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# THE BOUNDARY CONDITIONS DESCRIPTION OF TYPE I DOMAINS

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**Abstract.** Type I domains are the domains of the self-adjoint operators determined by the weak formulation of formally self-adjoint differential expressions $\ell$. This class of operators is defined by the requirement that the sesquilinear form $q(u, v)$ obtained from $\ell$ by integration by parts agrees with the inner product $\langle \ell u, v\rangle$. A complete characterisation of the boundary conditions assumed by functions in these domains for second-order differential expressions is given in this paper. In the singular case, the boundary conditions are stated in terms of certain 'boundary condition' functions and in the regular case they are given in terms of classical function values.

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**1. Introduction.** In this paper, we give a complete characterisation of the boundary conditions of functions belonging to *Type I domains* [**4**] that are associated with the differential expression

$$\ell u(x) = \frac{1}{w(x)}(-(p(x)u'(x))' + g(x)u(x)), \tag{1}$$

where $x \in I = (a, b)$, with $-\infty \leq a < b \leq \infty$. The expression $\ell$ gives rise to the formal sesquilinear form

$$q(u, v) = \int_I pu'\overline{v'} + gu\overline{v}$$

in addition to the form

$$\langle \ell u, v\rangle = \int_I (-(pu')' + gu)\overline{v}.$$

The equality

$$q(u, v) = \langle \ell u, v\rangle \tag{2}$$

requires the vanishing of the boundary term

$$-pu'\overline{v}\Big]_a^b, \tag{3}$$

which is the most general condition for (2) to hold. Possible sufficient conditions for the vanishing of this boundary term are $pu'\overline{v}\,(a) = 0 = pu'\overline{v}\,(b)$ or simply $pu'\overline{v}\,(a) = pu'\overline{v}\,(b)$ provided that the expressions involved are defined. The former case is referred to as separated boundary conditions, and the latter case is referred to as coupled boundary conditions. Each one of these two classes of boundary conditions include more specific possibilities such as $v\,(a) = 0 = pu'\,(b)$ or $pu'\,(a) = pu'\,(b)\,,\ \overline{v}\,(a) = \overline{v}\,(b)$ among many others. The natural question to ask then is which of these possible combinations of boundary conditions give rise to Type I operators. In this paper, we give a complete characterisation of such boundary conditions. We should also point out that the class of Type I operators includes, as a special case the Friedrichs Extension [**6**] which satisfies Dirichlet (i.e. separated) boundary conditions [**8**, **10**] in the regular case (see the next section). Since the Dirichlet boundary conditions are but a special form of the more general separated boundary conditions mentioned above, the Friedrichs Extension is a special case of Type I operators. Our work in this paper will establish that other separated boundary conditions such as $u\,(a) = u'\,(b) = 0$ give rise to Type I operators which are therefore different from the Friedrichs Extension.

All self-adjoint operators associated with the expression $\ell$ are realised through the requirement

$$\langle \ell u, v\rangle = \langle u, \ell v\rangle$$

which, in turn, requires the vanishing of the more general boundary term

$$-pu'\overline{v} + pu\overline{v'}\Big]_a^b. \tag{4}$$

Type I operators are a special class of these operators in the requirement that

$$\langle \ell u, v\rangle = q\,(u, v) = \langle u, \ell v\rangle \tag{5}$$

and (consequently) the vanishing of the boundary terms

$$-pu'\overline{v}\Big]_a^b,\ \ -pu\overline{v'}\Big]_a^b. \tag{6}$$

Equation (5), or equivalently equation (6), illustrates the specific attribute of Type I operators. It does not hold for a general self-adjoint operator associated with $\ell$, even in the regular case. For example, the expression $\ell u = -u'' + u$ defined on (0, 1) and the boundary conditions $u\,(0) + u'\,(0) = u\,(1) + u'\,(1) = 0$ give rise to a self-adjoint operator in $L^2\,(I)$. The function $u\,(x) = -3x^3 + 4x^2$ is in the domain of this operator but $-pu'\overline{v}]_0^1 \neq 0$. In this paper, it will be clear why this is so.

The study of self-adjoint operators associated with $\ell$ is not new (see [**7**, **9**, **12**, **13**] and the references therein), while the study of boundary conditions associated with them can be found in (see [**3**, **5**, **7**, **13**]). The study of Type I operators appeared in [**4**] and in a sense, this paper is a sequel to the work that started in the above cited reference.

This paper consists of three sections in addition to the introduction. In Section 2 we present some preliminary material that includes definitions, theorems and discussions needed for the rest of the paper. It is designed to be, more or less, self-contained and should help the reader to better follow the terminology used in connection with singular operators. In Section 3 we characterise the boundary conditions of Type I domains. In Section 4 we specialise to regular operators.

**2. Preliminaries.** In this section, we introduce notation, definitions and discussions that are necessary for this work. The main definitions and theorems can be found in [**4**, **7**, **9**, **12**, **13**]. We assume that $I = (a, b)$, $-\infty \le a < b \le \infty$,

$$1/p, g, w \in L_{\text{loc}}(I)$$

and that $w > 0$ almost everywhere in $I$. We formally define on $I$ the self-adjoint differential expression

$$\ell u = \frac{1}{w}[(-pu')' + gu],$$

the sesquilinear form

$$q(u, v) = \int_a^b pu'\overline{v}' + gu\overline{v},$$

the half-Lagrangian

$$\{u, v\}(x) := -u^{[1]}\overline{v}(x) \tag{7}$$

and the Lagrangian

$$[u, v](x) := \{u, v\}(x) - \{\overline{v}, \overline{u}\}(x), \; x \in I,$$

where $u^{[1]} := pu'$. For $\alpha, \beta \in I$ the notation $\{u, v\}_\alpha^\beta$ will mean $\{u, v\}(\beta) - \{u, v\}(\alpha)$ and $[u, v]_\alpha^\beta$ will mean $[u, v](\beta) - [u, v](\alpha)$.

Denote by $H$ the space $L_w^2(I)$ of complex valued square integrable functions with respect to the weight $w$, by $\langle \cdot, \cdot \rangle$ its inner product and by $\|\cdot\|$ its norm. The maximal operator $L$ generated by the expression $\ell$ in $H$ is defined by

$$D(L) = D = \{u \in H : \ell(u) \in H\},$$
$$Lu = \ell(u), \quad u \in D.$$

$L$ is closed and densely defined and its adjoint $L_0 := L^*$ with domain $D_0 := D(L_0)$ is called the minimal operator generated by $\ell$. $L_0$ is symmetric and it is known [**9**] that $L_0 \subset L = L_0^*$. Therefore, $L_0$ is a symmetric closed operator and, any self-adjoint extension $\widehat{L}$ of $L_0$ satisfies $L_0 \subset \widehat{L} = \widehat{L}^* \subset L_0^* = L$.

For $u, v \in D$ the limits

$$\lim_{x \to a^+}[u, v](x), \qquad \lim_{x \to b^-}[u, v](x)$$

both exist and are finite. We denote these limits by $[u, v](a)$, $[u, v](b)$, respectively. For all $u \in D_0$, $v \in D$, $[u, v]_a^b = 0$.

Our main assumption on $q$ is the following:

**(A)** $q$ is bounded below: $q(u) := q(u, u) \ge M\|u\|^2$ for some $M \in \mathbb{R}$.

This assumption guarantees (see [**4**]) that, for any value of $d$, the domain $D$ always contains a Type I domain. However, it excludes cases where one end point is LP but not strongly LP (see [**5**]).

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We let $V$ be the (dense) subspace of functions $u \in H$ for which $q(u) < \infty$. $V$ can be given the structure of a Hilbert space if equipped with the norm induced by $q + \lambda$ where $\lambda > \max\{M, 0\}$. Define the space $\widetilde{D}$ by

$$\widetilde{D} = \{u \in V : q(u, \cdot) \text{ is continuous on } D_0 \text{ with respect to the norm in } H\}. \tag{8}$$

It turns out that $D_0 \subset \widetilde{D} \subseteq D$. For $u, v \in \widetilde{D}$, the limits

$$\lim_{x\to a^+} \{u, v\}(x), \quad \lim_{x\to b^-} \{u, v\}(x)$$

both exist and are finite. We denote these limits by $\{u, v\}(a)$, $\{u, v\}(b)$, respectively. For $u, v \in \widetilde{D}$,

$$\langle \ell u, v\rangle = \{u, v\}_a^b + q(u, v). \tag{9}$$

For $u \in D_0,\ v \in D,\ \{u, v\}_a^b = 0$. It will also be convenient to list the values of the half-Lagrangians $\{u, v\}$ and $\{v, u\}$ at the endpoints $a$ and $b$ (if they exist) in a table which will be called the *multiplication table* of $u$ and $v$ (see e.g. (12)) below.

The number $d := \dim(D \bmod D_0)$ is called the deficiency index of $L_0$. The number $\delta := \dim\left(D \bmod \widetilde{D}\right)$ is called the *co-deficiency index* of $L_0$. In our case $d \in \{0, 1, 2\}$ and $\delta \in \{0, \ldots, d\}$. In the following sections, the investigation of the boundary values of functions will split into several cases depending on the values of $d$ and $\delta$. The various cases will be denoted by a pair $(d, \delta)$ or by a triple $(d, \delta, n)$ if the case splits into subcases. Our starting point for the analysis will always be the characterisation given in [**4**].

The endpoint $a$ is *regular* if $1/p, g, w \in L(a, c)$ for some (and hence all) $c \in I$; is *limit circle* (LC) if all solutions of

$$\ell u = 0 \tag{10}$$

are in $L_w^2(a, c)$ for some $c \in I$; is *limit point* (LP) if it is not LC. Similar definitions hold at $b$. An endpoint is *singular* if it is not regular. $d = 0$ if and only if both $a$ and $b$ are LP, $d = 1$ if and only if one end point is LP and the other is LC and $d = 2$ if and only if both $a$ and $b$ are LC. For convenience, whenever $d = 1$, we will always assume that $a$ is LC and $b$ is LP.

If the formal operator $\ell$ is in the case $(d, \delta)$ then we can select a set of $2d$ real functions $\psi_1, \ldots, \psi_{2d} \in D \bmod D_0$ (empty if $d = 0$) of which $2d - \delta$ are in $\widetilde{D} \bmod D_0$ and the rest are in $D \bmod \widetilde{D}$. In the case $d = 1$, $\psi_1, \psi_2$ can be selected so that they are identically zero near $b$ and $[\psi_1, \psi_2](\cdot) = -1$ near $a$. In the case $d = 2$, $\psi_1, \psi_2, \psi_3, \psi_4$ can be selected so that $\psi_1, \psi_2$ are identically zero near $b$, $[\psi_1, \psi_2](\cdot) = -1$ near $a$, $\psi_3, \psi_4$ are identically zero near $a$ and $[\psi_1, \psi_2](\cdot) = -1$ near $b$. We have

$$D = D_0 \dotplus \operatorname{span}\{\psi_1, \ldots, \psi_{2d}\}. \tag{11}$$

If $\psi_1, \psi_2 \in \widetilde{D} \bmod D_0$ (such as when $(d, \delta) = (1, 0)$), it was shown in [**4**] that they can be chosen so that

$$\{\psi_1, \psi_1\}(a) = \lambda_a > 0, \{\psi_2, \psi_2\}(a) = -\sigma_a,\ \lambda_a\sigma_a = \frac{1}{4},$$
$$[\psi_1, \psi_2](a) = -1,\ \{\psi_1, \psi_2\}(a) + \{\psi_2, \psi_1\}(a) = 0.$$

The last two equations give $\{\psi_1, \psi_2\}(a) = -\frac{1}{2},\ \{\psi_2, \psi_1\}(a) = \frac{1}{2}$. Replacing $\psi_1, \psi_2$ by $\psi_1/\sqrt{\lambda_a}, \psi_2/\sqrt{\sigma_a}$, respectively, we obtain the following multiplication tables for

$\psi_1, \psi_2$

$$\begin{array}{c|cc} (a) & \psi_1 & \psi_2 \\ \hline \psi_1 & 1 & -1 \\ \psi_2 & 1 & -1 \end{array}, \quad \begin{array}{c|cc} (b) & \psi_1 & \psi_2 \\ \hline \psi_1 & 0 & 0 \\ \psi_2 & 0 & 0 \end{array}. \tag{12}$$

Observe that we still have $[\psi_1, \psi_2](\cdot) = -1$ near $a$. Similarly, if $\psi_3, \psi_4 \in \widetilde{D} \bmod D_0$ they will be assumed to have the following multiplication table.

$$\begin{array}{c|cc} (a) & \psi_3 & \psi_4 \\ \hline \psi_3 & 0 & 0 \\ \psi_4 & 0 & 0 \end{array}, \quad \begin{array}{c|cc} (b) & \psi_3 & \psi_4 \\ \hline \psi_3 & 1 & -1 \\ \psi_4 & 1 & -1 \end{array}. \tag{13}$$

A *symmetric* (*self-adjoint*) domain $D^\dagger \subset D$ is the domain of a symmetric (self-adjoint) extension $L^\dagger$ of $L_0$. A symmetric domain $D^\dagger \subset D$ is a self-adjoint domain if and only if it is a $d$-dimensional extension of $D_0$. Consequently, a self-adjoint domain $\widehat{D} \subset \widetilde{D}$ is a Type I domain.

**3. Boundary conditions in the singular case.** As discussed in Section 2, the general requirement for a domain $\widehat{D} \subset \widetilde{D}$ to be a Type I domain is that $\{u, v\}_a^b = 0$ for all $u, v \in \widehat{D}$. This general boundary condition may be classified further as separated:

$$\{u, v\}(a) = \{u, v\}(b) = 0 \; \forall u, v \in \widehat{D} \tag{14}$$

or coupled:

$$\begin{gathered} \{u, v\}(a) = \{u, v\}(b) \;\; \forall u, v \in \widehat{D} \\ \text{and } \{u, v\}(a) \neq 0 \text{ for at least one pair } u, v \in \widehat{D}. \end{gathered} \tag{15}$$

In this section, we present a description of of Type I domains in terms of both types of boundary conditions.

For $d = 0$ the only self-adjoint extension of $L_0$ is $L_0$ itself. In this case $L_0$ is a Type I operator and $D_0$ satisfies separated boundary conditions (see [**4**]).

**3.1. The limit point case.** In this subsection, we assume that the deficiency index $d = 1$ and, without loss of generality, that $a$ is LC and $b$ is LP. Select $\psi_1, \psi_2 \in D \bmod D_0$ satisfying the properties discussed in Section 2. That is so that they are identically zero near $b$ and $[\psi_1, \psi_2](\cdot) = -1$ near $a$. We know from [**4**] that all Type I domains have the separated boundary condition (14). Depending on the values of $(d, \delta)$ we have the following cases:

*Case* (1,1) In this case $\psi_1, \psi_2$ can be selected so that $\psi_1 \in \widetilde{D} \bmod D_0$, $\psi_2 \in D \bmod \widetilde{D}$ and $\{\psi_1, \psi_1\}(a) = 0$.

*Case* (1,0)) In this case $\psi_1, \psi_2 \in \widetilde{D} \bmod D_0$ and we may assume the multiplication tables (12).

THEOREM 1 (Type I domains in the limit point case). *Assume the endpoint $a$ is LC and the endpoint $b$ is LP.*

(a) *If $\delta = 1$ then there exists a function $\eta \in \widetilde{D} \bmod D_0$ and a function $\xi \in D \bmod \widetilde{D}$ such that $\{\eta, \eta\}(a) = 0$, $[\eta, \xi](x) = -1$ near $a$ and $\xi(x) = \eta(x) = 0$ near $b$.*

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*The domain* $D_1$ *defined by*

$$D_1 = \{u \in D : \{u, \eta\}(a) = \{\eta, u\}(a) = 0\} \tag{16}$$

*is a Type I domain. Conversely, if* $\widehat{D}$ *is a Type I domain then* $\widehat{D}$ *is given by* (16).

(b) *If* $\delta = 0$ *then there exist two functions* $\eta_1, \eta_2 \in \widetilde{D} \bmod D_0$ *such that* $\{\eta_i, \eta_i\}(a) = 0$, $i = 1, 2$, $[\eta_1, \eta_2](x) = -1$ *near* $a$ *and* $\eta_1(x) = \eta_2(x) = 0$ *near* $b$.
*The domains* $D_1$ *and* $D_2$ *defined by*

$$D_1 = \{u \in D : \{\eta_1, u\}(a) = 0\}, \tag{17}$$

$$D_2 = \{u \in D : \{u, \eta_2\}(a) = 0\} \tag{18}$$

*are Type I domains. Conversely, if* $\widehat{D}$ *is a Type I domain then* $\widehat{D}$ *is given either by* (17) *or* (18).

*Proof.* (a): Let $\psi_1, \psi_2$ be the functions described in Case (1,1). The first statement in this part follows upon putting $\eta := \psi_1$ and $\xi = \psi_2$. To show that the set $D_1$ given by (16) is a Type I domain, let $u \in D_1$ and write $u = u_0 + \alpha_1\xi + \alpha_2\eta$. The conditions $\{u, \eta\}(a) = \{\eta, u\}(a) = 0$ give $[u, \eta](a) = 0$. Therefore,

$$0 = [u, \eta](a) = [u_0, \eta](a) + \alpha_1 [\xi, \eta](a) + \alpha_2 [\eta, \eta](a) = -\alpha_1$$

and

$$u = u_0 + \alpha_1\eta, \tag{19}$$

which means that $u \in \widetilde{D}$. Thus, $D_1 \subset \widetilde{D}$. Furthermore, since $D_1$ is a symmetric one-dimensional extension of $D_0$, $D_1$ is a Type I domain. To prove the converse statement assume $\widehat{D}$ is a Type I domain. Since $\widehat{D} \subset \widetilde{D}$, $\xi \notin \widehat{D}$. Therefore, any $u \in \widehat{D}$ has the form $u = u_0 + \alpha_1\eta$, which agrees with the characterisation (19) of elements in $D_1$. Thus, $\widehat{D} \subset D_1$. Since both sets are one-dimensional extensions of $D_0$, $\widehat{D} = D_1$.

(b): Let $\psi_1, \psi_2$ be the functions described in Case (1,0). The first statement in this part follows upon putting $\eta_1 := \psi_1 + \psi_2$ and $\eta_2 = \psi_1 - \psi_2$. From (12) we get the following multiplication tables for $\eta_1, \eta_2$:

| $(a)$ | $\eta_1$ | $\eta_2$ |
|---|---|---|
| $\eta_1$ | 0 | 4 |
| $\eta_2$ | 0 | 0 |

,

| $(b)$ | $\eta_1$ | $\eta_2$ |
|---|---|---|
| $\eta_1$ | 0 | 0 |
| $\eta_2$ | 0 | 0 |

.

Since $\{\eta_1, \eta_1\}(a) = 0$ and $\{\eta_1, \eta_2\}(a) \neq 0$, $\eta_1 \in D_1$ and $\eta_2 \notin D_1$. Therefore, $D_1$ is a one-dimensional extension of $D_0$ (any $u \in D_1$ has the representation $u = u_0 + \alpha\eta_1$). Furthermore, it is easy to check that $\{u, v\}(a) = \{u, v\}(b) = 0$ for all $u, v \in D_1$. Therefore, $D_1$ is a Type I domain. Similarly, we can show that $D_2$ is a Type I domain. To show the converse statement suppose that $\widehat{D}$ is a Type I domain that is not given by (18). We claim that $\eta_2 \notin \widehat{D}$. If not then let $u \in \widehat{D}$ and write $u = u_0 + \alpha\eta_1 + \beta\eta_2$. Since $\widehat{D}$ is a Type I domain, $\{u, \eta_2\}(a) = 0$. It follows that

$$0 = \{u, \eta_2\}(a) = \alpha\{\eta_1, \eta_2\}(a) = 4\alpha.$$

Hence, $u = u_0 + \beta\eta_2$ and $u \in D_2$, which means that $\widehat{D} \subset D_2$; a contradiction. Thus any $u \in \widehat{D}$ has the representation $u = u_0 + \alpha\eta_1$, which gives $\widehat{D} \subset D_1$. Since $D_1$ is a Type I domain, $\widehat{D} = D_1$. □

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REMARK 2. In special cases the boundary conditions stated in (16), (17) or (18) may reduce to the classical ones $u(a) = 0$ or $u^{[1]}(a) = 0$. For example, in the case $\delta = 1$, if $\psi_1(a)$ is finite and non-zero, $D_1$ is described by the boundary condition $u^{[1]}(a) = 0$. This is because for any $u \in D_1$, we can write $u = u_0 + \alpha\psi_1$ and note that $\{u_0, \psi_1\}(a) = 0$ and $\{\psi_1, \psi_1\}(a) = 0$ give also $u_0^{[1]}(a) = 0$ and $\psi_1^{[1]}(a) = 0$. Similarly, if $\psi_1^{[1]}(a)$ is finite and non-zero, $D_1$ is described by the boundary condition $u(a) = 0$. In the case $\delta = 0$, if $\psi_2^{[1]}(a)$ is finite and non-zero then $D_2$ is described by the boundary condition $u(a) = 0$ and, since $\{\psi_2, \psi_1\}(a) > 0$, $\psi_1(a)$ is finite and non-zero. This means that $D_1$ is described by the boundary condition $u^{[1]}(a) = 0$ while $D_2$ is described by the boundary condition $u(a) = 0$.

The following two examples illustrate Theorem 1.

EXAMPLE . Let $\gamma > 0$ and consider the operator $\ell(y) = \frac{1}{x^{\gamma+1}}(-(x^2y')' + \frac{\gamma^2-1}{4}y)$ defined on $(0, \infty)$. The two functions $\varphi(x) = x^{(-1+\gamma)/2}$, $\theta(x) = x^{(-1-\gamma)/2}$ are solutions for $\ell(y) = 0$. Let $\psi_1, \psi_2 \in D$ be such that

$$\psi_1(x) = \begin{cases} \varphi(x) \text{ near } 0 \\ 0 \text{ near } \infty \end{cases}, \quad \psi_2(x) = \begin{cases} \theta(x) \text{ near } 0 \\ 0 \text{ near } \infty \end{cases}. \tag{20}$$

Since $[\psi_1, \psi_2](x) = -\gamma$ for all $x$ near 0, we conclude that $\psi_1, \psi_2$ are linearly independent modulo $D_0$. Since $\{\psi_1, \psi_1\}(0) = 0$ and $\{\psi_2, \psi_2\}(0)$ does not exist, $\psi_1 \in \widetilde{D}$ and $\psi_2 \in D \bmod \widetilde{D}$. Hence, $(d, \delta) = (1, 1)$. Therefore, there is only one Type I domain $D_1$ determined by the boundary condition $\{u, \psi_1\}(0) = \{\psi_1, u\}(0) = 0$.

EXAMPLE . Consider the operator $\ell(y) = -y'' - y$ defined on $(0, \infty)$. The two functions $\theta(x) = \cos x, (d, \delta)$, $\varphi(x) = \sin x$ are solutions for $\ell(y) = 0$. Select $\psi_1, \psi_2 \in D$ as in (20). Then $(d, \delta) = (1, 0)$. The two Type I domains corresponding to this case are

$$D_1 = \{u \in D : u(0) = 0\},$$
$$D_2 = \{u \in D : u'(0) = 0\}.$$

**3.2. The limit circle case.** In this subsection, we assume that the deficiency index $d = 2$ and that functions $\psi_1, \psi_2, \psi_3, \psi_4 \in D \bmod D_0$ have been selected so that $\psi_1, \psi_2$ are identically zero near $b$, $[\psi_1, \psi_2](\cdot) = -1$ near $a$, $\psi_3, \psi_4$ are identically zero near $a$ and $[\psi_3, \psi_4](\cdot) = -1$ near $b$.

*Case* (2,2) In this case we may assume that $\psi_1, \psi_3 \in \widetilde{D} \bmod D_0$ and $\psi_2, \psi_4 \in D \bmod \widetilde{D}$ (see [**4**, comment after Lemma 17]) and $\{\psi_1, \psi_1\}(a) = \{\psi_3, \psi_3\}(b) = 0$. $\widetilde{D}$ is itself a Type I domain.

*Case* (2,1) In this case $\psi_1, \psi_2, \psi_3, \psi_4 \in D$ can be selected so that either

$$\psi_1, \psi_2, \psi_3 \in \widetilde{D} \bmod D_0, \ \ \psi_4 \in D \bmod \widetilde{D}$$

or

$$\psi_2, \psi_3, \psi_4 \in \widetilde{D} \bmod D_0, \ \ \psi_1 \in D \bmod \widetilde{D}.$$

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*Subcase* (2,1,1) If $\psi_1, \psi_2, \psi_3 \in \widetilde{D} \bmod D_0$ then we may assume the multiplication tables (12) and $\{\psi_3, \psi_3\}(b) = 0$. To describe Type I domains (necessarily satisfying separated boundary conditions), put

$$\eta_1 = \psi_1 + \psi_2,\ \eta_2 = \psi_1 - \psi_2,$$
$$\eta_3 = \psi_3,\ \eta_4 = \psi_4.$$

Then $\eta_1, \eta_2, \eta_3, \eta_4$ have the following properties:

$$\begin{gathered} \eta_1, \eta_2, \eta_3, \eta_4 \in D \bmod D_0, \\ \eta_1, \eta_2, \eta_3 \in \widetilde{D},\ \eta_4 \in D \bmod \widetilde{D}, \\ [\eta_3, \eta_4](\cdot) = -1 \text{ near } b, \\ \begin{array}{c|ccc} (a) & \eta_1 & \eta_2 & \eta_3 \\ \hline \eta_1 & 0 & 0 & 0 \\ \eta_2 & 4 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 \end{array}, \quad \begin{array}{c|ccc} (b) & \eta_1 & \eta_2 & \eta_3 \\ \hline \eta_1 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 \end{array}. \end{gathered} \tag{21}$$

Using Theorem 18 in [**4**] we get that there are two Type I domains described by

$$D_1 = D_0 + \text{span}\{\eta_1, \eta_3\},$$
$$D_2 = D_0 + \text{span}\{\eta_2, \eta_3\}.$$

*Subcase* (2,1,2) If $\psi_2, \psi_3, \psi_4 \in \widetilde{D} \bmod D_0$ we similarly define the four functions $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ by

$$\zeta_1 = \psi_1,\ \zeta_2 = \psi_2 + \psi_3,$$
$$\zeta_3 = \psi_2 - \psi_3,\ \zeta_4 = \psi_4.$$

Then $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ have the following properties:

$$\begin{gathered} \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in D \bmod D_0, \\ \zeta_2, \zeta_3, \zeta_4 \in \widetilde{D},\ \zeta_1 \in D \bmod \widetilde{D}, \\ [\zeta_1, \zeta_2](\cdot) = -1 \text{ near } a, \\ \begin{array}{c|ccc} (a) & \zeta_2 & \zeta_3 & \zeta_4 \\ \hline \zeta_2 & 0 & 0 & 0 \\ \zeta_3 & 0 & 0 & 0 \\ \zeta_4 & 0 & 0 & 0 \end{array}, \quad \begin{array}{c|ccc} (b) & \zeta_2 & \zeta_3 & \zeta_4 \\ \hline \zeta_2 & 0 & 0 & 0 \\ \zeta_3 & 0 & 0 & 0 \\ \zeta_4 & 0 & 4 & 0 \end{array}. \end{gathered} \tag{22}$$

We also have two Type I domains described by

$$D_3 = D_0 + \text{span}\{\zeta_2, \zeta_3\},$$
$$D_4 = D_0 + \text{span}\{\zeta_2, \zeta_4\}.$$

*Case* (2,0) In this case we may assume the multiplication table (12) for $\psi_1, \psi_2$ and the multiplication table (13) for $\psi_3, \psi_4$. To describe Type I domains satisfying separated boundary conditions put

$$\eta_1 = \psi_1 + \psi_2, \eta_2 = \psi_1 - \psi_2,$$
$$\eta_3 = \psi_3 + \psi_4, \eta_4 = \psi_3 - \psi_4.$$

Then $\eta_1, \eta_2, \eta_3, \eta_4$ have the following properties:

$$\eta_1, \eta_2, \eta_3, \eta_4 \in \widetilde{D} \bmod D_0,$$

$$\begin{array}{c|cccc} (a) & \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \hline \eta_1 & 0 & 0 & 0 & 0 \\ \eta_2 & 4 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 0 \\ \eta_4 & 0 & 0 & 0 & 0 \end{array}, \quad \begin{array}{c|cccc} (b) & \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \hline \eta_1 & 0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 0 \\ \eta_4 & 0 & 0 & 4 & 0 \end{array}. \tag{23}$$

Using Theorem 18 in [**4**] we get that there are four Type I domains satisfying separated boundary conditions. These domains are described by

$$D_1 = D_0 + \operatorname{span}\{\eta_1, \eta_3\},\ D_2 = D_0 + \operatorname{span}\{\eta_1, \eta_4\},$$
$$D_3 = D_0 + \operatorname{span}\{\eta_2, \eta_3\},\ D_4 = D_0 + \operatorname{span}\{\eta_2, \eta_4\}.$$

To describe Type I domains satisfying coupled boundary conditions put

$$\xi_1(t) = \psi_1 + \cosh t\psi_3 + \sinh t\psi_4,\ \xi_2(t) = \psi_2 + \sinh t\psi_3 + \cosh t\psi_4, \tag{24}$$
$$\xi_3(t) = -\psi_1 + \cosh t\psi_3 - \sinh t\psi_4,\ \xi_4(t) = \psi_2 + \sinh t\psi_3 - \cosh t\psi_4, \tag{25}$$

$t \in \mathbb{R}$. Then, for all $t \in \mathbb{R}$, $\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)$ have the following properties:

$$\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t) \in \widetilde{D} \bmod D_0,$$

$$\begin{array}{c|cccc} (a) & \xi_1(t) & \xi_2(t) & \xi_3(t) & \xi_4(t) \\ \hline \xi_1(t) & 1 & -1 & -1 & -1 \\ \xi_2(t) & 1 & -1 & -1 & -1 \\ \xi_3(t) & -1 & 1 & 1 & 1 \\ \xi_4(t) & 1 & -1 & -1 & -1 \end{array}, \quad \begin{array}{c|cccc} (b) & \xi_1(t) & \xi_2(t) & \xi_3(t) & \xi_4(t) \\ \hline \xi_1(t) & 1 & -1 & e^{2t} & e^{2t} \\ \xi_2(t) & 1 & -1 & e^{2t} & e^{2t} \\ \xi_3(t) & e^{-2t} & -e^{-2t} & 1 & 1 \\ \xi_4(t) & -e^{-2t} & e^{-2t} & -1 & -1 \end{array}. \tag{26}$$

Using Theorem 18 in [**4**] we get that there are two one-parameter families of Type I domains satisfying coupled boundary conditions. These domains are described by

$$D_1(t) = D_0 + \operatorname{span}\{\xi_1(t), \xi_2(t)\},$$
$$D_2(t) = D_0 + \operatorname{span}\{\xi_3(t), \xi_4(t)\},$$

$t \in \mathbb{R}$.

THEOREM 3 (Type I domains in the limit circle case). *Assume both endpoints are LC. The boundary values of functions belonging to Type I domains are described as follows*:

(a) *If* $\delta = 2$ *then there exist two functions* $\eta_1, \eta_2 \in \widetilde{D} \bmod D_0$ *such that* $\{\eta_i, \eta_j\}(a) = \{\eta_i, \eta_j\}(b) = 0,\ i, j = 1, 2$ *and two functions* $\xi_1, \xi_2 \in D \bmod \widetilde{D}$ *such that* $[\eta_1, \xi_1](x) = -1$ *near* $a$, $[\eta_1, \xi_1](\cdot) = 0$ *near* $b$, $[\eta_2, \xi_2](\cdot) = 0$ *near* $a$, $[\eta_2, \xi_2](\cdot) = -1$ *near* $b$. *The domain* $D_1$ *defined by*

$$D_1 = \{u \in D : \{u, \eta_1\}(a) = \{\eta_1, u\}(a) = \{u, \eta_2\}(b) = \{\eta_2, u\}(b) = 0\} \tag{27}$$

*is a Type I domain. Conversely, if* $\widehat{D}$ *is a Type I domain then it is given by* (27).

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(b) *If* $\delta = 1$ *then precisely one of the following situations holds*:

(i) *there exist four functions* $\eta_1, \eta_2, \eta_3, \eta_4 \in D \bmod D_0$ *with the properties listed in* (21), *or*

(ii) *there exist four functions* $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in D \bmod D_0$ *with the properties listed in* (22).

*If (i) holds then the two domains* $D_1, D_2$ *defined by*

$$D_1 = \{u \in D : \{u, \eta_1\}(a) = \{u, \eta_3\}(b) = \{\eta_3, u\}(b) = 0\}, \tag{28}$$
$$D_2 = \{u \in D : \{\eta_2, u\}(a) = \{\eta_3, u\}(b) = \{\eta_3, u\}(b) = 0\} \tag{29}$$

*are Type I domains. Conversely, if* $\widehat{D}$ *is a Type I domain then it is described by either* (28) *or* (29).

*If (ii) holds then the two domains* $D_3, D_4$ *defined by*

$$D_3 = \{u \in D : \{u, \zeta_2\}(a) = \{\zeta_2, u\}(a) = \{\zeta_3, u\}(b) = \{u, \zeta_3\}(b) = 0\}, \tag{30}$$
$$D_4 = \{u \in D : \{u, \zeta_2\}(a) = \{\zeta_2, u\}(a) = \{u, \zeta_4\}(b) = \{\zeta_4, u\}(b) = 0\} \tag{31}$$

*are Type I domains. Conversely, if* $\widehat{D}$ *is a Type I domain then it is described by either* (30) *or* (31).

(c) *If* $\delta = 0$ *then there are four functions* $\eta_1, \eta_2, \eta_3, \eta_4 \in \widetilde{D} \bmod D_0$ *with the multiplication tables given by* (23), *and for any* $t \in \mathbb{R}$ *there exist four functions* $\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t) \in \widetilde{D} \bmod D_0$ *with the multiplication table given by* (26).

(i) *The four domains*

$$D_1 = \{u \in D : \{u, \eta_1\}(a) = \{u, \eta_3\}(b) = 0\}, \tag{32}$$
$$D_2 = \{u \in D : \{u, \eta_1\}(a) = \{\eta_4, u\}(b) = 0\}, \tag{33}$$
$$D_3 = \{u \in D : \{\eta_2, u\}(a) = \{\eta_3, u\}(b) = 0\}, \tag{34}$$
$$D_4 = \{u \in D : \{\eta_2, u\}(a) = \{\eta_4, u\}(b) = 0\} \tag{35}$$

*are Type I domains satisfying separated boundary conditions. Conversely, if* $\widehat{D}$ *is a Type I domain satisfying separated boundary conditions then it is described by precisely one of the equations* (32)–(35).

(ii) *The two one-parameter families of domains*

$$D_1(t) = \{u \in D : \{u, \xi_i(t)\}(a) = \{u, \xi_1(t)\}(b), \{\xi_2(t), u\}(a) = \{\xi_2(t), u\}(b)\}, \tag{36}$$
$$D_2(t) = \{u \in D : \{u, \xi_3(t)\}(a) = \{u, \xi_3(t)\}(b), \{\xi_4(t), u\}(a) = \{\xi_4(t), u\}(b)\} \tag{37}$$

*are Type I domains satisfying coupled boundary conditions. Conversely, if* $\widehat{D}$ *is a Type I domain satisfying coupled boundary conditions then it is described by either* (36) *or* (37).

*Proof.* (a): Let $\psi_1, \psi_2, \psi_3, \psi_4$ be as defined in Case (2,2). The first assertion in this part follows by letting $\eta_1 = \psi_1, \eta_2 = \psi_3, \xi_1 = \psi_2, \xi_2 = \psi_4$. To show that the domain $D_1$ defined by (27) is a Type I domain, we let $u \in D_1$ and write $u = u_0 + \alpha_1\eta_1 + \alpha_2\eta_2 +$

$\beta_1\xi_1 + \beta_2\xi_2$. The conditions $\{u, \eta_1\}(a) = \{\eta_1, u\}(a)$ give $[\eta_1, u](a) = 0$. Therefore

$$\begin{aligned} 0 &= [u, \eta_1](a) \\ &= [u_0, \eta_1](a) + \alpha_1 [\eta_1, \eta_1](a) + \alpha_2 [\eta_2, \eta_1](a) + \beta_1 [\xi_1, \eta_1] + \beta_2 [\xi_2, \eta_1] \\ &= -\beta_1. \end{aligned}$$

We similarly show that $\beta_2 = 0$. Therefore, $D_1 \subset \widetilde{D}$. Since $D_1$ is also a symmetric two-dimensional of $D_0$, $D_1$ is a self-adjoint domain. The converse statement also follows in the same way as in the proof of Part (a) of Theorem 1.

(b): We prove Part (i) only. The first assertion in this part follows by choosing $\eta_1, \eta_2, \eta_3, \eta_4$ as described in Subcase (2, 1, 1). The conditions $\{\eta_3, u\}(b) = \{u, \eta_3\}(b) = 0$ serve to take $\eta_4$ out of $D_1$. The condition $\{u, \eta_1\}(a) = 0$ serves to take $\eta_2$ out of $D_1$. To see this, let $u \in D_1$ and write $u = u_0 + \alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3$. Using the multiplication table (21), we get

$$0 = \{u, \eta_1\}(a) = \alpha_2\{\eta_2, \eta_1\}(a) = 4\alpha_2.$$

Moreover, we automatically have

$$\{\eta_1, u\}(a) = \{\eta_1, u_0 + \alpha_1\eta_1 + \alpha_3\eta_3\}(a) = 0.$$

We can then show (as in Part (a)) that $D_1$ is a Type I domain. Similar arguments work for $D_2$. To prove the converse statement, suppose $\widehat{D}$ is a Type I domain such that $\widehat{D} \neq D_2$. Since $\widehat{D} \subset \widetilde{D}$, $\eta_4 \notin \widehat{D}$. We claim that $\eta_2 \notin \widehat{D}$. If not, then for any $u \in \widehat{D}$ we may write $u = u_0 + \alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3$. Using the multiplication tables in (21) and the condition $\{u, \eta_2\}(a) = 0$, which is satisfied for any two functions in $\widehat{D}$, we get

$$0 = \alpha_1\{\eta_1, \eta_2\}(a) = 4\alpha_1.$$

Hence, $\alpha_1 = 0$ and $u \in D_2$; resulting in $\widehat{D} \subset D_2$ contrary to the assumption. Thus, any $u \in \widehat{D}$ has the representation $u = u_0 + \alpha_1\eta_1 + \alpha_3\eta_3$, which means that $\widehat{D} \subset D_1$. However, since both domains are two-dimensional extensions of $D_0$, we get $\widehat{D} = D_1$.

(c): (i) Showing that $D_i$, $i = 1, 2, 3, 4$ are Type I domains is by now a standard procedure. To show the converse statement suppose $\widehat{D}$ is a Type I domain satisfying separated boundary conditions. Assume $\widehat{D} \neq D_2$. We claim that either $\eta_1 \notin \widehat{D}$ or $\eta_4 \notin \widehat{D}$. If not, then both $\eta_1, \eta_4 \in \widehat{D}$. For any $u \in \widehat{D}$ we may write

$$u = u_0 + \alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3 + \alpha_4\eta_4.$$

Using the multiplication tables in (23) and the conditions $\{u, \eta_1\}(a) = \{u, \eta_3\}(b) = 0$, we get

$$4\alpha_2 = 0,\ 4\alpha_4 = 0.$$

Therefore, $u \in D_1$ and, by a similar argument as above, $\widehat{D} = D_1$. Similarly, we can show that if $\widehat{D} \neq D_4$ then $\widehat{D} = D_3$.

(c): (ii) The direct statement is again straightforward to show. To prove the converse statement, assume that $\widehat{D}$ is a Type I domain satisfying coupled boundary conditions. If $\widehat{D} = D_2(t)$ for some $t \in \mathbb{R}$, then there is nothing to prove. So, suppose that $\widehat{D} \neq D_2(t)$ for all $t \in \mathbb{R}$. Fix a $t \in \mathbb{R}$. We claim that $\xi_3(t) \notin \widehat{D}$. If not then $\xi_3(t) \in \widehat{D}$ . For any

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$u \in \widehat{D}$ we can write

$$u = u_0 + \alpha_1 \xi_1(t) + \alpha_2 \xi_2(t) + \alpha_3 \xi_3(t) + \alpha_4 \xi_4(t).$$

The multiplication tables in (26) together with the conditions $\{\xi_3(t), u\}(a) = \{\xi_3(t), u\}(b)$, $\{u, \xi_3(t)\}(a) = \{u, \xi_3(t)\}(b)$ yield

$$-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = e^{-2t}\alpha_1 - e^{-2t}\alpha_2 + \alpha_3 + \alpha_4,$$
$$-\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = e^{2t}\alpha_1 + e^{2t}\alpha_2 + \alpha_3 - \alpha_4.$$

These two equations give $\alpha_1 = \alpha_2 = 0$. Therefore, $u \in D_2(t)$ and $\widehat{D} \subset D_2(t)$, which is a contradiction. We can similarly show that $\xi_4(t) \notin \widehat{D}$. Thus, $\widehat{D} = D_1(t)$. □

Reflecting back on the proof of the last part of Theorem 3 we see that if $\widehat{D}$ is a Type I domain satisfying coupled boundary conditions then either $\widehat{D} = D_1(t)$ for all $t \in \mathbb{R}$ or $\widehat{D} = D_2(t)$ for all $t \in \mathbb{R}$. In other words, the domains $D_1(t)$ and $D_2(t)$ defined in (36) and (37) are independent of $t \in \mathbb{R}$. This fact could also have been observed from the multiplication tables (26) because, for $i, j = 1, 2$ or $i, j = 3, 4$, the multiplication tables for $\{\xi_i(t), \xi_j(t)\}(\cdot)$ at the endpoints are equal and independent of $t$. We then have the following corollary, which sharpens the results in [**4**].

Corollary 4. *Assume both endpoints are LC. If* $\delta = 0$, *then there are precisely two Type I domains satisfying coupled boundary conditions. These domains are described by* (36) *and* (37) *for any (and hence, all)* $t \in \mathbb{R}$.

The following examples will serve to illustrate Theorem 3.

Example. Let $\ell(y) = \frac{1}{w}[-(x^2y')' + m(m+1)y]$, $m > 0$ defined on $(0, \infty)$, where

$$w(x) = \min\left\{\frac{1}{x^{2m+2}}, x^{2m+2}\right\}.$$

The equation $\ell(y) = 0$ has two solutions $\theta(x) := x^{-(m+1)}$, $\varphi(x) := x^m$ for which $[\theta, \varphi](\cdot) = 2m + 1 > 0$ so that $\theta, \varphi$ are linearly independent modulo $D_0$. Since $\theta, \varphi \in L^2_w(0, \infty)$, $d = 2$. Therefore, both 0 and $\infty$ are LC. Also, $\theta^{[1]}\theta(x) \sim x^{-(2m+1)}$, $\varphi^{[1]}\varphi(x) \sim x^{2m+1}$, $\theta^{[1]}\varphi(x) \sim 1$, $\varphi^{[1]}\theta(x) \sim 1$ (here $\sim$ means equality up to a multiplicative constant). Choose $\eta_1, \eta_2 \in \widetilde{D} \bmod D_0$ and $\xi_1, \xi_2 \in D \bmod \widetilde{D}$ such that

$$\eta_1 = \begin{cases} \varphi \text{ near } 0, \\ 0 \text{ near } \infty, \end{cases} \quad \eta_2 = \begin{cases} 0 \text{ near } 0, \\ \theta \text{ near } \infty, \end{cases}$$
$$\xi_1 = \begin{cases} 0 \text{ near } 0, \\ \varphi \text{ near } \infty, \end{cases} \quad \xi_2 = \begin{cases} \theta \text{ near } 0, \\ 0 \text{ near } \infty. \end{cases}$$

Then we are in Case (2,2). The only Type I domain in this case is described by the boundary conditions

$$D_1 = \{u \in D : \left(x^{m+2}u'\right)(0) = \left(x^{m+1}u\right)(0) = \left(x^{-m+1}u'\right)(\infty) = \left(x^{-m}u\right)(\infty) = 0\}.$$

Example . Consider the operator $\ell(y) = -((1 - x^2)y')'$ defined on $(0, 1)$. The equation $\ell(y) = 0$ has two solutions $\varphi(x) = 1, \theta(x) = \log\frac{1+x}{1-x}$. Choose functions

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$\eta_1, \eta_2, \eta_3 \in \widetilde{D} \bmod D_0$ and $\xi \in D \bmod \widetilde{D}$ such that

$$\eta_1 = \begin{cases} \varphi \text{ near } 0 \\ 0 \text{ near } 1 \end{cases}, \quad \eta_2 = \begin{cases} \theta \text{ near } 0 \\ 0 \text{ near } 1 \end{cases},$$
$$\eta_3 = \begin{cases} 0 \text{ near } 0 \\ \varphi \text{ near } 1 \end{cases}, \quad \xi = \begin{cases} 0 \text{ near } 0 \\ \theta \text{ near } 1 \end{cases}.$$

Then we have a Case (2,1). The two Type I domains $D_1, D_2$ in this case are given by the boundary conditions

$$D_1 = \left\{u \in D : u^{[1]}(0) = u^{[1]}(1) = 0\right\},$$
$$D_2 = \left\{u \in D : u(0) = u^{[1]}(1) = 0\right\}.$$

EXAMPLE. To provide an example for the Case (2,0) consider the operator $\ell(y) = -\left(x^{1/3}y'\right)' + \frac{1}{3}x^{-2/3}y$ defined on $(0, 1)$. The equation $\ell(y) = 0$ has two series solutions $\varphi(x) = x^{2/3}\sigma(x)$ and $\theta(x)$, where

$$\sigma(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\, 5 \cdot 8 \cdots (3n+1)},$$
$$\theta(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\, 1 \cdot 4 \cdots (3n-3)}.$$

It can be easily checked that $\varphi(x), \theta(x) \in \widetilde{D} \bmod D_0$. Choose functions $\eta_i \in \widetilde{D} \bmod D_0$, $1 \le i \le 4$ such that

$$\eta_1(x) = \begin{cases} \varphi(x) \text{ near } 0, \\ 0 \text{ near } 1, \end{cases} \quad \eta_2(x) = \begin{cases} \theta(x) \text{ near } 0, \\ 0 \text{ near } 1, \end{cases}$$
$$\eta_3(x) = \begin{cases} 0 \text{ near } 0, \\ \widetilde{\varphi}(x) \text{ near } 1, \end{cases} \quad \eta_4(x) = \begin{cases} 0 \text{ near } 0, \\ \widetilde{\theta}(x) \text{ near } 1, \end{cases}$$

where $\widetilde{\varphi}, \widetilde{\theta}$ are linear combinations of $\varphi, \theta$ such that $\widetilde{\varphi}(1) = 0 = \widetilde{\theta}'(1)$, $\widetilde{\varphi}'(1) = \widetilde{\theta}(1) = 1$. Then $\{\widetilde{\varphi}, \widetilde{\varphi}\}(1) = \{\widetilde{\theta}, \widetilde{\theta}\}(1) = 0$. The four Type I domains with separated boundary conditions are

$$D_1 = \left\{u \in D : u^{[1]}(0) = u(1) = 0\right\},$$
$$D_2 = \left\{u \in D : u^{[1]}(0) = u'(1) = 0\right\},$$
$$D_3 = \{u \in D : u(0) = u(1) = 0\},$$
$$D_4 = \left\{u \in D : u(0) = u'(1) = 0\right\}.$$

To obtain Type I domains with coupled boundary conditions we choose functions $\psi_i \in \widetilde{D} \bmod D_0$, $1 \le i \le 4$ such that

$$\psi_1(x) = \begin{cases} \theta(x) - \varphi(x) \text{ near } 0, \\ 0 \text{ near } 1, \end{cases} \quad \psi_2(x) = \begin{cases} \theta(x) + \varphi(x) \text{ near } 0, \\ 0 \text{ near } 1, \end{cases}$$
$$\psi_3(x) = \begin{cases} 0 \text{ near } 0, \\ \widehat{\varphi}(x) \text{ near } 1, \end{cases} \quad \psi_4(x) = \begin{cases} 0 \text{ near } 0, \\ \widehat{\theta}(x) \text{ near } 1, \end{cases}$$

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where $\widehat{\varphi}, \widehat{\theta}$ are linear combinations of $\varphi, \theta$ such that $\widehat{\varphi}(1) = \widehat{\varphi}'(1) = -1$, $\widehat{\theta}(1) = -1 = \widetilde{\theta}'(1) = 1$. Defining the functions $\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)$ as in (24), (25) we get the multiplication tables (26). The two 'one-parameter families of' domains in this case are given by

$$D_1 = \left\{u \in D : u(0) = u(1),\ u^{[1]}(0) = u^{[1]}(1)\right\},$$
$$D_2 = \left\{u \in D :\ u(0) = -u(1),\ u^{[1]}(0) = -u^{[1]}(1)\right\},$$

which are in agreement with Corollary 4.

**4. Boundary conditions in the regular case.** When the formally self-adjoint expression $\ell$ is regular, $a$ and $b$ are finite and $1/p$, $q$ are integrable on $(a, b)$. Any function $u \in D$ is absolutely continuous as well as its pseudo-derivative on $[a, b]$. Furthermore, $u$ and $u^{[1]}$ attain arbitrary complex values at $a$ and $b$. Hence, in the regular case $\widetilde{D} = D$, $d = 2$ and $\delta = 0$. All these properties make it possible to obtain boundary conditions describing Type I domains directly in terms of the boundary values of $u$ and $u^{[1]}$. The goal of this section is to specialise the results in Subsection 3.2 to the regular case.

For Type I operators with separated boundary conditions, choose real functions $\eta_1, \eta_2, \eta_3, \eta_4 \in D$ such that

$$\eta_1(a) = 1,\ \eta_1^{[1]}(a) = 0,\ \eta_1(b) = 0,\ \eta_1^{[1]}(b) = 0,$$
$$\eta_2(a) = 0,\ \eta_2^{[1]}(a) = 1,\ \eta_2(b) = 0,\ \eta_2^{[1]}(b) = 0,$$
$$\eta_3(a) = 0,\ \eta_3^{[1]}(a) = 0,\ \eta_3(b) = 1,\ \eta_3^{[1]}(b) = 0,$$
$$\eta_4(a) = 0,\ \eta_4^{[1]}(a) = 0,\ \eta_4(b) = 0,\ \eta_4^{[1]}(b) = 1.$$

THEOREM 5 (Type I domains in the regular case). *Assume both a and b are regular. Then the boundary values of functions belonging to Type I domains are described as follows*:

(1) *The four domains given by*

$$D_1 = \left\{u \in D : u^{[1]}(a) = u^{[1]}(b) = 0\right\},$$
$$D_2 = \left\{u \in D : u^{[1]}(a) = u(b) = 0\right\},$$
$$D_3 = \left\{u \in D : u(a) = u^{[1]}(b) = 0\right\},$$
$$D_4 = \{u \in D : u(a) = u(b) = 0\}$$

*are Type I domains with separated boundary conditions. Conversely, if* $\widehat{D}$ *is a Type I domain with separated boundary conditions then* $\widehat{D}$ *is equal to one of the above four domains.*

(2) *The two domains given by*

$$D_1 = \left\{u \in D : u(a) = u(b),\ u^{[1]}(a) = u^{[1]}(b)\right\},$$
$$D_2 = \left\{u \in D :\ u(a) = -u(b),\ u^{[1]}(a) = -u^{[1]}(b)\right\}$$

*are Type I domains with coupled boundary conditions. Conversely, if* $\widehat{D}$ *is a Type I domain with coupled boundary conditions then* $\widehat{D}$ *is equal to one of the above two domains.*

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