 9}$. The remaining components of $C_{\mu\nu}$ come from $C_{\mu\nu 9}$, and so on.

Of course, these relations should be translated into rules for T-dualising the spacetime fields in the supergravity actions (7.41) and (7.42). The NS–NS sector fields' transformations are the same as those shown in equations (5.4),(5.6), while for the R–R potentials[77]:

$$\begin{aligned}
\tilde{C}^{(n)}_{\mu\cdots\nu\alpha 9} &= C^{(n-1)}_{\mu\cdots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\cdots\nu|9}G_{|\alpha]9}}{G_{99}} \\
\tilde{C}^{(n)}_{\mu\cdots\nu\alpha\beta} &= C^{(n+1)}_{\mu\cdots\nu\alpha\beta 9} + nC^{(n-1)}_{[\mu\cdots\nu\alpha}B_{\beta]9} + n(n-1)\frac{C^{(n-1)}_{[\mu\cdots\nu|9}B_{|\alpha|9}G_{|\beta]9}}{G_{99}}.
\end{aligned} \tag{8.2}$$

8.1.2 T-duality of type I superstrings

Just as in the case of the bosonic string, the action of T-duality in the open and unoriented open superstring theory produces D-branes and orientifold planes. Having done it once (say on X^9 with radius R), we get a T_9-dual theory on the line interval $S^1/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as the reflection $X^9 \to -X^9$. The S^1 has radius $R' = \alpha'/R$). There are 16 D8-branes and their mirror images (coming from the 16 D9-branes), together with two orientifold O8-planes located at $X^9 = 0, \pi R'$. This is called the 'type I′' theory (and sometimes the 'type IA' theory, and then the usual open string is 'type IB'), about which we will have more to say later as well.

Starting with the type IB theory, i.e. 16 D9-branes and one O9-plane, we can carry this out n times on n directions, giving us 16 D$(9-n)$ and their

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mirror images through 2^n O$(9-n)$-planes arranged on the hypercube of fixed points of $T^n/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts as a reflection in the n directions. If n is odd, the bulk theory away from the planes and branes is type IIA string theory, while we are back in type IIB otherwise.

Let us focus here on a single D-brane, taking a limit in which the other D-branes and the O-planes are very far away and can be ignored. Away from the D-brane, only closed strings propagate. The local physics is that of the type II theory, with two gravitinos. This is true even though we began with the unoriented type I theory which has only a single gravitino. The point is that the closed string begins with two gravitinos, one with the spacetime supersymmetry on the right-moving side of the world-sheet and one on the left. The orientation projection of the type I theory leaves one linear combination of these. However in the T-dual theory, the orientation projection does not constrain the local state of the string, but relates it to the state of the (distant) image gravitino. Locally there are two independent gravitinos, with equal chiralities if n, (the number of dimensions on which we dualised) is even and opposite if n is odd.

This is all summarised nicely by saying that while the type I string theory comes from projecting the type IIB theory by Ω, the T-dual string theories come from projecting type II string theory compactified on the torus T^n by $\Omega\prod_m[R_m(-1)^F]$, where the product over m is over all the n directions, and R_m is a reflection in the mth direction. This is indeed a symmetry of the theory and hence a good symmetry with which to project. So we have that T-duality takes the orientifold groups into one another:

$$\{\Omega\} \leftrightarrow \{1, \Omega\textstyle\prod_m[R_m(-1)^F]\}. \tag{8.3}$$

This is a rather trivial example of an orientifold group, since it takes type II strings on the torus T^n and simply gives a theory which is simply related to type I string theory on T^n by n T-dualities. Nevertheless, it is illustrative of the general constructions of orientifold backgrounds made by using more complicated orientifold groups. This is a useful piece of technology for constructing string backgrounds with interesting gauge groups, with fewer symmetries, as a starting point for phenomenological applications.

8.1.3 T-duality for the heterotic strings

As we noticed in section 7.4, there is a T-duality equivalence between the heterotic strings once we compactify on a circle. Let us uncover it carefully.

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We can begin by compactifying the $SO(32)$ string on a circle of radius R, with Wilson line:

$$A_9^i = \frac{1}{2\pi R}\mathrm{diag}\left\{\frac{1}{2},\dots\frac{1}{2},0,\dots,0\right\}, \tag{8.4}$$

with eight $\frac{1}{2}$s and eight 0s breaking down the gauge group to $SO(16) \times SO(16)$. We can compute the mass spectrum of the nine dimensional theory which results from this reduction, in the presence of the Wilson line. This is no harder than the computations which we did in chapter 4. The Wilson line simply shifts the contribution to the spectrum coming from the p^i_{L} momenta. We can focus on the sector which is uncharged under the gauge group, i.e. we switch off the p^i_{L}. The mass formula is:

$$p_{\substack{\mathrm{L}\\\mathrm{R}}} = \frac{(n+2m)}{R} \pm \frac{2mR}{\alpha'},$$

where we see that the allowed windings (coming in units of two) are controlled by the integer m, and the momenta are controlled by m and n in the combination $n+2m$.

We could instead have started from the $E_8 \times E_8$ string on a circle of radius R', with Wilson line

$$A_9^i = \frac{1}{2\pi R'}\mathrm{diag}\{1,0\dots0,1,0,\dots,0\}, \tag{8.5}$$

again in two equal blocks of eight. This also breaks down the gauge group to $SO(16)\times SO(16)$. A computation of the spectrum of the neutral states gives:

$$p'_{\substack{\mathrm{L}\\\mathrm{R}}} = \frac{(n'+2m')}{R'} \pm \frac{2m'R'}{\alpha'},$$

for integers n' and m'. We see that if we exchange $n+2m$ with m' and m with $n'+2m'$ then the spectrum is invariant if we do the right handed parity identification $p_{\mathrm{L}} \leftrightarrow p'_{\mathrm{L}}$, $p_{\mathrm{R}} \leftrightarrow -p'_{\mathrm{R}}$, provided that the circles' radii are inversely related as $R' = \alpha'/(2R)$.

We shall see that this relation will result in some very remarkable connections between non-perturbative string vacua much later, in chapters 12 and 16.

8.2 D-branes as BPS solitons

Let us return to the type II strings, and the D-branes which we can place in them. While there is type II string theory in the bulk (i.e. away from the

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branes and orientifolds), notice that the open string boundary conditions are invariant under only one supersymmetry. In the original type I theory, the left-moving world-sheet current for spacetime supersymmetry $j_\alpha(z)$ flows into the boundary and the right-moving current $\tilde{j}_\alpha(\bar{z})$ flows out, so only the total charge $Q_\alpha + \tilde{Q}_\alpha$ of the left- and right-movers is conserved. Under T-duality this becomes

$$Q_\alpha + \left(\prod_m P_m\right) \tilde{Q}_\alpha, \tag{8.6}$$

where the product of reflections P_m runs over all the dualised dimensions, that is, over all directions orthogonal to the D-brane. Closed strings couple to open, so the general amplitude has only one linearly realised supersymmetry. That is, the vacuum without D-branes is invariant under $N = 2$ supersymmetry, but the state containing the D-brane is invariant under only $N = 1$: *it is a BPS state*[265, 93].

BPS states must carry conserved charges. In the present case there is only one set of charges with the correct Lorentz properties, namely the antisymmetric R–R charges. The world volume of a p-brane naturally couples to a $(p+1)$-form potential $C_{(p+1)}$, which has a $(p+2)$-form field strength $G_{(p+2)}$. This identification can also be made from the g_s^{-1} behaviour of the D-brane tension: this is the behaviour of an R–R soliton[94, 96] as will be developed further later.

The IIA theory has Dp-branes for $p = 0$, 2, 4, 6, and 8. The vertex operators (7.27) describe field strengths of all even ranks from zero to ten. The n-form and $(10 - n)$-form field strengths are Hodge dual to one another*, so a p-brane and $(6 - p)$-brane are sources for the same field, but one magnetic and one electric. The field equation for the ten-form field strength allows no propagating states, but the field can still have a physically significant energy density [265, 97, 98].

The IIB theory has Dp-branes for $p = -1$, 1, 3, 5, 7, and 9. The vertex operators (7.27) describe field strengths of all odd ranks from one to nine, appropriate to couple to all but the nine-brane. The nine-brane does couple to a non-trivial *potential,* as we will see below.

A (-1)-brane is a Dirichlet instanton, defined by Dirichlet conditions in the time direction as well as all spatial directions[99]. Of course, it is not clear that T-duality in the time direction has any meaning, but one can argue for the presence of (-1)-branes as follows. Given zero-branes in the IIA theory, there should be virtual zero-brane world-lines that wind in a purely spatial direction. Such world-lines are required by quantum mechanics, but note that they are essentially instantons, being localised in time. A T-duality in the winding direction then gives a (-1)-brane. One

* This works at the level of vertex operators via a Γ-matrix identity.

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of the first clues to the relevance of D-branes[25], was the observation that D-instantons, having action g_s^{-1}, would contribute effects of order e^{-1/g_s} as expected from the behaviour of large orders of string perturbation theory[100].

The D-brane, unlike the fundamental string, carries R–R charge. This is consistent with the fact that they are BPS states, and so there must be a conserved charge. A more careful argument, involving the R–R vertex operators, can be used to show that they *must* couple thus, and furthermore that fundamental strings cannot carry R–R charges (see also insert 8.1).

8.3 The D-brane charge and tension

The discussion of section 5.3 will supply us with the world-volume action (5.21) for the bosonic excitations of the D-branes in this supersymmetric context. Now that we have seen that Dp-branes are BPS states, and couple to R–R sector $(p+1)$-form potential, we ought to compute the values of their charges and tensions.

Focusing on the R–R sector for now, supplementing the spacetime supergravity action with the D-brane action we must have at least (recall that the dilaton will not appear here, and also that we cannot write this for $p=3$):

$$S = -\frac{1}{2\kappa_0^2}\int G_{(p+2)}{}^{*}G_{(p+2)} + \mu_p \int_{\mathcal{M}_{p+1}} C_{(p+1)}, \tag{8.7}$$

where μ_p is the charge of the Dp-brane under the $(p+1)$-form $C_{(p+1)}$. $\mathcal{M}_{p+1}$ is the world-volume of the Dp-brane.

Now the same vacuum cylinder diagram as in the bosonic string, as we did in chapter 6. With the fermionic sectors, our trace must include a sum over the NS and R sectors, and furthermore must include the GSO projection onto even fermion number. Formally, therefore, the amplitude looks like[265]:

$$\mathcal{A} = \int_0^\infty \frac{dt}{2t}\mathrm{Tr}_{\mathrm{NS+R}}\left\{\frac{1+(-1)^F}{2}e^{-2\pi t L_0}\right\}. \tag{8.8}$$

Performing the traces over the open superstring spectrum gives

$$\begin{aligned}\mathcal{A} = 2V_{p+1}\int \frac{dt}{2t}\,(8\pi^2\alpha' t)^{-(p+1)/2}e^{-t\frac{Y^2}{2\pi\alpha'}}\\ \frac{1}{2}f_1^{-8}(q)\left\{-f_2(q)^8+f_3(q)^8-f_4(q)^8\right\},\end{aligned} \tag{8.9}$$

where again $q=e^{-2\pi t}$, and we are using the definitions given in chapter 4, when we computed partition functions of various sorts. Insert 14.1, p. 327, uncovers more of the properties of the f-functions.

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Insert 8.1. A summary of forms and branes

Common to both type IIA and IIB are the NS–NS sector fields

$$\Phi, G_{\mu\nu}, B_{\mu\nu}.$$

The latter is a rank two antisymmetric tensor potential, and we have seen that the fundamental closed string couples to it electrically by the coupling

$$\nu_1 \int_{\mathcal{M}_2} B_{(2)},$$

where $\nu_1 = (2\pi\alpha')^{-1}$, $\mathcal{M}_2$ is the world sheet, with coordinates ξ^a, $a = 1, 1$. $B_{(2)} = B_{ab} d\xi^a d\xi^b$, and B_{ab} is the pullback of $B_{\mu\nu}$ via (5.8). By ten dimensional Hodge duality, we can also construct a six form potential $B_{(6)}$, by the relation $dB_{(6)} = *dB_{(2)}$. There is a natural electric coupling $\nu_5 \int_{\mathcal{M}_6} B_{(6)}$, to the world-volume $\mathcal{M}_6$ of a five dimensional extended object. This NS–NS charged object, which is commonly called the 'NS5-brane' is the magnetic dual of the fundamental string[72, 73]. It is in fact, in the ten dimensional sense, the monopole of the $U(1)$ associated to $B_{(2)}$. We shall be forced to discuss it by strong coupling considerations in section 12.3.

The string theory has other potentials, from the R–R sector:

$$\begin{aligned} \text{type IIA}: &\quad C_{(1)}, C_{(3)}, C_{(5)}, C_{(7)} \\ \text{type IIB}: &\quad C_{(0)}, C_{(2)}, C_{(4)}, C_{(6)}, C_{(8)} \end{aligned}$$

where in each case the last two are Hodge duals of the first two, and $C_{(4)}$ is self dual. (A p-form potential and a rank q-form potential are Hodge dual to one another in D dimensions if $p + q = D - 2$.)

Dp-branes are the basic p-dimensional extended sources which couple to all of these via an electric coupling of the form:

$$\mu_p \int_{\mathcal{M}_{p+1}} C_{(p+1)}$$

to their $p+1$-dimensional world volumes $\mathcal{M}_{p+1}$.

The three terms in the braces come from the open string R sector with $\frac{1}{2}$ in the trace, from the NS sector with $\frac{1}{2}$ in the trace, and the NS sector with $\frac{1}{2}(-1)^F$ in the trace; the R sector with $\frac{1}{2}(-1)^F$ gives no net contribution. In fact, these three terms sum to zero by Jacobi's abstruse identity (*'aequatio identico satis abstrusa'*, see insert 14.2, p. 328) as they

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ought to since the open string spectrum is supersymmetric, and we are computing a vacuum diagram.

What does this result mean? Recall that this vacuum diagram also represents the exchange of closed strings between two identical branes. the result $\mathcal{A} = 0$ is simply a restatement of the fact that D-branes are BPS states: the net forces from the NS–NS and R–R exchanges cancel. $\mathcal{A} = 0$ has a useful structure, nonetheless, and we can learn more by identifying the separate NS–NS and R–R pieces. This is easy, if we look at the diagram afresh in terms of closed string: In the terms with $(-1)^F$, the world-sheet fermions are *periodic* around the cylinder thus correspond to R–R exchange. Meanwhile the terms without $(-1)^F$ have *antiperiodic* fermions and are therefore NS–NS exchange.

Obtaining the $t \to 0$ behaviour as before (use the limits in insert 6.2 (p. 145)) gives

$$\begin{aligned}\mathcal{A}_{\rm NS} \ = \ -\mathcal{A}_{\rm R} &\sim \frac{1}{2}V_{p+1}\int\frac{dt}{t}(2\pi t)^{-(p+1)/2}(t/2\pi\alpha')^4 e^{-t\frac{Y^2}{8\pi^2\alpha'^2}}\\ &= V_{p+1}2\pi(4\pi^2\alpha')^{3-p}G_{9-p}(Y^2).\end{aligned} \tag{8.10}$$

Comparing with field theory calculations runs just as it did in chapter 6, with the result[265]:

$$2\kappa_0^2\mu_p^2 = 2\kappa^2\tau_p^2 = 2\pi(4\pi^2\alpha')^{3-p}. \tag{8.11}$$

Finally, using the explicit expression (7.44) for κ in terms of string theory quantities, we get an extremely simple form for the charge:

$$\mu_p = (2\pi)^{-p}\alpha'^{-\frac{(p+1)}{2}}, \quad \text{and} \quad \tau_p = g_{\rm s}^{-1}\mu_p. \tag{8.12}$$

(For consistency with the discussion in the bosonic case, we shall still use the symbol T_p to mean $\tau_p g_{\rm s}$, in situations where we write the action with the dilaton present. It will be understood then that $e^{-\Phi}$ contains the required factor of $g_{\rm s}^{-1}$.)

It is worth updating our bosonic formula (5.27) for the coupling of the Yang–Mills theory which appears on the world-volume of Dp-branes with our superstring result above, to give:

$$g_{{\rm YM},p}^2 = \tau_p^{-1}(2\pi\alpha')^{-2} = (2\pi)^{p-2}\alpha'^{(p-3)/2}, \tag{8.13}$$

a formula we will use a lot in what is to follow.

Note that our formula for the tension (8.12) gives for the D1-brane

$$\tau_1 = \frac{1}{2\pi\alpha' g_{\rm s}}, \tag{8.14}$$

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which sets the ratios of the tension of the fundamental string, $\tau_1^{\mathrm{F}} \equiv T = (2\pi\alpha')^{-1}$, and the D–string to be simply the string coupling g_{s}. This is a very elegant normalisation and is quite natural.

D-branes that are not parallel feel a net force since the cancellation is no longer exact. In the extreme case, where one of the D-branes is rotated by π, the coupling to the dilaton and graviton is unchanged but the coupling to the R–R tensor is reversed in sign. So the two terms in the cylinder amplitude add, instead of cancelling, as Jacobi cannot help us. The result is:

$$\mathcal{A} = V_{p+1} \int \frac{dt}{t} (2\pi t)^{-(p+1)/2} e^{-t(Y^2 - 2\pi\alpha')/8\pi^2\alpha'^2} f(t), \tag{8.15}$$

where $f(t)$ approaches zero as $t \to 0$. Differentiating this with respect to Y to extract the force per unit world-volume, we get

$$F(Y) = Y \int \frac{dt}{t} (2\pi t)^{-(p+3)/2} e^{-t(Y^2 - 2\pi\alpha')/8\pi^2\alpha'^2} f(t). \tag{8.16}$$

The point to notice here is that the force diverges as $Y^2 \to 2\pi\alpha'$. This is significant. One would expect a divergence, of course, since the two oppositely charged objects are on their way to annihilating[101]. The interesting feature it that the divergence begins when their separation is of order the string length. This is where the physics of light fundamental strings stretching between the two branes begins to take over. Notice that the argument of the exponential is tU^2, where $U = Y/(2\alpha')$ is the energy of the lightest open string connecting the branes. A scale like U will appear again, as it is a useful guide to new variables to D-brane physics at 'substringy' distances[102, 103, 104] in the limit where α' and Y go to zero.

8.4 The orientifold charge and tension

Orientifold planes also break half the supersymmetry and are R–R and NS–NS sources. In the original type I theory the orientation projection keeps only the linear combination $Q_\alpha + \tilde{Q}_\alpha$. In the T-dualised theory this becomes $Q_\alpha + (\prod_m P_m)\tilde{Q}_\alpha$ just as for the D-branes. The force between an orientifold plane and a D-brane can be obtained from the Möbius strip as in the bosonic case; again the total is zero and can be separated into NS–NS and R–R exchanges. The result is similar to the bosonic result (6.18),

$$\mu'_p = \mp 2^{p-5} \mu_p, \qquad \tau'_p = \mp 2^{p-5} \tau_p, \tag{8.17}$$

where the plus sign is correlated with $SO(n)$ groups and the minus with $USp(n)$. Since there are 2^{9-p} orientifold planes, the total O-plane charge is $\mp 16\mu_p$, and the total fixed-plane tension is $\mp 16\tau_p$.

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8.5 Type I from type IIB, revisited

A non-zero total tension represents a source for the graviton and dilaton, for which the response is simply a time dependence of these background fields[105]. A non-zero total R–R source is more serious, since this would mean that the field equations are inconsistent: there is a violation of Gauss's Law, as R–R flux lines have no place to go in the compact space T^{9-p}. So our result tells us that on T^{9-p}, we need exactly 16 D-branes, with the SO projection, in order to cancel the R–R $G_{(p+2)}$ form charge. This gives the T-dual of $SO(32)$, completing our simple orientifold story.

The spacetime anomalies for $G \neq SO(32)$ (see also section 7.1.3) are thus accompanied by a divergence[107] in the full string theory, as promised, with inconsistent field equations in the R–R sector: as in field theory, the anomaly is related to the ultra-violet limit of a (open string) loop graph. But this ultraviolet limit of the annulus/cylinder ($t \to \infty$) is in fact the infrared limit of the closed string tree graph, and the anomaly comes from this infrared divergence. From the world-sheet point of view, as we have seen in the bosonic case, inconsistency of the field equations indicates that there is a conformal anomaly that cannot be cancelled. This is associated to the presence of a 'tadpole' which is simply an amplitude for creating quanta out of the vacuum with a one-point function, which is a sickness of the theory which must be cured.

The prototype of all of this is the original $D = 10$ type I theory[31]. The N D9-branes and single O9-plane couple to an R–R ten-form, and we can write its action formally as

$$(32 \mp N)\frac{\mu_{10}}{2}\int C_{10}. \qquad (8.18)$$

The field equation from varying C_{10} is just $G = SO(32)$.

8.6 Dirac charge quantisation

We are of course studying a quantum theory, and so the presence of both magnetic and electric sources of various potentials in the theory should give some cause for concern. We should check that the values of the charges are consistent with the appropriate generalisation of[114] the Dirac quantisation condition. The field strengths to which a Dp-brane and D$(6-p)$-brane couple are dual to one another, $G_{(p+2)} = *G_{(8-p)}$.

We can integrate the field strength $*G_{(p+2)}$ on an $(8-p)$-sphere surrounding a Dp-brane, and using the action (8.7), we find a total flux $\Phi = \mu_p$. We can write $*G_{(p+2)} = G_{(8-p)} = dC_{(7-p)}$ everywhere except on a Dirac 'string' (see also insert 9.2; here it is really a sheet), at the end of

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which lives the D$(6-p)$ 'monopole'. Then

$$\Phi = \frac{1}{2\kappa_0^2}\int_{S_{8-p}} *G_{(p+2)} = \frac{1}{2\kappa_0^2}\int_{S_{7-p}} C_{(7-p)}, \tag{8.19}$$

where we perform the last integral on a small sphere surrounding the Dirac string. A $(6-p)$-brane circling the string picks up a phase $e^{i\mu_{6-p}\Phi}$. The condition that the string be invisible is

$$\mu_{6-p}\Phi = 2\kappa_0^2\mu_{6-p}\mu_p = 2\pi n. \tag{8.20}$$

The D-branes' charges (8.11) satisfy this condition with the minimum quantum $n = 1$.

While this argument does not apply directly to the case $p = 3$, as the self-dual five-form field strength has no covariant action, the result follows by the T-duality recursion relation (5.11) and the BPS property.

8.7 D-branes in type I

As we saw in section 7.1.3, the only R–R potentials available in type I theory are the two-form and its dual, the 6-form, and so we can have D1-branes in the theory, and D5-branes, which are electromagnetic duals of each other. The overall 16 d9-branes carry an $SO(32)$ gauge group, as we have seen from many points of view. Let us remind ourselves of how this gauge group came about, since there are important subtleties of which we should be mindful[132].

The action of Ω has representation γ_Ω, which acts on the Chan–Paton indices, as discussed in chapter 4:

$$\Omega: \quad |\psi, ij\rangle \longrightarrow (\gamma_\Omega)_{ii'}|\Omega\psi, j'i'\rangle(\gamma_\Omega^{-1})_{j'j},$$

where ψ represents the vertex operator which makes the state in question, and $\Omega\psi$ is the action of Ω on it. The reader should recall that we transposed the indices because Ω exchanges the endpoints of the string. We can consider the square of Ω:

$$\Omega^2: \qquad |\psi, ij\rangle \longrightarrow \left[\gamma_\Omega(\gamma_\Omega^T)^{-1}\right]_{ii'}|\psi, i'j'\rangle\left[\gamma_\Omega^T\gamma_\Omega\right]_{j'j}, \tag{8.21}$$

and so we see that we have the choice

$$\gamma_\Omega^T = \pm\gamma_\Omega.$$

If γ_Ω is symmetric, the with n branes we can write it as $\mathbf{I}_{2n}$, the $2n\times 2n$ identity matrix. Since the 99 open string vertex operator is $\partial_t X^\mu$, it has

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(as we have seen a lot in chapter 4) $\Omega = -1$. Therefore we do have the symmetric choice since, as we tacitly assumed in equation (8.21) $\Omega^2 = 1$, and so we conclude that the Chan–Paton wavefunction is antisymmetric. Since $n = 16$, we have gauge group $SO(32)$.

If γ_Ω was antisymmetric, then we could have written it as

$$\gamma_\Omega = \begin{pmatrix} 0 & i\mathbf{I}_n \\ -i\mathbf{I}_n & 0 \end{pmatrix},$$

and we would have been able to have gauge group $USp(2n)$. In fact, we shall have to make this choice for D5-branes. Let us see why. Let us place the D5-branes so that they are pointlike in the directions X^m, $m = 6, 7, 8, 9$, and aligned in the directions X^μ, $\mu = 0, 1, \ldots, 5$.

Consider the 5–5 sector, i.e. strings beginning and ending on D5-branes. Again we have $\Omega = -1$ for the vectors $\partial_t X^\mu$, and the opposite sign for the transverse scalars $\partial_n X^m$. In general, other sectors can have different mode expansions. Generically the mode for a fermion is ψ_r and Ω acts on this as $\pm(-1)^r = \pm e^{i\pi r}$ (see chapter 11 for more discussion of these possible modings). In the NS sector they are half-integer and since GSO requires them to act in pairs in vertex operators, their individual $\pm i$s give $\Omega = \pm 1$, with a similar result in the R sector by supersymmetry.

The 59 sector is more subtle[132]. The X^m are now half-integer moded and the ψ^m are integer moded. The ground states of the latter therefore form a representation of the Clifford algebra and we can bosonise them into a spin field, as we did in chapter 7 in a similar situation: $e^{iH_3} \sim \psi^6 + i\psi^7$, and $e^{iH_4} \sim \psi^8 + i\psi^9$. In fact, the vertex operator (the part of it relevant to this discussion) in that sector is

$$V_{59} \sim e^{i(H_3+H_4)/2}.$$

Now consider the square of this operator. It has parts which are either in the 55 sector or the 99 sector, and is of the form

$$V_{59}^2 \sim e^{i(H_3+H_4)} \sim (\psi^6 + i\psi^7)_{-1/2}(\psi^8 + i\psi^9)_{-1/2}|0\rangle.$$

So it has $\Omega = -1$, since each $\psi_{-1/2}$ gives $\pm i$. So $\Omega^2 = -1$ for V_{59} for consistency.

Returning to our problem of the choices to make for the Chan–Paton factors we see that we have an extra sign in equation (8.21), and so must choose the antisymmetric condition $\gamma_\Omega^T = -\gamma_\Omega$. Therefore, in type I string theory, n D5-branes have gauge group $USp(2n)$. Notice that this means that a single one has $SU(2)$, and the Chan–Paton wavefunction can be chosen as the Pauli matrices. The Chan–Paton wavefunction for

https://doi.org/10.1017/9781009401371.009 Published online by Cambridge University Press

the scalars for transverse motion must simply be δ^{ij}, since we have another sign. This simply means that the two D5-branes (corresponding to the two index choices) are forced to move with each other as one unit.

Notice that this fits rather nicely with our charge quantisation computation of the previous section[132]. The orientifold projection will halve the force between D1-branes and between D5-branes in the charge calculation, and so their effective charges would be reduced by $\sqrt{2}$, violating the Dirac quantisation condition by a factor of a half. However, the fact that the D5-branes are forced to move as a pair restores a factor of two in the quantisation condition, and so we learn that D-branes are still the smallest consistent charge carries of the R–R sector.

We can augment the argument above for Dp branes in type I in general, and obtain[132]

$$\Omega^2 = (\pm i)^{\frac{9-p}{2}}.$$

For $p = 3$ and $p = 7$, we see that simply gives an inconsistency, which is itself consistent with the fact that there is no R–R four-forms or eight-form for a stable D3-brane or D7-brane to couple to. For $p = 1$ we recover the naively expected result that they have an $SO(2n)$ gauge group.

In chapter 14 we shall see that when we combine the orientifold action with other spacetime orbifold symmetries, we can recover extra phase factors by means analogous to what we have uncovered here in order to discover other choices for D5- and D9-brane gauge groups.

https://doi.org/10.1017/9781009401371.009 Published online by Cambridge University Press