COLLISIONAL RELAXATION : A NEW APPROACH

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ABSTRACT. Recent numerical simulations for 1-dimensional systems have shown that the relaxation time due to encounters is far shorter than the generally accepted estimate. To account for this, a new approach to the theory is necessary. The analysis of encounters presented here is characterized by the retention of periodic trajectories in the mean field. The kinetic equation obtained yields a relaxation time scale in qualitative agreement with the simulations. The analysis can be extended to the 3-dimensional case, and preliminary results predict here also a reduction of the relaxation time.

Our present understanding of the relaxation process in gravitational systems has not significantly progressed with respect to Chandrasekhar's original formulation (1) in 1941. The various refinements and reformulations introduced over the years have left his results essentially unchanged. All existing theories have in common one important approximation : the influence of the mean gravitational field is not taken into account in the analysis of encounters. These are described as perturbations of straight trajectories, and one obtains, after the ad hoc suppression of a logarithmic divergence, a relaxation time t_R proportional to N t_D/log N, where N is the number of particles in the system and t_D a characteristic dynamical time (crossing time).

Applied to 1-dimensional systems, this approach yields $N^2 t_D$ as minimum estimate for t_R : indeed in 1-d, simple binary encounters cannot give rise to energy exchange, as a result of the constraints imposed by the conservation laws. Recent numerical simulations (2), (3), have however established that collisional relaxation develops on a far shorter time scale $t_R < N\ t_D$. A typical example is provided by Fig. 1, giving the evolution in a 2-mass system of the ratio k of the average kinetic energies for the 2 masses : one sees that k(t) effectively relaxes towards the equipartition value k = 1 on a time scale much shorter than N t_D. Moreover, the results exhibited Fig. 2 suggest that the relaxation time for equipartition increase linearly with N.

These simulation results make it imperative to reconsider the classical analysis of encounters. For the 1-d problem one is dealing

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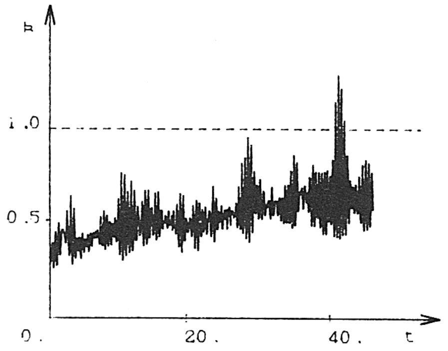

Fig. 1 Evolution of $k = \langle m_1 v_1^2/2\rangle \langle m_h v_h^2/2\rangle$, with $m_1/m_h = 1/3$, $N_1/N_h = 3$ and $N = N_1 + N_h = 200$. The time t is measured in units t_D.

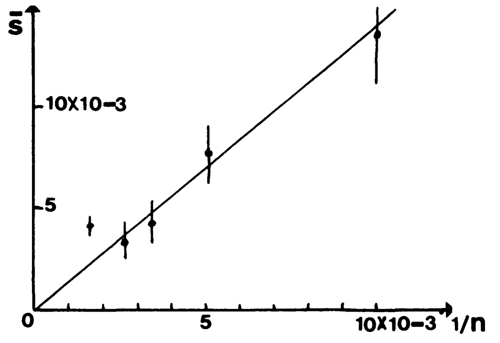

Fig. 2 The slope s of the linear fit to the k(t) plots, in terms of 1/N. Since $s = (t_D/t_R)(m_h-m_1)/m_1$, t_R varies linearly with N.

with a system of mass sheets, interacting through binary forces $ma_{12}=-2\pi Gm^2\,\mathrm{sgn}(x_1-x_2)$; for simplicity, we take particles (sheets) of the same mass (density) m. The particles are confined in the self-consistent mean field mA(x,t), which, contrary to the classical analysis, we retain as a dominant effect. The feasibility of the problem results from the fact that, once the violent relaxation phase is over, mA(x,t) can quite realistically be modelled by the constant harmonic oscillator field $-m\omega^2 x$, with $\omega=2\pi t_D^{-1}$.

The dynamical analysis for the N-body system proceeds from the BBGKY hierarchy and, with the usual approximations on the higher order and initial correlation functions, leads, see ref. (4), to the following equation for the 1-particle distribution function $f_1(t)\equiv f(x_1,v_1,t)$:

$$(\partial_t+L_1)f_1(t) = \int dx_2 dv_2 \int_o^t d\tau\, \partial_{12} a(x_{12}) a(x_{12}\cos\omega\tau-\omega^{-1}v_{12}\sin\omega\tau)$$
$$[\partial_{12}\cos\omega\tau + \omega^{-1}\nabla_{12}\sin\omega\tau]\,[\exp-(L_1+L_2)\tau]\,f_1(t-\tau)f_2(t-\tau) \quad (2)$$

$$L_i = v_i\,\partial/\partial x_i - \omega^2 x_i\,\partial/\partial v_i,$$

$$x_{12}=x_1-x_2,\; v_{12}=v_1-v_2,\; \partial_{12}=\partial/\partial v_1-\partial/\partial v_2,\; \nabla_{12}=\partial/\partial x_1-\partial/\partial x_2.$$

To focus attention on the time integration, we rewrite Eq.(2) as

$$(\partial_t+L_1)f_1(t) = \int_o^t d\tau F(\tau)\,[\exp-(l_1+L_2)\tau]\,f_1(t-\tau)f_2(t-\tau). \quad (3)$$

F(τ) is a periodic function of τ (and an operator on position and velocity). The exponential operator advances f(t−τ) along the unperturbed orbit so that one can approximate

$$[\exp-(L_1+L_2)\tau]\; f_1(t-\tau)f_2(t-\tau) \cong f_1(t)f_2(t) \quad (4)$$

as long as the perturbation due to the encounters remain small. If we ignore this restriction and take the usual time limit $t \to \infty$, we obtain, writing the integral as a sum over successive periods $t_D = 2\pi\omega^{-1}$,

$$(\partial_t + L_1) f_1(t) = \sum_{n=0}^{p\to\infty} \int_{nt_D}^{(n+1)t_D} d\tau \; F(\tau) f_1(t) f_2(t) \tag{5}$$

$$= \lim_{p\to\infty} p \left[\int_0^{t_D} d\tau F(\tau) \right] f_1(t) f_2(t) \tag{6}$$

The manifest divergence of this result comes from the approximation of Eq. (4), which must break down over times τ of the order of the relaxation time t_R. However the assumption of perfect harmonic motion clearly becomes unrealistic well before that : non-isochronic trajectories, as also fluctuations of the mean field, will introduce a progressive loss of coherence between the successive contributions to Eq. (5). Heuristically, we can proceed by retaining only the first p terms of the sum, with $pt_D < t_R$, thus obtaining for Eq.(2) :

$$(\partial_t + L_1) f_1(t) = 4\pi G^2 m^2 p \int dx_2 dv_2 \partial_{12} \operatorname{sgn}(x_{12} v_{12})$$
$$\left[\sin\omega\tau_{12} \partial_{12} - \omega^{-1} (\cos\omega\tau_{12}) \nabla_{12} \right] f_1(t) f_2(t) \tag{7}$$

with $\tau_{12} = \operatorname{Arc\,tan} \omega x_{12}/v_{12}$.

Eq. (7) is a generalized Fokker-Planck equation : in the collision integral, velocity diffusion is described by the $\partial_{12}\partial_{12}$ contribution and gives rise to a monotonous growth in entropy, while the ∇_{12} term ensures the stationary of the equilibrium distribution $\exp -m(v^2 + \omega^2 x^2)/2$. In order of magnitude, Eq. (7) gives for the relaxation time : $t_R^{-1} \propto p.G^2 m^2 N/\langle v^2 \rangle$. On the other hand, one has $\omega^2 = 4\pi GmN/L$, with $L \cong \sqrt{\langle v^2 \rangle}/\omega$. Since $t_D \propto \omega^{-1}$, it follows that $t_R \propto Nt_D/p$ with the coherence factor p to be determined from simulations. This result constitutes a radical reduction, by a factor Np, with respect to the standard estimate, and is in qualitative agreement with our simulations, refs. (2), (3).

The 1-d analysis can be extended (4) to three dimensions, but here the time integration has not yet been worked out. Nevertheless it is clear that there is no longer a divergence at large particle separations and that, in qualitative agreement with recent 3-d simulations (5), a moderate reduction of the relaxation time (by the coherence factor p) is to be expected.

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