THE b-FUNCTIONS AND HOLONOMY DIAGRAMS OF IRREDUCIBLE REGULAR PREHOMOGENEOUS VECTOR SPACES

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Introduction

The purpose of this paper is to investigate the micro-local structure and to calculate, by constructing the holonomy diagrams, the b-functions (See [2]) of irreducible regular prehomogeneous vector spaces (See [1]).

Since we know the relation of b-functions with respect to castling transformations (See § 12), it is enough to calculate them only when they are reduced. In this paper, we shall deal with twenty of all twenty nine reduced regular P.V.'s in the Table in [1]. Together with other articles, this completes the list of b-functions of irreducible reduced regular prehomogeneous vector spaces (See § 12) except $(SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$ which is the most complicated case (See I. Ozeki [11]). This paper consists of the following twelve sections and one Appendix with I. Ozeki.

Appendix with I. Ozeki. $(GL(1) \times \mathrm{Spin}\ (14),\ \square \otimes$ half-spin rep., $V(1) \otimes V(64))$

In § 1, we shall review the main results of [2] which will be used later. From § 2 to § 11, we do the classification of the orbits, construction of the holonomy diagrams and calculation of the b-functions. In § 12, we shall give the list of b-functions for irreducible reduced regular P.V.'s. Some of them have been already calculated by M. Sato and the author using the different method (See [7]). The holonomy diagrams in § 2, § 8 and § 10 are first obtained by M. Sato. The author would like to express his hearty thanks to Professors Mikio Sato and Masaki Kashiwara for their invaluable advice and encouragement.

§ 1. Preliminaries

Let (G, ρ, V) be an irreducible regular prehomogeneous vector space (abbrev. P.V.) with the singular set S. Then S is the zeros of the relative invariant $f(x) : S = \{x \in V; f(x) = 0\}$, $f(\rho(g)x) = \chi(g)f(x)$ for all $g \in G$ and $x \in V$. We shall consider the micro-differential equations $\mathfrak{M} = \mathscr{E}f^s$ where $\mathscr{E}$ is the sheaf of micro-differential operators of finite order on the cotangent bundle $T^*V = V \times V^*$ (See [2]). Note that the group G acts on T^*V by $(x, y) \mapsto (\rho(g)x, \rho^*(g)y)$ for $x \in V$, $y \in V^*$ and $g \in G$ where ρ^* denotes the contragredient representation of ρ. Let Λ be the Zariski-closure of a conormal bundle of some G-orbit $\rho(G) \cdot x_0 (x_0 \in V)$. Since we consider only *the Zariski-closure* of a conormal bundle, we shall omit the word "the Zariski-closure" for simplicity. Assume that Λ is G-prehomogeneous and is contained in $W = \overline{\{(x, \mathrm{grad}\ \log f(x)^s); x \in V - S, s \in C\}}$. In this case, Λ is called a good holonomic variety. It is an irreducible component of the characteristic variety of $\mathfrak{M}$. We can show that there exists a local b-function $b_\Lambda(s)$ which is unique up to a constant multiple (See [2]). We have $b_{V \times \{0\}}(s) = 1$ and $b_{\{0\} \times V^*}(s) = b(s)$ where $b(s)$ denotes the b-function of this P.V. When two good holonomic varieties Λ_0 and Λ_1 intersect with codimension one, we have the relation between $b_{\Lambda_0}(s)$ and $b_{\Lambda_1}(s)$ as follows (See [2]).

THEOREM 1–1 ([2] Theorem 7–5). *Let Λ_0 and Λ_1 be good holonomic varieties whose intersection is of codimension one with the intersection exponent $(\mu : \nu)$. Assume that $\mathfrak{M} = \mathscr{E}f^s$ is a simple holonomic system with support $\Lambda_0 \cup \Lambda_1$ and $\Lambda_0 \cap \Lambda_1 \not\subset \overline{\mathrm{supp}\ \mathfrak{M} - (\Lambda_0 \cup \Lambda_1)}$. Assume that $m_0 > m_1$ where*

$\operatorname{ord}_{\Lambda_i} f^s = -m_i s - \mu_i/2$ $(i = 0, 1)$. *Then we have, up to a constant multiple,*

(1.1)
$$b_{\Lambda_0}(s)/b_{\Lambda_1}(s) = \prod_{k=0}^{\nu} \left[\frac{1}{\nu+1}(\operatorname{ord}_{\Lambda_1} f^s - \operatorname{ord}_{\Lambda_0} f^s) + \frac{\mu + 2k}{2(\nu+\mu)} \right]^{(m_0-m_1)/(\nu+1)}$$
$$\text{where } [\alpha]^k = \alpha(\alpha + 1) \cdots (\alpha + k - 1).$$

Here we denote by $\operatorname{ord}_{\Lambda} f^s$ the order of f^s at Λ (See [2]). Note that m_0 and m_1 are non-negative integers, and $(\mu : \nu) = (1 : 0)$ or $(\mu : \nu)$ is a pair of positive integers satisfying $\mu \geq 2$, $\nu \geq 1$, and $(m_0 - m_1)$ is a multiple of $(\nu + 1)$.

COROLLARY 1–2 ([2] Corollary 7–6). *If Λ_0 and Λ_1 intersect regularly, i.e., $\mu = 1$ and $\nu = 0$, we have*

(1.2)
$$b_{\Lambda_0}(s)/b_{\Lambda_1}(s) = \prod_{k=1}^{m_0-m_1} \left((m_0 - m_1)s + \frac{\mu_0 - \mu_1 - 1}{2} + k \right)$$
$$\text{where } \operatorname{ord}_{\Lambda_i} f^s = -m_i s - \frac{\mu_i}{2} \quad (i = 0, 1).$$

Let Λ be a good holonomic variety. Then $\Lambda = \overline{G(x_0, y_0)}$ for some $x_0 \in V$, $y_0 \in V^*$ where $G(x_0, y_0) = \{(\rho(g)x_0, \rho^*(g)y_0); g \in G\}$. In this case, we can calculate the order $\operatorname{ord}_{\Lambda} f^s$ by the following proposition.

PROPOSITION 1–3 ([2] Proposition 4–14). *Let A_0 be an element of the Lie algebra $\mathfrak{g}$ of G satisfying $d\rho(A_0)x_0 = 0$ and $d\rho^*(A_0)y_0 = y_0$. Then we have*

(1.3)
$$\operatorname{ord}_{\Lambda} f^s = s\delta\chi(A_0) - \operatorname{tr}_{V_{x_0}^*} d\rho_{x_0}(A_0) + \tfrac{1}{2} \dim V_{x_0}^*$$

where $V_{x_0}^*$ denotes the conormal vector space $(d\rho(\mathfrak{g}) \cdot x_0)^\perp$, and $d\rho_{x_0}$ denotes the representation of $\mathfrak{g}_{x_0} = \{A \in \mathfrak{g}; d\rho(A)x_0 = 0\}$ induced by $d\rho^*$.

Now let $\Lambda_0 = \overline{G(x_0, y_0)}$ and $\Lambda_1 = \overline{G(x_1, y_1)}$ be good holonomic varieties such that $(x_0, y_1) \in \Lambda_0 \cap \Lambda_1$ and $\dim G(x_0, y_1) = \dim V - 1$. In this case, the intersection exponent $(\mu : \nu)$ is given by the following proposition.

PROPOSITION 1–4 ([2] Proposition 6–5). *Let A_1 be an element of $\mathfrak{g}$ satisfying $d\rho(A_1)x_0 = 0$ and $d\rho^*(A_1)y_1 = y_1$. Then A_1 acts on the one-dimensional vector space $\tilde{V} = V_{x_0}^*$ modulo $d\rho_{x_0}(\mathfrak{g}_{x_0})y_1$. Let β be its eigenvalue, i.e., $\beta = \operatorname{tr}_{\tilde{V}} A_1$. Then μ and ν are given by $\beta = \mu/(\mu + \nu)$, $(\mu, \nu) = 1$. If β is not determined uniquely, i.e., β depends on A_1, then we have $\mu = 1$, $\nu = 0$, and Λ_0, Λ_1 intersect regularly.*

Let $\Lambda = \overline{T(\rho(G)x_0)}^{\perp}$ be a conormal bundle of a G-orbit $\rho(G)x_0$. Then G acts on Λ prehomogeneously if and only if the colocalization $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ at x_0 is a P.V. We shall consider some sufficient conditions that $\Lambda \subset W$, i.e., Λ is a good holonomic variety.

PROPOSITION 1–5 ([2] Proposition 6–6). *Let Λ_0 and Λ_1 be two conormal bundles of some G-orbits. Assume that $\dim \mathfrak{g}_0 \cdot p = \dim V - 1$ for some $p \in \Lambda_0 \cap \Lambda_1$ where $\mathfrak{g}_0 = \{A \in \mathfrak{g}; \delta\chi(A) = 0\}$. Assume that Λ_0 (or Λ_1) $\subset W$. Then we have $\Lambda_0 \cup \Lambda_1 \subset W$. Moreover W is non-singular and $W = \{(x, y) \in V \times V^*; \langle d\rho(A)x, y \rangle = 0$ for all $A \in \mathfrak{g}_0\}$ near p.*

Let $V_{x_0} = V \bmod d\rho(\mathfrak{g})x_0$ be the normal vector space. Then the isotropy subgroup G_{x_0} acts on V_{x_0}. We denote this action by $\tilde{\rho}_{x_0}$. Let f_{x_0} be the localization of $f(x)$ at x_0 (See [2]). This is a relative invariant of $(G_{x_0}, \tilde{\rho}_{x_0}, V_{x_0})$ corresponding to $\chi | G_{x_0}$. Let S_{x_0} be the singular set of $(G_{x_0}, \tilde{\rho}_{x_0}, V_{x_0})$.

PROPOSITION 1–6 ([2] Proposition 6–9). *If $\operatorname{grad} \log f_{x_0}: V_{x_0} - S_{x_0} \to V_{x_0}^*$ is generically surjective, then $\Lambda_0 = \overline{T(\rho(G)x_0)}^{\perp} \subset W$, i.e., Λ_0 is a good holonomic variety.*

COROLLARY 1–7 ([2] Corollary 6–10). *Assume that the colocalization $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ of (G, ρ, V) at x_0 ($\in V$) is a regular P.V. If $\delta\chi|\mathfrak{g}_{x_0}$ is a non-degenerate element, then the conormal bundle $\Lambda_0 = \overline{T(\rho(G)x_0)}^{\perp}$ of the G-orbit $\rho(G)x_0$ is a good holonomic variety.*

COROLLARY 1–8 ([2] Corollary 6–11). *Assume that the colocalization $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ of (G, ρ, V) at x_0($\in V$) is an irreducible regular P.V. Then the conormal bundle $\Lambda_0 = \overline{T(\rho(G)x_0)}^{\perp}$ of the orbit $\rho(G)x_0$ is a good holonomic variety.*

PROPOSITION 1–9 ([1] Proposition 14 in § 4).
(1) *For $d = \deg f$ and $n = \dim V$, we have $d | 2n$ and $\chi(g)^{2n/d} = \det_V \rho(g)^2$ for $g \in G$.*
(2) $\delta\chi(A) = (d/n) \operatorname{tr} d\rho(A)$ *for $A \in \mathfrak{g}$.*

Remark 1–10. Let (G, ρ, V) be an irreducible regular P.V. with finitely many orbits. Let $\mathscr{L} = \{\Lambda, \Lambda', \cdots, \Lambda''\}$ be the set of all conormal bundles in W, of some G-orbits in V. The holonomy diagram is, by definition, given as follows.

If $\dim \Lambda \cap \Lambda' = \dim V - 1$, and $\Lambda \cap \Lambda' \not\subset \Lambda''$ for any other Λ'' in $\mathscr{L}$, then we write the diagram as in Figure 1–1. Moreover, if Λ and Λ' are good
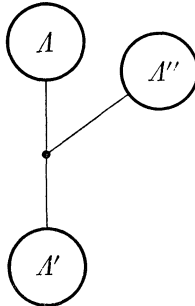
Figure 1-1.   Figure 1-2.   Figure 1-3. $(m_{\Lambda'} > m_{\Lambda})$
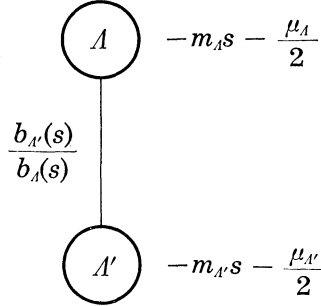
holonomic varieties, we write the orders $\mathrm{ord}_{\Lambda} f^s = - m_{\Lambda}s - \mu_{\Lambda}/2$ for Λ and Λ', and the ratio of the b-functions as in Figure 1–3. If $\dim \Lambda \cap \Lambda' = \dim V - 1$ and $\Lambda \cap \Lambda' \subset \Lambda''$ for some Λ'', then we write the diagram as in Figure 1–2 (e.g. Figure 11–1). Although some general theory for such cases has been established, it is not published yet and hence in this paper we avoid to argue this case. Actually, only in § 11, such case will appear and to calculate the b-function in § 11, we can use another part of the holonomy diagram. Although usually we do not write the conormal bundles outside W (e.g. Figure 3–2), sometimes we write them (e.g. Figure 4–1, Figure 11–1). Since G is reductive, we have $(G, \rho, V) \cong (G, \rho^*, V^*)$ and we identify them.

We sometimes write as Λ Λ' when T and T' are the dual orbits of each other (See § 11) where Λ and Λ' are the conormal bundles of T and T' respectively. If $T = T'$, we write as Λ (e.g. Figure 4–1 and Figure 11–1).

§ 2. Regular P.V.'s related with $GL(n)$

We shall use the same notations as in [1].

2-1. $(\tilde{G} \times GL(m), \tilde{\rho} \otimes \Lambda_1, V(m) \otimes V(m))$ where $\tilde{\rho} : \tilde{G} \to GL(V(m))$ is an m-dimensional irreducible representation of a connected semi-simple algebraic group $\tilde{G}$ (or $\tilde{G} = \{1\}$ and $m = 1$)

The representation space $V = V(m) \otimes V(m)$ can be identified with the totality of $m \times m$ matrices $M(m, C)$. Then the action $\rho = \tilde{\rho} \otimes \Lambda_1$ is given by $\rho(g)X = \tilde{\rho}(g_1)X\,{}^t g_2$ for $g = (g_1, g_2) \in G = \tilde{G} \times GL(m)$, $X \in M(m, C)$. The relative invariant $f(X)$ is given by the determinant: $f(X) = \det X$. Since we are concerned with relative invariants, we may assume that $\tilde{G} = SL(m)$ and $\tilde{\rho} = \Lambda_1$. It is well-known that there exist $(m + 1)$-orbits

$$\rho(G)X_\mu = \{X \in M(m, C); \operatorname{rank} X = \mu\}$$

where $\quad X_\mu = \begin{pmatrix} 1 & & & & & \\ & \ddots & \mu & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & 0 & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$ for $\mu = 0, 1, \cdots, m$.

We identify the dual V^* of V with $V = M(m, C)$ by $\langle X, Y \rangle = \operatorname{tr} {}^t XY$ for $X, Y \in M(m, C)$. Then the dual ρ^* of ρ is given by $\rho^*(g)Y = {}^t g_1^{-1} Y g_2^{-1}$ for $g = (g_1, g_2) \in G = SL(m) \times GL(m)$, $Y \in M(m, C)$.

Since $d\rho(\tilde{A})X_\mu = AX_\mu + X_\mu {}^t B = \left(\begin{array}{c|c} A_1 + {}^t B_1 & {}^t B_3 \\ \hline A_3 & 0 \end{array} \right)$ for $\tilde{A} = (A, B) \in \mathfrak{g}$ with $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, the conormal vector space $V^*_{X_\mu} = (d\rho(\mathfrak{g})X_\mu)^\perp$ is given by $V^*_{X_\mu} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Y_\mu \end{pmatrix}; Y_\mu \in M(m - \mu, C) \right\}$. The isotropy subalgebra $\mathfrak{g}_{X_\mu} = \{\tilde{A} \in \mathfrak{g}; d\rho(\tilde{A})X_\mu = 0\}$ is given by

(2.1)
$$\mathfrak{g}_{X_\mu} = \left\{ \left(\begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}, \begin{pmatrix} -{}^t A_1 & B_2 \\ 0 & B_4 \end{pmatrix} \right) \in \mathfrak{g}; A_1 \in M(\mu, C), A_2, B_2 \right.$$
$$\left. \in M(\mu, m - \mu, C), A_4, B_4 \in M(m - \mu, C) \right\} .$$

This $\mathfrak{g}_{X_\mu}$ acts on $V^*_{X_\mu}$ as $d\rho_{X_\mu}(\tilde{A})Y_\mu = -{}^t A_4 Y_\mu - Y_\mu B_4$ for $\tilde{A} \in \mathfrak{g}_{X_\mu}$. Therefore we have $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu}) \cong (SL(m - \mu) \times GL(m - \mu), \Lambda_1 \otimes \Lambda_1, V(m - \mu) \otimes V(m - \mu))$. Put $Y_\mu = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-\mu} \end{pmatrix}$ where $I_{m-\mu}$ denotes the unit matrix of size $m - \mu (\mu = 0, 1, \cdots, m)$. Then Y_μ is a generic point of the colocalization $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu})$, and $Y_{\mu+1}$ is a point of the one-codimensional orbit $(\mu \leq m - 1)$. We denote by Λ_μ the conormal bundle of $\rho(G)X_\mu$ $(0 \leq \mu \leq m)$. Then, we have $\dim \Lambda_\mu \cap \Lambda_{\mu+1} = \dim V - 1$. Note that the colocalization $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu})$ $(\mu = 0, 1, \cdots, m)$ has finitely many orbits with the unique one-codimensional orbit, it is clear that we have obtained all one-codimensional intersections among Λ_μ $(\mu = 0, 1, \cdots, m)$. Since $\mathfrak{g}_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(m)$, we have $\dim \mathfrak{g}_0(X_\mu, Y_{\mu+1}) = m^2 - 1$ for $\mu = 0, 1, \cdots, m - 1$, and hence by Proposition 1–5, we have $\Lambda_\mu \subset W$, i.e., Λ_μ is a good holonomic variety $(0 \leq \mu \leq m)$. Note that $\Lambda_m = V \times \{0\}$ is always a good holonomic variety. We shall calculate the intersection exponent $(\tilde{\mu} : \tilde{\nu})$ of Λ_μ and $\Lambda_{\mu+1}$ by using Proposition 1–4. For any $\beta \in C$, put $A_\mu^\beta = \left((0), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{array}{c} -\beta \\ -I_{m-\mu-1} \end{array} \end{array} \right) \right) \in \mathfrak{g}$. Then we have $d\rho(A_\mu^\beta)X_\mu = 0$, $d\rho^*(A_\mu^\beta)Y_{\mu+1} = Y_{\mu+1}$, and $\beta = \operatorname{tr} A_\mu^\beta$ where tr denotes

the trace of A_μ^β on $V_{X_\mu}^*$ modulo $d\rho_{X_\mu}(\mathfrak{g}_{X_\mu})Y_{\mu+1}$ since $V_{X_\mu}^*$ modulo $d\rho_{X_\mu}(\mathfrak{g}_{X_\mu})Y_{\mu+1}$
$\cong \{yE_{\mu+1,\mu+1} \in M(m, C); y \in C\}$ where E_{ij} denotes the matrix unit. Therefore
we have $\tilde{\mu} = 1$ and $\tilde{\nu} = 0$, i.e., they intersect regularly. Now by Proposition
1–3, we shall calculate the order $\operatorname{ord}_{\Lambda_\mu} f^s$ of $\mathfrak{m} = \mathscr{E} f^s$ at Λ_μ where $f(X) =$
$\det X$.

Put $A_\mu = \left((0), \begin{pmatrix} 0 & 0 \\ 0 & -I_{m-\mu} \end{pmatrix}\right) \in \mathfrak{g}$. Then $d\rho(A_\mu)X_\mu = 0$ and $d\rho^*(A_\mu)Y_\mu =$
Y_μ $(0 \le \mu \le m)$. The character $\delta\chi$ corresponding to $f(X) = \det X$ is given
by $\delta\chi(\tilde{A}) = \operatorname{tr} B$ for $\tilde{A} = (A, B) \in \mathfrak{g} = \mathfrak{sl}(m) \oplus \mathfrak{gl}(m)$. Since $\dim V_{X_\mu}^* = (m - \mu)^2$
and $\operatorname{tr}_{V_{X_\mu}^*} d\rho_{X_\mu}(A_\mu) = (m - \mu)^2$, we have $\operatorname{ord}_{\Lambda_\mu} f^s = s\delta\chi(A_\mu) - \operatorname{tr}_{V_{X_\mu}^*} d\rho_{X_\mu}(A_\mu) +$
$(1/2) \dim V_{X_\mu}^* = -(m - \mu)s - ((m - \mu)^2/2)$. Thus we obtain the holonomy
diagram (Figure 2–1).

By Corollary 1–2, we have $b_{\Lambda_\mu}(s)/b_{\Lambda_{\mu+1}}(s) = s + (m - \mu) (0 \le \mu \le m - 1)$.
Hence

$$b(s) = b_{\Lambda_0}(s) = b_{\Lambda_m}(s) \cdot \prod_{\mu=0}^{m-1} b_{\Lambda_\mu}(s)/b_{\Lambda_{\mu+1}}(s)$$

$$= \prod_{\mu=0}^{m-1} (s + m - \mu) = (s + 1)(s + 2) \cdots (s + m).$$

Note that $b_{\Lambda_m}(s) = 1$.
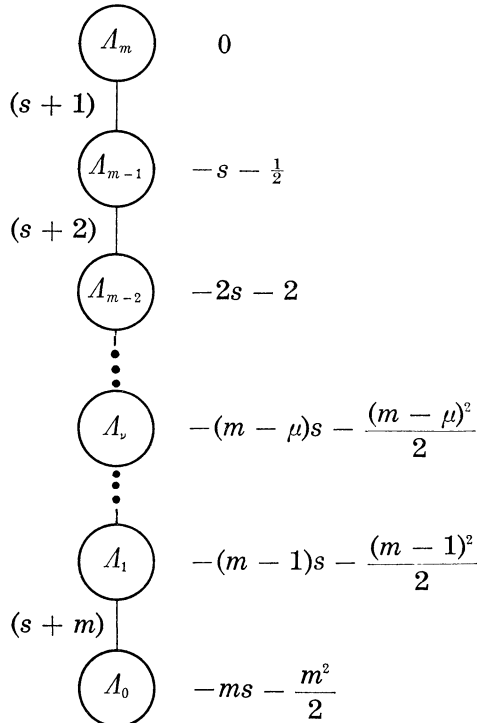
Figure 2–1. Holonomy diagram of $(SL(m) \times GL(m),\ \Lambda_1 \otimes \Lambda_1,\ V(m) \otimes V(m))$

Remark 2–1. The *b*-function of **2–1** is classically known by using Capelli's identity (See H. Weyl: Classical Groups).

2–2. $(GL(n), 2\Lambda_1, V(\frac{1}{2}n(n+1)))$ $(n \geq 2)$

The representation space can be identified with the totality of $n \times n$ symmetric matrices $V = \{X \in M(n, \boldsymbol{C}); {}^t X = X\}$. Then the action $\rho = 2\Lambda_1$ of $GL(n)$ on V is given by $\rho(g)X = gX{}^t g$ for $g \in GL(n)$, $X \in V$. It is well-known that there exist $(n+1)$-orbits $\rho(G)X_\nu = \{X \in V; \text{rank } X = \nu\}$ where

$$
X_\nu = \begin{pmatrix} 1 & & & & & & \\ & \ddots & \nu & & 0 & & \\ & & 1 & & & & \\ & & & 0 & & & \\ & 0 & & & \ddots & & \\ & & & & & 0 \end{pmatrix} \text{ for } \nu = 0, 1, \cdots, n \ .
$$

The relative invariant is given by the determinant $f(X) = \det X$. If we identify the dual V^* of V with V by $\langle X, Y \rangle = \operatorname{tr} XY$, we have $\rho^*(g)Y = {}^t g^{-1} Y g^{-1}$ for $g \in GL(n)$. We have

$$
(2.2) \qquad d\rho(A)X_\nu = AX_\nu + X_\nu {}^t A = \left(\begin{array}{c|c} A_1 + {}^t A_1 & {}^t A_3 \\ \hline A_3 & 0 \end{array} \right)
$$

$$
\text{for} \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathfrak{gl}(n) \ .
$$

Therefore we have

$$
(2.3) \qquad \mathfrak{g}_{X_\nu} = \left\{ \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}; {}^t A_1 = -A_1, A_1 \in M(\nu), A_2 \in M(\nu, n-\nu), \right.
$$

$$
\left. A_4 \in M(n-\nu) \right\} \ .
$$

Since $\dim \mathfrak{g}_{X_\nu} = n(n-\nu) + (\nu(\nu-1)/2)$, we have $\dim \rho(G)X_\nu = n\nu - (\nu(\nu-1)/2)$. The conormal vector space $V^*_{X_\nu}$ is given by

$$
(2.4) \qquad V^*_{X_\nu} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & W_\nu \end{pmatrix}; {}^t W_\nu = W_\nu \in M(n-\nu) \right\} \ .
$$

Since $d\rho^*\left(\begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \right) \begin{pmatrix} 0 & 0 \\ 0 & W_\nu \end{pmatrix} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -{}^t A_4 W - W A_4 \end{array} \right)$, the colocalization $(G_{X_\nu}, \rho_{X_\nu}, V^*_{X_\nu})$ at X_ν is isomorphic to $(GL(n-\nu), 2\Lambda_1, V(\frac{1}{2}(n-\nu)(n-\nu+1)))$. Put $Y_\nu = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-\nu} \end{pmatrix} (\nu = 0, 1, \cdots, n)$. Then Y_ν is a generic point of the

colocalization at X_ν and $Y_{\nu+1}$ is a point of the unique one-codimensional orbit. Thus we have $\dim \varLambda_\nu \cap \varLambda_{\nu+1} = \dim V - 1$, where $\varLambda_\nu$ denotes the conormal bundle of $\rho(G)X_\nu$. Since $\dim d\rho^*(\mathfrak{g}_{x_\nu} \cap \mathfrak{g}_0)Y_{\nu+1} = \dim d\rho^*(\mathfrak{g}_{x_\nu})Y_{\nu+1}$, we have $\dim \mathfrak{g}_0(X_\nu, Y_{\nu+1}) = (n(n+1)/2) - 1$, and hence $\varLambda_\nu$ is a good holonomic variety by Proposition 1–5 $(\nu = 0, 1, \cdots, n)$.

Put $A_\nu^\beta = \begin{pmatrix} 0 & & \\ \hline & -\beta & \\ \hline & & -\frac{1}{2}I_{n-\nu-1} \end{pmatrix} \in \mathfrak{gl}(n)$. Then we have $d\rho(A_\nu^\beta)X_\nu = 0$,
$\underbrace{\quad}_{\nu}\underbrace{\quad}_{1}\underbrace{\quad}_{n-\nu-1}$

$d\rho^*(A_\nu^\beta)Y_{\nu+1} = Y_{\nu+1}$ and $2\beta = \operatorname{tr} A_\nu^\beta$ where tr denotes the trace of A_ν^β on $V_{X_\nu}^*$ modulo $d\rho_{x_\nu}(\mathfrak{g}_{x_\nu})Y_{\nu+1}$. Hence, $\varLambda_\nu$ and $\varLambda_{\nu+1}$ intersect regularly, i.e., the intersection exponent of $\varLambda_\nu$ and $\varLambda_{\nu+1}$ equals $(1:0)$. We shall calculate the order $\operatorname{ord}_{\varLambda_\nu} f^s$ by Proposition 1–3. Put $A_\nu = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}I_{n-\nu} \end{pmatrix}$ $(0 \leq \nu \leq n)$. Then $d\rho(A_\nu)X_\nu = 0$ and $d\rho^*(A_\nu)Y_\nu = Y_\nu$. Since $\delta\chi(A_\nu) = 2\operatorname{tr} A_\nu = -(n-\nu)$, and $\operatorname{tr}_{V_{X_\nu}^*} d\rho_{x_\nu}(A_\nu) = \dim V_{X_\nu}^* = \frac{1}{2}(n-\nu)(n-\nu+1)$, we have $\operatorname{ord}_{\varLambda_\nu} f^s = s\delta\chi(A_\nu) - \operatorname{tr}_{V_{X_\nu}^*} d\rho_{x_\nu}(A_\nu) + \frac{1}{2}\dim V_{X_\nu}^* = -(n-\nu)s - \frac{1}{4}(n-\nu)(n-\nu+1)$.

Thus we obtain the holonomy diagram (Figure 2–2). By Corollary
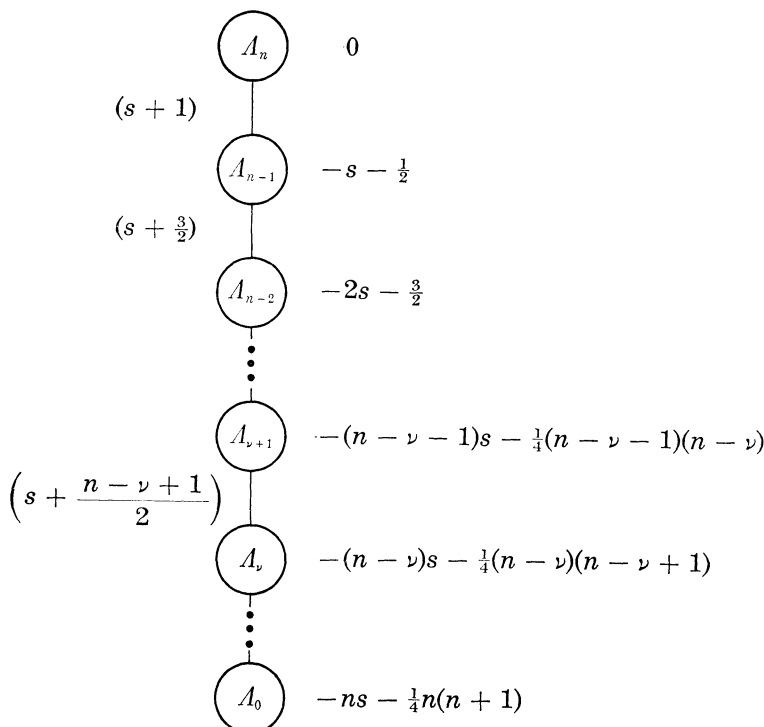
Figure 2-2. Holonomy diagram of $(GL(n), 2\varLambda_1, V(\frac{1}{2}n(n+1)))$ $(n \geq 2)$

1–2, we have $b_{\Lambda_\nu}(s)/b_{\Lambda_{\nu+1}}(s) = s + ((n - \nu + 1)/2)$ $(0 \leq \nu \leq n - 1)$, and hence $b(s) = b_{\Lambda_0}(s) = \prod_{\nu=1}^{n}(s + (\nu + 1)/2)$.

Remark. The b-function of **2–2** is also already known. It can be obtained by using Capelli's identity or by a direct calculation of the Fourier transform of $f(x)^s$.

2–3. $(GL(2m), \Lambda_2, V(m(2m - 1)))$ $(m \geq 3)$

The representation space can be identified with $V_m = \{X \in M(2m, C) \mid {}^t X = - X\}$. Then the action $\rho = \Lambda_2$ is given by $\rho(g)X = gX {}^t g$ for $g \in GL(2m)$, $X \in V_m$. The relative invariant $f(X)$ is the Pfaffian of X. It is well-known that there exists $(m + 1)$-orbits $\rho(G)X_\mu = \{X \in V_m ; \mathrm{rank}\, X = 2\mu\}$ where $X_\mu = \begin{bmatrix} 0 & 0 & I_\mu & 0 \\ 0 & 0 & 0 & 0 \\ -I_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $(0 \leq \mu \leq m)$.

By simple calculation, we have

$$
(2.5) \qquad d\rho(\tilde{A})X_\mu = \left[\begin{array}{c|c|c|c} \overbrace{{}^t B_1 - B_1}^{\mu} & \overbrace{{}^t B_3}^{m-\mu} & \overbrace{A_1 + {}^t D_1}^{\mu} & \overbrace{{}^t D_3}^{m-\mu} \\ -B_3 & & A_3 & \\ \hline -D_1 - {}^t A_1 & -{}^t A_3 & C_1 - {}^t C_1 & -{}^t C_3 \\ -D_3 & & C_3 & \end{array}\right]
$$

where $\tilde{A} = \left[\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ \hline C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{array}\right]$

and hence,

$$
(2.6) \qquad \mathfrak{g}_{X_\mu} = \left\{ \left[\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ 0 & A_4 & 0 & B_4 \\ \hline C_1 & C_2 & -{}^t A_1 & D_2 \\ 0 & C_4 & 0 & D_4 \end{array}\right] ; {}^t B_1 = B_1, {}^t C_1 = C_1 \right\}
$$

$$
V_{X_\mu}^* = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & X & 0 & Y \\ 0 & 0 & 0 & 0 \\ 0 & -{}^t Y & 0 & Z \end{bmatrix} ; \begin{array}{l} {}^t X = - X \\ {}^t Z = - Z \end{array} \right\} \simeq \left\{ \left(\begin{array}{c|c} X & Y \\ \hline {}^t Y & Z \end{array}\right) ; \begin{array}{l} {}^t X = - X \\ {}^t Z = - Z \end{array} \right\} = V_{m-\mu}.
$$

Since $\mathfrak{g}_{X_\mu}$ acts on $V_{X_\mu}^*$ as $\tilde{X} \mapsto - {}^t \tilde{A}_4 \tilde{X} - \tilde{X}\tilde{A}_4$ where $\tilde{A}_4 = \left(\begin{array}{c|c} A_4 & B_4 \\ \hline C_4 & D_4 \end{array}\right)$ and

$\tilde{X} = \left(\begin{array}{c|c} X & Y \\ \hline -\,^t Y & Z \end{array} \right)$ with $^t X = -\,X$, $^t Z = -\,Z$, the colocalization $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu})$ at X_μ is isomorphic to $(GL(2m - 2\mu),\ \Lambda_2,\ V((m - \mu)(2m - 2\mu - 1)))$. Here we identified the dual V^*_m of V_m with V_m by $\langle X, Y \rangle = \mathrm{tr}\, XY$.

$$\text{Put } Y_\mu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m-\mu} \\ 0 & 0 & 0 & 0 \\ 0 & -I_{m-\mu} & 0 & 0 \end{bmatrix} \quad (0 \le \mu \le m)\,.$$

Then Y_μ is a generic point of the colocalization $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu})$ at X_μ, and $Y_{\mu+1}$ is a point of the one-dimensional orbit and hence we have $\dim \Lambda_\mu \cap \Lambda_{\mu+1} = \dim V - 1$ $(0 \le \mu \le m - 1)$ where Λ_μ denotes the conormal bundle of $\rho(G)X_\mu$.

By (2.6), we have $\mathfrak{g}_{X_\mu} \not\subset \mathfrak{g}_0$ for $\mu \ne m$, and hence $\dim d\rho(\mathfrak{g})X_\mu = \dim d\rho(\mathfrak{g}_0)X_\mu$ for $\mu \ne m$. Applying this fact to the colocalization at X_μ, we have $\dim \mathfrak{g}_0(X_\mu, Y_{\mu+1}) = \dim \mathfrak{g}(X_\mu, Y_{\mu+1}) = m(2m - 1) - 1$. This implies that Λ_μ is a good holonomic variety by Proposition 1–5.

$$\text{Put } A^\beta_\mu = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix} -\beta \\ & -\frac{1}{2}I_{m-\mu-1} \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \begin{matrix} -\beta \\ & -\frac{1}{2}I_{m-\mu-1} \end{matrix} \end{array} \right] \quad \text{for } \beta \in C\,.$$

Then we have $d\rho(A^\beta_\mu)X_\mu = 0$, $d\rho^*(A^\beta_\mu)Y_{\mu+1} = Y_{\mu+1}$ and $\mathrm{tr}\, A^\beta_\mu = 2\beta$ where tr denotes the trace of A^β_μ on $V^*_{X_\mu}$ modulo $d\rho(\mathfrak{g}_{X_\mu})Y_{\mu+1}$, and hence by Proposition 1–4, Λ_μ and $\Lambda_{\mu+1}$ intersect regularly, i.e., the intersection exponent of Λ_μ and $\Lambda_{\mu+1}$ equals $(1:0)$. We shall calculate the order $\mathrm{ord}_{\Lambda_\mu} f^s$.

$$\text{Put } A_0 = \left[\begin{array}{c|c|c} \begin{matrix} \\ -\frac{1}{2}I_{m-\mu} \\ \\ \end{matrix} & & \\ \hline & & \\ \hline & & -\frac{1}{2}I_{m-\mu} \end{array} \right]. \quad \text{Then we have } d\rho(A_0)X_\mu = 0 \text{ and}$$

$d\rho^*(A_0)Y_\mu = Y_\mu$. Since $\delta\chi(A_0) = -(m - \mu)$, $\mathrm{tr}_{V^*_{X_\mu}} A_0 = \dim V^*_{X_\mu} = (m - \mu)$ $(2m - 2\mu - 1)$, we have $\mathrm{ord}_{\Lambda_\mu} f^s = s\delta\chi(A_0) - \mathrm{tr}_{V^*_{X_\mu}} A_0 + \frac{1}{2}\dim V^*_{X_\mu} = -(m - \mu)s - \frac{1}{2}(m - \mu)(2m - 2\mu - 1)$.

By Corollary 1–2, we have $b_{\Lambda_\mu}(s)/b_{\Lambda_{\mu+1}}(s) = s + 2(m - \mu) - 1$ $(0 \le \mu \le m - 1)$. Hence we obtain the holonomy diagram (Figure 2–3) and b-function $b(s) = \prod_{\mu=0}^{m-1}(s + 2(m - \mu) - 1) = \prod_{k=1}^{m}(s + 2k - 1)$.

$$\begin{array}{ll} \textcircled{$\varLambda_m$} & 0 \\[2mm] (s+1) & \\[2mm] \textcircled{$\varLambda_{m-1}$} & -s-\frac{1}{2} \\[2mm] (s+3) & \\[2mm] \textcircled{$\varLambda_{m-2}$} & -2s-3 \\[2mm] \vdots & \\[2mm] \textcircled{$\varLambda_{\mu+1}$} & \\[2mm] (s+2m-2\mu-1) & \\[2mm] \textcircled{$\varLambda_{\mu}$} & -(m-\mu)s-\frac{1}{2}(m-\mu)(2m-2\mu-1) \\[2mm] \vdots & \\[2mm] \textcircled{$\varLambda_0$} & -ms-\dfrac{m(m-1)}{2} \end{array}$$
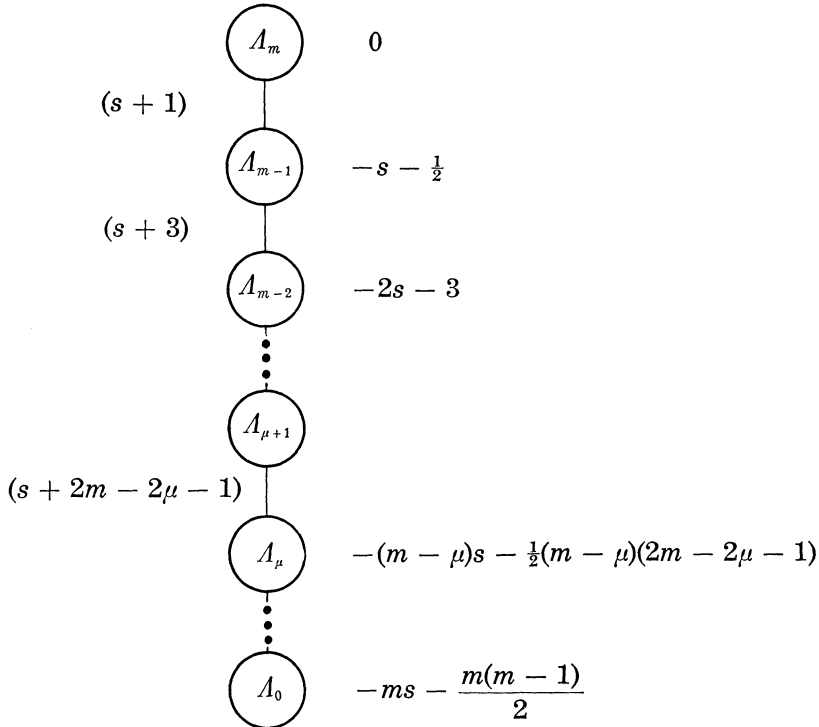
Figure 2-3. Holonomy diagram of $(GL(2m),\ \varLambda_2,\ V(m(2m-1)))$ $(m \geq 3)$.

Remark. These three P.V.'s have many common properties: (1) $\left(GL(m),\ 2\varLambda_1,\ V\left(\binom{m}{2}\ell+m\right)\right)$ with $\ell \doteq 1$ (2) $\left(SL(m) \times GL(m),\ \varLambda_1 \otimes \varLambda_1,\ V\left(\binom{m}{2}\ell+m\right)\right)$ with $\ell = 2$ (3) $\left(GL(2m),\ \varLambda_2,\ V\left(\binom{m}{2}\ell+m\right)\right)$ with $\ell = 4$. They have $(m+1)$-orbits and their relative invariants are of degree m of $\binom{m}{2}\ell+m$ variables. We denote $\textcircled{$\varLambda$}$ by $\textcircled{$\mu$}$ if $\varLambda$ is the conormal bundle of a μ-codimensional orbit. Then their holonomy diagrams are as in Figure 2–4.

§3. $(Sp(n) \times GL(2m),\ \varLambda_1 \otimes \varLambda_1,\ V(2n) \otimes V(2m))$ with $n \geq 2m$

The representation space V can be identified with the totality of $2n \times 2m$ matrices. Then the action $\rho = \varLambda_1 \otimes \varLambda_1$ is given by $\rho(g)X = g_1 X\,{}^t g_2$ for $g = (g_1, g_2) \in G = Sp(n) \times GL(2m)$, $X \in V$. Let X be an element of V such that rank $X = \nu$ and rank ${}^t XJX = 2\mu(2m \geqq \nu \geqq 2\mu \geqq 0)$ where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Then by the action of $GL(2m)$, we may assume that $X = (X', 0)$ with $X' \in M(2n, \nu)$ satisfying ${}^t X'JX' = \begin{pmatrix} 0 & 0 & I_\mu \\ 0 & 0 & 0 \\ -I_\mu & 0 & 0 \end{pmatrix}$. Put $X_{\nu, 2\mu}$ as follows.

Diagram 1 (top left):

$(s+1)$
circle 0 — 0
circle 1 — $-s-\frac{1}{2}$
$\left(s+\frac{\ell}{2}+1\right)$
circle $\ell+2$ — $-2s-\frac{\ell+2}{2}$
$\left(s+\frac{2\ell}{2}+1\right)$
circle $3\ell+3$ — $-3s-\frac{3\ell+3}{2}$
$\left(s+\frac{3\ell}{2}+1\right)$
circle $6\ell+4$ — $-4s-\frac{6\ell+4}{2}$
$\left(s+\frac{4\ell}{2}+1\right)$
$\left(s+\frac{(m-1)\ell}{2}+1\right)$
circle $\binom{m}{2}\ell+m$ — $-ms-\frac{\frac{1}{2}m(m+1)}{2}$

$(m+1)$-orbits   deg $f=m$ dim $V=\binom{m}{2}\ell+m$

Diagram 2 (top right):

circle 0 — 0
$\left(s+\frac{2}{2}\right)$
circle 1 — $-s-\frac{1}{2}$
$\left(s+\frac{3}{2}\right)$
circle 3 — $-2s-\frac{3}{2}$
$\left(s+\frac{4}{2}\right)$
circle 6 — $-3s-\frac{6}{2}$
$\left(s+\frac{5}{2}\right)$
circle 10 — $-4s-\frac{10}{2}$
$\left(s+\frac{6}{2}\right)$
$\left(s+\frac{m+1}{2}\right)$
circle $\frac{1}{2}m(m+1)$ — $-ms-\frac{\binom{m}{2}\ell+m}{2}$

(1) $\ell=1$

Diagram 3 (bottom left):

circle 0 — 0
$(s+1)$
circle 1 — $-s-\frac{1}{2}$
$(s+2)$
circle 4 — $-2s-\frac{4}{2}$
$(s+3)$
circle 9 — $-3s-\frac{9}{2}$
$(s+4)$
circle 16 — $-4s-\frac{16}{2}$
$(s+5)$
$(s+m)$
circle m^2 — $-ms-\frac{m^2}{2}$

(2) $\ell=2$

Diagram 4 (bottom right):

circle 0 — 0
$(s+1)$
circle 1 — $-s-\frac{1}{2}$
$(s+3)$
circle 6 — $-2s-\frac{6}{2}$
$(s+5)$
circle 15 — $-3s-\frac{15}{2}$
$(s+7)$
circle 28 — $-4s-\frac{28}{2}$
$(s+9)$
$(s+2m-1)$
circle $m(2m-1)$ — $-ms-\frac{m(2m-1)}{2}$

(3) $\ell=4$

Figure 2-4

(3.1)

$$
X_{\nu,2\mu} =
\begin{pmatrix}
\boxed{I_\mu} & & & \\
& \boxed{I_{\nu-2\mu}} & & \\
& & & \\
& & \boxed{I_\mu} & \\
& & & \\
& & &
\end{pmatrix}
\begin{matrix} \\ \\ \Big\} n \\ \\ \Big\} n \end{matrix}
$$

Then $X_{\nu,2\mu}$ satisfies the same condition as X and hence there exists an element g_1 of $Sp(n)$ satisfying $g_1 X = X_{\nu,2\mu}$. This implies that $S_{\nu,2\mu} = \{X \in V;$ rank $X = \nu$, rank ${}^t X J X = 2\mu\}$ $(2m \geqq \nu \geqq 2\mu \geqq 0)$ consists of a single G-orbit, and we complete the orbital decomposition of this space. Put $A \in \mathfrak{sp}(n)$ and $D \in \mathfrak{gl}(2m)$ as follows:

(3.2)

$$
A =
\left(
\begin{array}{ccc|ccc}
A_1 & A_{12} & A_{13} & B_1 & B_{12} & B_{13} \\
A_{21} & A_2 & A_{23} & {}^t B_{12} & B_2 & B_{23} \\
A_{31} & A_{32} & A_3 & {}^t B_{13} & {}^t B_{23} & B_3 \\
\hline
C_1 & C_{12} & C_{13} & -{}^t A_1 & -{}^t A_{21} & -{}^t A_{31} \\
{}^t C_{12} & C_2 & C_{23} & -{}^t A_{12} & -{}^t A_2 & -{}^t A_{32} \\
{}^t C_{13} & {}^t C_{23} & C_3 & -{}^t A_{13} & -{}^t A_{23} & -{}^t A_3
\end{array}
\right),
$$

$$
\underbrace{}_{\mu} \underbrace{\phantom{A_{12}}}_{\nu-2\mu} \underbrace{\phantom{A_{13}}}_{n-\nu+\mu} \underbrace{}_{\mu} \underbrace{\phantom{B_{12}}}_{\nu-2\mu} \underbrace{\phantom{B_{13}}}_{n-\nu+\mu}
$$

$$
D =
\left(
\begin{array}{cccc}
D_1 & D_{12} & D_{13} & D_{14} \\
D_{21} & D_2 & D_{23} & D_{24} \\
D_{31} & D_{32} & D_3 & D_{34} \\
D_{41} & D_{42} & D_{43} & D_4
\end{array}
\right)
$$

$$
\underbrace{}_{\mu} \underbrace{\phantom{D_{12}}}_{\nu-2\mu} \underbrace{\phantom{D_{13}}}_{\mu} \underbrace{\phantom{D_{14}}}_{2m-\nu}
$$

where ${}^t B_i = B_i$, ${}^t C_i = C_i$ $(i = 1, 2, 3)$.

Then, for $\tilde{A} = A \oplus D \in \mathfrak{g}$, we have

(3.3)

$$
d\rho(\tilde{A}) X_{\nu,2\mu} = A X_{\nu,2\mu} + X_{\nu,2\mu} {}^t D
$$

$$
=
\left(
\begin{array}{cc|cc}
A_1 + {}^t D_1 & A_{12} + {}^t D_{21} & B_1 + {}^t D_{31} & {}^t D_{41} \\
A_{21} + {}^t D_{12} & A_2 + {}^t D_2 & {}^t B_{12} + {}^t D_{32} & {}^t D_{42} \\
A_{31} & A_{32} & {}^t B_{13} & 0 \\
\hline
C_1 + {}^t D_{13} & C_{12} + {}^t D_{23} & -{}^t A_1 + {}^t D_3 & {}^t D_{43} \\
{}^t C_{12} & C_2 & -{}^t A_{12} & 0 \\
{}^t C_{13} & {}^t C_{23} & -{}^t A_{13} & 0
\end{array}
\right)
$$

and hence the isotropy subalgebra $\mathfrak{g}_{X_{\nu,2\mu}}$ is given as follows:

$$\mathfrak{g}_{X_\nu,2\mu} = \left\{ \tilde{A} = \begin{bmatrix} A_1 & 0 & 0 & B_1 & B_{12} & 0 \\ A_{21} & A_2 & A_{23} & {}^tB_{12} & B_2 & B_{23} \\ 0 & 0 & A_3 & 0 & {}^tB_{23} & B_3 \\ \hline C_1 & 0 & 0 & -{}^tA_1 & -{}^tA_{21} & 0 \\ 0 & 0 & 0 & 0 & -{}^tA_2 & 0 \\ 0 & 0 & C_3 & 0 & -{}^tA_{23} & -{}^tA_3 \end{bmatrix} \right.$$

$$\oplus \begin{bmatrix} -{}^tA_1 & -{}^tA_{21} & -C_1 & D_{14} \\ 0 & -{}^tA_2 & 0 & D_{24} \\ -B_1 & -B_{12} & A_1 & D_{34} \\ 0 & 0 & 0 & D_4 \end{bmatrix} \left. \right\}$$

(3.4)

$$\cong \left\{ \begin{bmatrix} A_2 & A_{21} & {}^tB_{12} & A_{23} & B_{23} & B_2 \\ \hline & A_1 & B_1 & & & B_{12} \\ 0 & C_1 & -{}^tA_1 & & 0 & -{}^tA_{21} \\ \hline & & & A_3 & B_3 & {}^tB_{23} \\ 0 & & 0 & C_3 & -{}^tA_3 & -{}^tA_{23} \\ \hline 0 & & 0 & & 0 & -{}^tA_2 \end{bmatrix} \right.$$

$$\oplus \begin{bmatrix} A_1 & B_1 & -B_{12} & D_{34} \\ C_1 & -{}^tA_1 & {}^tA_{21} & -D_{14} \\ \hline & & -{}^tA_2 & D_{24} \\ 0 & & 0 & D_4 \end{bmatrix} \left. \right\}$$

$$\cong (\mathfrak{gl}(\nu - 2\mu) \oplus \mathfrak{gl}(2m - \nu) \oplus \mathfrak{sp}(\mu) \oplus \mathfrak{sp}(n - \nu + \mu)) \oplus \mathfrak{u}(k)$$

where $\mathfrak{u}(k)$ denotes the Lie algebra of a k-dimensional unipotent group with $k = \frac{1}{2}(4n + 1)(\nu - 2\mu) - \frac{3}{2}(\nu - 2\mu)^2 + \nu(2m - \nu)$. In this paper, we make a convention that the first (resp. second) $\oplus$ implies the direct sum as Lie algebras (resp. vector spaces) for $(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \oplus \mathfrak{g}_3$.

We identify the dual space V^* of V with V by $\langle X, Y \rangle = \operatorname{tr} X^tY$ for $X, Y \in V = M(2n, 2m)$, and hence we have $\rho^*(g)Y = {}^tg_1^{-1}Yg_2^{-1}$ for $g = (g_1, g_2) \in G$, $Y \in V$ and $d\rho^*(\tilde{A})Y = -{}^tAY - YD$ for $\tilde{A} = (A, D) \in \mathfrak{g}$. From (3.3), the conormal vector space $V^*_{X_\nu,2\mu}$ is given by

(3.5)
$$V^*_{X_\nu,2\mu} = \left\{ \tilde{Y} = \begin{bmatrix} & & & 0 \\ 0 & 0 & 0 & 0 \\ & & & Y \\ \hline & 0 & & 0 \\ 0 & X & 0 & Z \\ & 0 & & W \end{bmatrix} \begin{matrix} \\ \end{matrix} \begin{matrix} \}\mu \\ \}\nu - 2\mu \\ \}n - \nu + \mu \end{matrix} \quad ; {}^tX = -X \right\}$$
$$\underbrace{\quad}_{\mu} \underbrace{\quad}_{\nu - 2\mu} \underbrace{\quad}_{\mu} \underbrace{\quad}_{2m - \nu}$$

$$\cong \left\{ \tilde{Y}' = \left(\begin{array}{c|c} Y_1 & Y_2 \\ \hline 0 & Y_3 \end{array} \right)\!\!\begin{array}{l}{}^{\}\nu-2\mu} \\ {}^{\}2(n-\nu+\mu)}\end{array} \; ; \; {}^t Y_1 = -\, Y_1 \right\}.$$

$$\underbrace{}_{\nu-2\mu}\underbrace{}_{2m-\nu}$$

Here the isomorphism is obtained by putting $Y_1 = X$, $Y_2 = Z$ and $Y_3 = \begin{bmatrix} Y \\ W \end{bmatrix}$. Then the action $d\rho_{X_\nu, 2\mu}$ of $\mathfrak{g}_{X_\nu, 2\mu}$ on $V^*_{X_\nu, 2\mu}$ is given as follows.

$$d\rho_{X_\nu, 2\mu}(\tilde{A})\tilde{Y}' = \left(\begin{array}{c|cc} A_2 & -B_{23} & A_{23} \\ \hline 0 & -{}^t A_3 & -C_3 \\ & -B_3 & A_3 \end{array} \right)\left(\begin{array}{c|c} Y_1 & Y_2 \\ \hline 0 & Y_3 \end{array} \right) + \left(\begin{array}{c|c} Y_1 & Y_2 \\ \hline 0 & Y_3 \end{array} \right)\left(\begin{array}{c|c} {}^t A_2 & -D_{24} \\ \hline 0 & -D_4 \end{array} \right).$$

Thus the action on Y_1-space is isomorphic to $(GL(\nu - 2\mu), \Lambda_2, V(\frac{1}{2}(\nu - 2\mu) \times (\nu - 2\mu - 1)))$ and the action on Y_3-space is isomorphic to $(Sp(n - \nu + \mu) \times GL(2m - \nu), \Lambda_1 \otimes \Lambda_1, V(2n - 2\nu + 2\mu) \otimes V(2m - \nu))$. First we shall consider the case when ν is even, i.e., $\nu = 2\nu'$. Let $\tilde{Y}_0$ be an element of $V^*_{X_\nu, 2\mu}$ with $X = \begin{pmatrix} 0 & I_{\nu'-\mu} \\ -I_{\nu'-\mu} & 0 \end{pmatrix}$, $Y = \begin{pmatrix} I_{m-\nu'} & 0 \\ 0 & \end{pmatrix}$, $W = \begin{pmatrix} 0 & I_{m-\nu} \\ 0 & \end{pmatrix}$ and $Z = 0$ in (3.5). Then $\tilde{Y}_0$ is a generic point and $\tilde{Y}_0 \in S^*_{2m-2\mu, 2m-2\nu'}$, i.e., $\Lambda_{2\nu', 2\mu} = \Lambda^*_{2m-2\mu, 2m-2\nu'}$ where $\Lambda_{\nu, 2\mu}$ (resp. $\Lambda^*_{\nu, 2\mu}$) denotes the conormal bundle of $S_{\nu, 2\mu}$ (resp. $S^*_{\nu, 2\mu}$). We shall calculate the order $\mathrm{ord}_{\Lambda_{2\nu', 2\mu}} f^s$ where $f(X) = Pf\,{}^t XJX$. Let $\tilde{A}_0$ be the element of $\mathfrak{g}_{X_\nu, 2\mu}$ with $A_2 = \frac{1}{2} I_{2(\nu'-\mu)}$, $D_4 = -I_{2(m-\nu')}$, all remaining parts zero in (3.4). Then we have $d\rho(\tilde{A}_0)X_{\nu, 2\mu} = 0$ and $d\rho^*(\tilde{A}_0)\tilde{Y}_0 = \tilde{Y}_0$ Since $\delta\chi(\tilde{A}_0) = -(2m - \nu' - \mu)$, $\mathrm{tr}_{V^*_{X_\nu, 2\mu}}\tilde{A}_0 = (\nu' - \mu)(2\nu' - 2\mu - 1) + 4(m - \nu')(n - 2\nu' + \mu) + 6(m - \nu')(\nu' - \mu)$ and $\dim V^*_{X_\nu, 2\mu} = (\nu' - \mu)(2\nu' - 2\mu - 1) + 4(m - \nu')(n - 2\nu' + \mu) + 4(m - \nu')(\nu' - \mu)$, we have

$$(3.6) \qquad \mathrm{ord}_{\Lambda_{2\nu', 2\mu}} f^s = -(2m - \nu' - \mu)s - \tfrac{1}{2}(\nu' - \mu)(2\nu' - 2\mu - 1)$$
$$- 2(m - \nu')(n - 2\nu' + \mu) - 4(m - \nu')(\nu' - \mu).$$

Let $\tilde{Y}_1$ be the element of $V^*_{X_\nu, 2\mu}$ with $X = \begin{pmatrix} 0 & I_{\nu'-\mu} \\ -I_{\nu'-\mu} & 0 \end{pmatrix}$, $Y = \begin{pmatrix} I_{m-\nu'+1} & 0 \\ 0 & \end{pmatrix}$, $W = \begin{pmatrix} 0 & I_{m-\nu'-1} \\ 0 & \end{pmatrix}$ and $Z = 0$. Since $\tilde{Y}_1$ is a point of a one-codimensional orbit and $\tilde{Y}_1 \in S^*_{2m-2\mu, 2(m-\nu'-1)}$, we have $\Lambda_{2\nu', 2\mu} \cap \Lambda_{2(\nu'+1), 2\mu} = \dim V - 1$. They intersect regularly. By Corollary 1–2, we have

$$(3.7) \qquad b_{\Lambda_{2\nu', 2\mu}}(s)/b_{\Lambda_{2(\nu'+1), 2\mu}}(s) = s + 2n - 2\nu' \qquad (m - 1 \geqq \nu' \geqq 0).$$

Now let $\tilde{Y}_2$ be the element of $V^*_{X_\nu, 2\mu}$ with $X = \left(\begin{array}{cc|c} 0 & I_{\nu'-\mu-1} & 0 \\ -I_{\nu'-\mu-1} & 0 & \\ \hline 0 & & 0 \end{array} \right)$, $Y = \begin{pmatrix} I_{m-\nu'} & 0 \\ 0 & \end{pmatrix}$, $W = \begin{pmatrix} 0 & I_{m-\nu} \\ 0 & \end{pmatrix}$ and $Z = 0$. Since $\tilde{Y}_2$ is a point of the

other one-codimensional orbit and $\tilde{Y}_2 \in S^*_{2(m-\mu-1),2(m-\nu')}$, we have $\dim \Lambda_{2\nu',2\mu} \cap \Lambda_{2\nu',2(\mu+1)} = \dim V - 1$. They intersect regularly. By Corollary 1–2, we have

$$(3.8) \qquad b_{\Lambda_{2\nu',2\mu}}(s)/b_{\Lambda_{2\nu',2(\mu+1)}} = s + 2m - 2\mu - 1 \qquad (m - 1 \geq \mu \geq 0).$$

Now we shall show that $\Lambda_{\nu,2m}$ is not a good holonomic variety when ν is odd, i.e., $\nu = 2\nu' + 1$. Let $\tilde{Y}_0$ be the element of $V^*_{X_{\nu,2\mu}}$ with $X = \begin{pmatrix} 0 & I_{\nu'-\mu} & 0 \\ -I_{\nu'-\mu} & & \\ \hline 0 & & 0 \end{pmatrix}$, $Y = \begin{pmatrix} I_{m-\nu'-1} & 0 \\ 0 & \end{pmatrix}$, $W = \begin{pmatrix} 0 & I_{m-\nu} \\ 0 & \end{pmatrix}$ and $Z = 0$ in (3.5). Then it is a generic point of the conormal vector space. Let $\tilde{A}_0$ be the element of $\mathfrak{g}_{X_{\nu,2\mu}}$ with $A_2 = \begin{pmatrix} \frac{1}{2} I_{2(\nu'-\mu)} & 0 \\ 0 & \beta \end{pmatrix}$, $D_4 = -I_{2m-\nu}$, all remaining parts zero. Then we have $d\rho(\tilde{A}_0)X_{\nu,2\mu} = 0$ and $d\rho^*(\tilde{A}_0)\tilde{Y}_0 = \tilde{Y}_0$. Therefore, if $\Lambda_{\nu,2\mu}$ is a good holonomic variety, $m_{\Lambda_{\nu,2\mu}} = -\delta\chi(\tilde{A}_0) = 2m - \nu' - \mu - 1 + \beta$ is a non-negative integer which is a contradiction. Thus we obtain the following proposition.

PROPOSITION 3–1. *The irreducible regular P.V. $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ $(n \geqq 2m)$ has finitely many orbits $S_{\nu,2\mu} = \{X \in M(2n, 2m);$ rank $X = \nu$, rank ${}^t XJX = 2\mu\}$ $(2m \geqq \nu \geqq 2\mu \geqq 0)$. When ν is odd, the conormal bundle $\Lambda_{\nu,2\mu}$ of $S_{\nu,2\mu}$ is outside W, i.e., $\Lambda_{\nu,2\mu}$ is not a good holonomic variety. When ν is even $(\nu = 2\nu')$, $\Lambda_{\nu,2\mu}$ is a good holonomic variety and $\mathrm{ord}_{\Lambda_{\nu,2\mu}} f^s = -(2m - \nu' - \mu)s - \frac{1}{2}(\nu' - \mu)(2\nu' - 2\mu - 1) - 2(m - \nu')(n - 2\nu' + \mu) - 4(m - \nu')(\nu' - \mu)$. We have $\dim \Lambda_{\nu,2\mu} \cap \Lambda_{\nu,2(\mu+1)} = \dim \Lambda_{\nu,2\mu} \cap \Lambda_{\nu+2,2\mu} = \dim V - 1$. The b-function $b(s)$ is given by $b(s) = \prod_{k=1}^{m}(s + 2k - 1) \cdot \prod_{\ell=0}^{m-1}(s + 2n - 2\ell)$.*

Figure 3–1. (ν: even)

TATSUO KIMURA
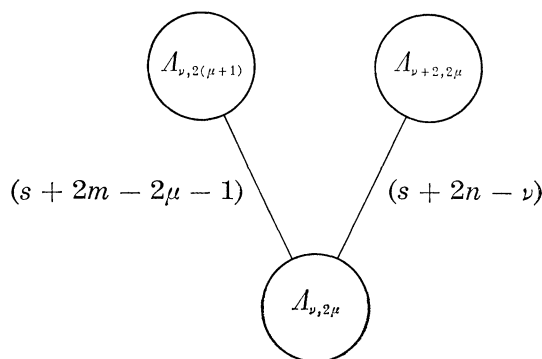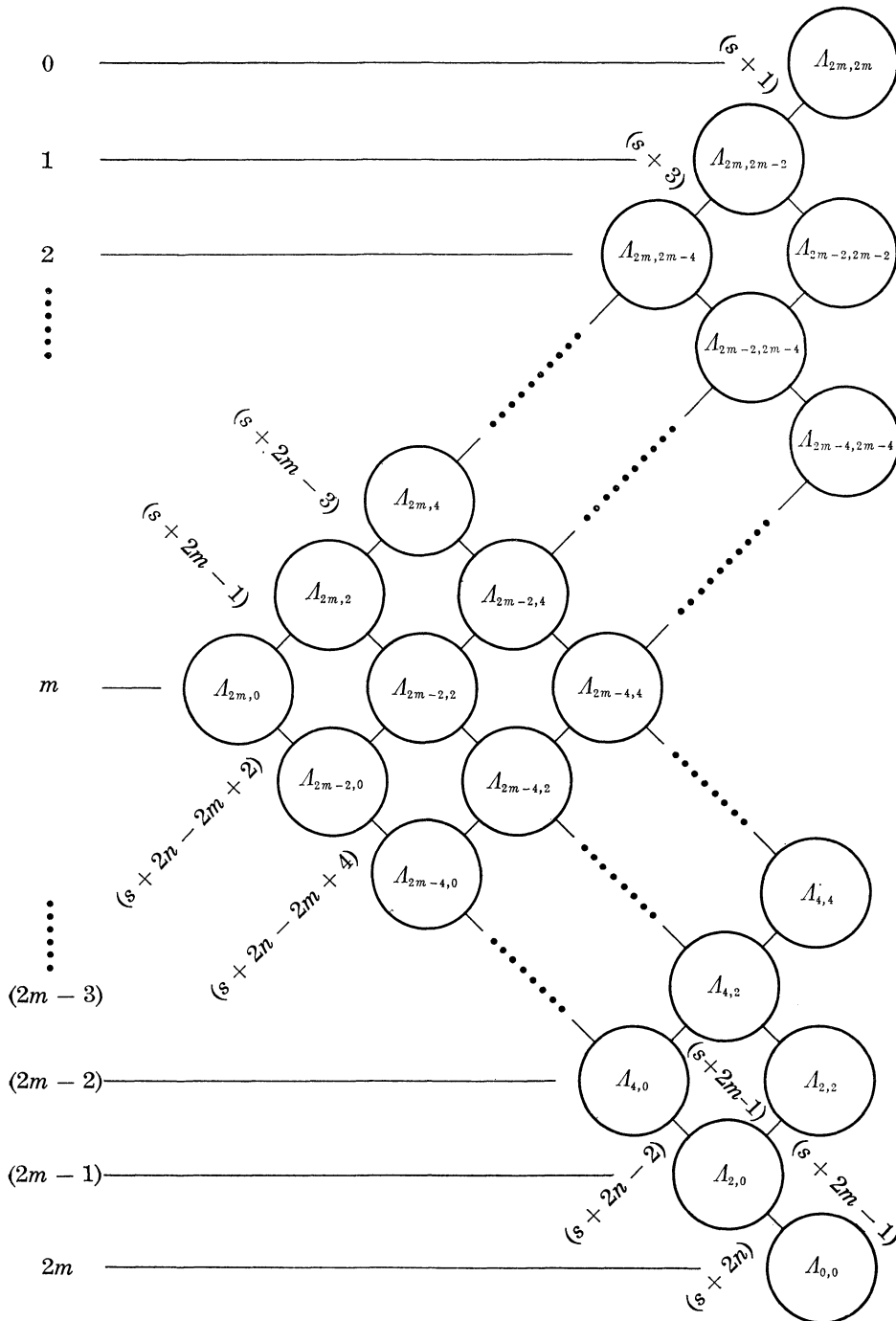
Figure 3-2. Holonomy diagram of $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ with $n \geqq 2m$.

§ 4.   (Spin(10) $\times$ GL(2), half-spin rep. $\otimes \Lambda_1$, $V(16) \otimes V(2)$)

The representation space $V(16) \otimes V(2)$ is identified with $V = V(16) \oplus V(16)$ where $V(16)$ is spanned by 1, $e_i e_j$, $e_k e_l e_m e_n$ $(1 \leq i < j \leq 5, 1 \leq k < \ell < m < n \leq 5)$ (See p. 110–112 in [1]). The action $\rho = \rho_1 \otimes \Lambda_1$ is given by $\rho(g)x = (\rho_1(g_1)X, \rho_1(g_1)Y){}^t g_2$ for $g = (g_1, g_2) \in \mathrm{Spin}(10) \times GL(2)$, $x = (X, Y) \in V = V(16) \oplus V(16)$ where ρ_1 denotes the even half-spin representation of $\mathrm{Spin}(10)$ on $V(16)$. First of all, we shall complete the orbital decomposition of this space. J.-I. Igusa completed the orbital decomposition of $(\mathrm{Spin}(10), \rho_1, V(16))$ (See [3]). There exist three orbits $S_m' = \rho_1(\mathrm{Spin}(10)) \cdot x_m'$ $(m = 0, 5, 16)$ where S_m' denotes the m-codimensional $\mathrm{Spin}(10)$-orbit and $x_0' = 1 + e_1 e_2 e_3 e_4$, $x_5' = 1$, $x_{16}' = 0$. If $\lambda \in C^\times$, for any index i satisfying $1 \leq i \leq 5$, we put $S_i(\lambda) = \lambda^{-1} + (\lambda - \lambda^{-1})e_i f_i$. Then $S_i(\lambda)$ is an element of $\mathrm{Spin}(10)$. For any two distinct indices i, j satisfying $1 \leq i, j \leq 10$, $j \neq i + 5$, $i \neq j + 5$, we put $S_{ij}(\lambda) = 1 + \lambda e_i e_j = \exp(\lambda e_i e_j)$ where $e_k = f_{k-5}$ for $6 \leq k \leq 10$ (See [1], [3]). Then $S_{ij}(\lambda)$ is an element of $\mathrm{Spin}(10)$ satisfying $S_{ij}(\lambda)S_{ji}(\lambda) = 1$.

PROPOSITION 4–1. *The triplet* (Spin(10) $\times$ GL(2), half-spin rep. $\otimes \Lambda_1$, $V(16) \otimes V(2)$) *has nine orbits* $S_m = \rho(G)x_m$ $(m = 0, 1, 4, 8, 9, 13, 15, 20, 32)$ *where S_m denotes the m-codimensional orbit.*

(1) $x_0 = (1 + e_1 e_2 e_3 e_4, e_1 e_5 + e_2 e_3 e_4 e_5)$
(2) $x_1 = (1 + e_1 e_2 e_3 e_4, e_1 e_2 + e_2 e_3 e_4 e_5)$
(3) $x_4 = (1, e_1 e_5 + e_2 e_3 e_4 e_5)$
(4) $x_8 = (1, e_1 e_2 e_3 e_4)$
(5) $x_9 = (1, e_1 e_2 + e_3 e_4)$
(6) $x_{13} = (1, e_1 e_2)$
(7) $x_{15} = (1 + e_1 e_2 e_3 e_4, 0)$
(8) $x_{20} = (1, 0)$
(9) $x_{32} = (0, 0)$

Proof. Let $\tilde{x} = (x, y)$ be a representative of one of the orbits of $V = V(16) \oplus V(16)$. Then we may assume that $x = 0$, 1, or $1 + e_1 e_2 e_3 e_4$ by the action of $\mathrm{Spin}(10)$. If $x = 0$, then we have also $y = 1 + e_1 e_2 e_3 e_4$, 1, 0, i.e., (7), (8), (9) respectively. Note that we can exchange x and y in $\tilde{x} = (x, y)$ by the action of $GL(2)$. Assume that $x = 1$. We may put $y = y_0 + y_2 + y_4 \neq 0$ where $y_0 = y_0 \cdot 1$, $y_2 = \sum y_{ij} e_i e_j$ and $y_4 = \sum y_{rstu} e_r e_s e_t e_u$. We may assume that $y_0 = 0$ by the action of $\begin{pmatrix} 1 & 0 \\ -y_0 & 1 \end{pmatrix}$. If $y = y_2 \neq 0$, we may assume that $y_{12} = 1$ by the action of some $S_{ij}(\lambda)$ $(i = 1, 2; j \geq 6)$ and $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ if necessary.

In this case, we have $y = e_1e_2 + y_{34}e_3e_4 + y_{35}e_3e_5 + y_{45}e_4e_5$ by $S_{j7}(-y_{1j})$ and $S_{j6}(y_{2j})$ for $j = 3, 4, 5$. If $y_{34} = y_{35} = y_{45} = 0$, we have (6), and otherwise we may assume that $y_{34} = 1$, $y_{35} = y_{45} = 0$ by the action of suitable elements of $\{S_{3,10}(\lambda),\ S_{4,10}(\lambda), S_{58}(\lambda), S_{59}(\lambda); \lambda \in \boldsymbol{C}\}$, i.e., (5). If $y_4 \neq 0$, we may assume that $y_4 = e_1e_2e_3e_4$. By the action of $S_{89}(y_{12})$ and $\begin{pmatrix} 1 & 0 \\ y_{12}y_{34} & 1 \end{pmatrix}$, we have $y_{12} = 0$. Similarly $y_{ij} = 0$ for $1 \leq i < j \leq 4$, and hence $y = \sum_{j=1}^{4} y_{j5}e_je_5 + e_1e_2e_3e_4$. If $y_{j5} = 0$ for all $j = 1, \cdots, 4$, we have (4). In the other case, we may assume that $y_{15} = 1$ and $y_{j5} = 0$ $(2 \leq j \leq 4)$. By the action of $S_{56}(-1)$ and $S_{1,10}(1)$, we have (3).

Finally assume that $x = 1 + e_1e_2e_3e_4$. We may put $y = y_2 + y_4$. If $y_4 \neq 0$, we may assume that $y_4 = e_2e_3e_4e_5$ or $y_4 = e_1e_2e_3e_4$. In the former case, if $y_{15} \neq 0$, we may assume that $y_{15} = 1$ by the action of $S_1(\lambda)S_5(\lambda)S_2(\lambda^{-1})$ and λI_2 where $\lambda^4 \cdot y_{15} = 1$. Then by the action of $S_{j10}(-y_{1j})$, $S_{j6}(-y_{j5})$ $(j = 2, 3, 4)$, $S_{9,10}(y_{23})$, $S_{8,10}(-y_{24})$ and $S_{7,10}(y_{34})$, we have (1). If $y_{15} = 0$, we may assume that $y_{35} = y_{45} = 0$ by $\{S_{28}(\lambda), S_{29}(\lambda), S_{37}(\lambda), S_{47}(\lambda); \lambda \in \boldsymbol{C}\}$. Then by $S_{8,10}(-y_{24})$ and $S_{9,10}(y_{23})$, we may assume that $y_{24} = y_{23} = 0$. By some $S_{39}(\lambda)$ and $S_{28}(\lambda)$, we may also assume that $y_{14} = 0$, i.e., $y = y_{12}e_1e_2 + y_{13}e_1e_3 + y_{34}e_3e_4 + y_{25}e_2e_5 + e_2e_3e_4e_5$. By the action of $S_{7,10}(y_{34})$, $\begin{pmatrix} 1 & 0 \\ y_{25}y_{34} & 1 \end{pmatrix}$ and $S_{1,10}(y_{25}y_{34})$, we have $y_{34} = 0$. By $S_{89}(y_{25})$ and $S_{12}(y_{25})$, we also have $y_{25} = 0$, i.e., $y = y_{12}e_1e_2 + y_{13}e_1e_3 + e_2e_3e_4e_5$, where we may assume that $y_{13} = 0$. If $y_{12} \neq 0$, we have (2). If $y_{12} = 0$, it is transferred to x_4 by $S_{12}(-1)$, $S_{89}(-1)$, $S_{34}(-1)$, $S_{67}(-1)$, $S_{17}(1)$, $S_{26}(-1)$, $S_{56}(-1)$, $S_{1,10}(1)$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $S_{6,10}(1)$, $S_{15}(1)$ and $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Now consider the latter case, i.e., $y_4 = e_1e_2e_3e_4$. If some of y_{j5} $(1 \leq j \leq 4)$ is not zero, we may assume that $y = e_1e_5 + y_{23}e_2e_3 + e_1e_2e_3e_4$. If $y_{23} = 0$, it is transferred to x_4 by $S_{15}(1)$, $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $S_{56}(-1)$ and $S_{1,10}(1)$. If $y_{23} \neq 0$, we may assume that $y_{23} = 1$. In this case, it is transferred to x_1 by $S_{69}(1)$, $S_{23}(1)$, $S_{4,10}(-1)$, $S_{78}(1)$, $S_{14}(1)$, $S_{46}(1)$, $S_{19}(-1)$, $S_{29}(-1)$, $S_{47}(1)$. When all $y_{j5} = 0$ for $1 \leq j \leq 4$, $y = \sum_{1 \leq i < j \leq 4} y_{ij}e_ie_j + e_1e_2e_3e_4$. If all $y_{ij} = 0$ for $1 \leq i < j \leq 4$, it is transferred to x_8 by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. In the other case, we may assume that $y = e_1e_2 + y_{34}e_3e_4 + e_1e_2e_3e_4$. By the action of $S_{67}(\lambda)$, $S_{34}(\lambda)$ and $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ with $\lambda^2 - \lambda - y_{34} = 0$, we have $y = e_1e_2 + (1 + 2\lambda)e_1e_2e_3e_4$. If $(1 + 2\lambda) \neq 0$, it is transferred to x_8 by $S_{89}(\mu)$, $S_{12}(\mu)$ and $\begin{pmatrix} 1 & -\mu \\ 0 & \mu \end{pmatrix}$ with $\mu = \dfrac{1}{1 + 2\lambda}$. If $(1 + 2\lambda) = 0$, it is equivalent to x_9 by $S_{67}(-1)$, $S_{12}(-1)$, $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $S_1(\sqrt{-1})$, and $\sqrt{-1}\ I_2$.

Finally consider the case $y = y_2$, i.e., $y_4 = 0$. Since $y \neq 0$, we may assume that $y = e_1e_2 + y_{34}e_3e_4 + \sum_{j=1}^{4} y_{j5}e_je_5$. If $y_{34} = y_{j5} = 0$ for $1 \leq j \leq 4$, it is equivalent to x_9 as we have already seen. If $y_{34} \neq 0$ and $y_{j5} = 0$ for $1 \leq j \leq 4$, it is transferred to x_8 by $S_{34}(\lambda)$, $\begin{pmatrix} 1 & -1/\lambda \\ 1 & 0 \end{pmatrix}$, $S_{12}(1/\lambda)$, $\begin{pmatrix} 1 & 0 \\ 0 & 1/2\lambda \end{pmatrix}$, $S_{67}(\lambda/2)$, $S_{89}(1/2\lambda)$, $\begin{pmatrix} 1 & 0 \\ 1/4 & 1 \end{pmatrix}$ with $\lambda^2 = y_{34}$. If some of y_{j5} $(1 \leq j \leq 4)$ is not zero, y is equivalent to an element of the form $e_1e_5 + y_{23}e_2e_3 + y_{24}e_2e_4 + y_{34}e_3e_4$. If $y_{ij} = 0$ $(2 \leq i < j \leq 4)$, it is equivalent to x_4 as we have already seen. In the other case, we have $y = e_1e_5 + e_3e_4$. By the action of $S_{26}(-1)$, $S_{17}(1)$, $S_{34}(-1)$, $S_{67}(-1)$, $S_{7,10}(1)$, $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $S_{89}(-1)$, $S_{12}(-1)$, $S_{1,10}(-1)$, it is equivalent to x_1. About the codimension of these orbits, we will see later.   Q.E.D.

By the degree formula (See Proposition 15, § 4 in [1]), we know that there exists a relatively invariant irreducible polynomial $f(x, y)$ of degree four which is unique up to a constant multiple. We shall give an explicit form of $f(x, y)$ after H. Kawahara's work (Master Thesis in Japanese, University of Tokyo, 1974).

For an element $x = x_0 + \sum_{i<j} x_{ij}e_ie_j + \sum_k x_k^*e_k^*$ of $V(16)$ where $e_ke_k^* = e_1e_2e_3e_4e_5$ for $1 \leq k \leq 5$, let $X = (x_{ij})$ be the skew-symmetric matrix of degree five determined by x_{ij}, and X_i the skew-symmetric matrix of degree four obtained from $(-1)^i X$ by crossing out its i-th line and column $(1 \leq i \leq 5)$. We denote by $\mathrm{Pf}(Y)$ the Pfaffian of the skew-symmetric matrix $Y = (y_{ij})$ of degree four, i.e., $\mathrm{Pf}(Y) = y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23}$. We define ten quadratic forms $Q_i(x)$ on $V(16)$ by $Q_i(x) = \sum_{j=1}^{5} x_{ij}x_j^*$ and $Q_{i+5}(x) = x_0x_i^* + \mathrm{Pf}(X_i)$ for $1 \leq i \leq 5$.

PROPOSITION 4–2 (H. Kawahara).

(1) $\rho_1(\mathrm{Spin}(10)) \cdot 1 = \{x \in V(16) ; Q_i(x) = 0 \ (1 \leq i \leq 10)\} - \{0\}$, where ρ_1 denotes the even half-spin representation. Moreover, this is the totality of pure spinors.

(2) The relative invariant $f(x, y)$ of $(\mathrm{Spin}(10) \times GL(2), \rho_1 \otimes \Lambda_1, V(16) \oplus V(16))$ is given by $f(x, y) = \sum_{i=1}^{5} B_i(x, y)B_{i+5}(x, y)$ for $(x, y) \in V(16) \oplus V(16)$ where $B_i(x, y) = Q_i(x + y) - Q_i(x) - Q_i(y)$ is the associated bilinear form of $Q_i(x)$ for $1 \leq i \leq 10$.

Proof. We shall use the same notation as in [4]. By simple calculation, we have $\beta_1(x, x) = (1/8)\sum_{i=1}^{10} Q_i(x)e_i$. Since $\beta_1(\rho_1(s)x, \rho_1(s)x) = \lambda(s) \cdot \zeta_1(\chi(s)) \cdot \beta_1(x, x)$ for $s \in \mathrm{Spin}(10)$ where ζ_1 is the representation Λ_1 of $SO(10) = \chi(\mathrm{Spin}(10))$ (See p. 90 in [4]), we have

$$(4.1) \qquad \sum_{i=1}^{10} Q_i(\rho_1(s)x)e_i = \lambda(s) \cdot \zeta_1(\chi(s)) \cdot \sum_{i=1}^{10} Q_i(x)e_i.$$

This implies that $W = \{x \in V(16)\,; Q_i(x) = 0, 1 \le i \le 10\}$ is a Spin(10)-invariant subspace. From the orbital decomposition, it is clear that $W = S_5' \cup S_{16}'$, i.e., $S_5' = W - \{0\}$. Since the totality of pure spinors in $V(16)$ is a single Γ^+-orbit where Γ^+ denotes the even Clifford group, and $\beta_1(x, x) = 0$ for a pure spinor x (See [4]), we have (1). From (4.1), $F(x) = \sum_{i=1}^{5} Q_i(x) Q_{i+5}(x)$ is invariant under the action ρ_1 of Spin(10) since $\tilde{f}(y) = \sum_{i=1}^{5} y_i y_{i+5}$ for $y = \sum_{i=1}^{10} y_i e_i$ is invariant under the action ζ_1 of $SO(10) = \chi(\text{Spin}(10))$. The triplet $(\text{Spin}(10), \rho_1, V(10))$ has no relative invariant (See [1]) and hence we have $F(x) \equiv 0$. By using (4.1), it is clear that $f(x, y)$ is invariant under the action of Spin(10). We shall show that $f(x, y)$ is relatively invariant under $GL(2)$. Assume that $Q_i(x)$ (resp. $Q_{i+5}(x)$) has a term $x_{i_1} x_{i_2}$ (resp. $x_{i_3} x_{i_4}$) ($1 \le i \le 5$). Since $F(x) \equiv 0$, we may assume that $Q_j(x)$ (resp. $Q_{j+5}(x)$) has a term $x_{i_1} x_{i_3}$ (resp. $x_{i_2} x_{i_4}$) for some j satisfying $1 \le j \le 5$. This implies that $f(x, y) = \sum_{i=1}^{5} B_i(x, y) B_{i+5}(x, y)$ is a linear combination of terms of the following form:

$$(4.2) \quad \begin{aligned} &(x_{i_1} y_{i_2} + y_{i_1} x_{i_2})(x_{i_3} y_{i_4} + y_{i_3} x_{i_4}) - (x_{i_1} y_{i_3} + y_{i_1} x_{i_3})(x_{i_2} y_{i_4} + y_{i_2} x_{i_4}) \\ &= \det \begin{pmatrix} x_{i_2} & y_{i_2} \\ x_{i_3} & y_{i_3} \end{pmatrix} \cdot \det \begin{pmatrix} x_{i_4} & y_{i_4} \\ x_{i_1} & y_{i_1} \end{pmatrix}. \end{aligned}$$

Hence it is clear that $f(x, y)$ is relatively invariant under $GL(2)$. Since $f(1 + e_1 e_2 e_3 e_4,\ e_1 e_5 + e_2 e_3 e_4 e_5) = 1$, it is not identically zero.   Q.E.D.

Now we shall consider the micro-differential equation $\mathfrak{M} = \mathscr{E}f(x, y)^s$ and by constructing its holonomy diagram, we shall calculate the b-function of this space.

Since $G = \text{Spin}(10) \times GL(2)$ is reductive, we have $(G, \rho^*, V^*) \cong (G, \rho, V)$ and hence the dual space V^* has also nine G-orbits $S_m^*(m = 0, 1, 4, 8, 9, 13, 15, 20, 32)$. We identify V and V^* by taking $(e_{i_1} \cdots e_{i_k}, e_{j_1} \cdots e_{j_\ell})$ ($k, \ell = 0, 2, 4$) as a dual basis, where $e_{i_1} \cdots e_{i_k} = 1$ for $k = 0$. We denote by Λ_m (resp. Λ_m^*) the conormal bundle of S_m (resp. S_m^*).

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at $x_0 = (1 + e_1 e_2 e_3 e_4, e_1 e_5 + e_2 e_3 e_4 e_5)$ is isomorphic to $(\mathfrak{g}_2) \oplus \mathfrak{sl}(2)$ (See (5.40) and (5.42) in [1]). Since $\Lambda_0 = V \times \{0\} = \Lambda_{32}^*$, Λ_0 is a good holonomic variety and we have $\text{ord}_{\Lambda_0} f^s = 0$.

(2) The isotropy subalgebra $\mathfrak{g}_{x_1}$ at $x_1 = (1 + e_1 e_2 e_3 e_4, e_1 e_2 + e_2 e_3 e_4 e_5)$ is isomorphic to $(\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(11)$ (See (5.43) in [1]). The conormal vector space $V_{x_1}^*$ is spanned by $(e_1 e_3 e_4 e_5,\ - e_1 e_5) = y_1 \in S_{13}^*$. Hence $\Lambda_1 =$

$\overline{G(x_1, y_1)} = \Lambda_{13}^*$ and $\Lambda_{13} = \Lambda_1^*$. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $d_{11} = d_{22} = -1/4$, all remaining parts zero in (5.43) of [1]. Then we have $d\rho(A_0)x_1 = 0$ and $d\rho^*(A_0)y_1 = y_1$. Since $\delta\chi(A_0) = 2(d_{11} + d_{22}) = -1$, $\mathrm{tr}_{V_{x_1}^*}A_0 = \dim V_{x_1}^* = 1$, we have $\mathrm{ord}_{\Lambda_1}f^s = -s - 1/2$. It is clear that Λ_0 and Λ_1 intersect regularly and G_0-prehomogeneously with codimension one. Hence we have $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s + 1)$. Note that $G_0 = \{g \in G ; \chi(g) = 1\}$.

(3) The isotropy subalgebra $\mathfrak{g}_{x_4}$ at $x_4 = (1, e_1e_5 + e_2e_3e_4e_5)$ is, by simple calculation using (5.38) in [1], given as follows:

$$(4.3) \qquad \mathfrak{g}_{x_4} = \left\{ \tilde{A} = \left(\begin{array}{c|c} A & 0 \\ \hline C & -{}^tA \end{array} \right) \oplus \left(\begin{array}{cc} 3\varepsilon - \eta & 0 \\ c & \eta \end{array} \right) ; A = \left(\begin{array}{c|c|c} 3\varepsilon & 0 & 0 \\ \hline * & \varepsilon I_3 + X & 0 \\ \hline * & * & -2n \end{array} \right), \right.$$

$$\left. X \in \mathfrak{sl}(3), {}^tC = -C \quad \text{with} \quad c_{i5} = 0, i = 1, \cdots, 4 \right\}.$$

Put $\omega_1 = (e_1e_3e_4e_5, 0)$, $\omega_2 = (-e_1e_2e_4e_5, 0)$, $\omega_3 = (e_1e_2e_3e_5, 0)$, and $\omega_4 = (e_1e_2e_3e_4, 0)$. Then the conormal vector space $V_{x_4}^*$ is spanned by $\omega_1, \cdots, \omega_4$. The action $d\rho_{x_4}$ of $\mathfrak{g}_{x_4}$ on $V_{x_4}^*$ is given as follows:

$$(4.4) \qquad d\rho_{x_4}(\tilde{A})(\omega_1, \cdots, \omega_4) = (\omega_1, \cdots, \omega_4) \left(\begin{array}{c|c} (2\eta - 5\varepsilon)I_3 + X & 0 \\ \hline * \quad * \quad * & -6\varepsilon \end{array} \right).$$

Since ω_1 is a generic point, we have $\Lambda_4 = \Lambda_{20}^*$ and $\Lambda_{20} = \Lambda_4^*$. Let A_0 be an element of $\mathfrak{g}_{x_4}$ with $2\eta - 5\varepsilon = 1$, all remaining parts zero except ε and η in (4.3). Then $d\rho(A_0)x_4 = 0$ and $d\rho^*(A_0)\omega_1 = \omega_1$. However we have $\delta\chi(A_0) = 6\varepsilon$ which is not definite. If Λ_4 is a good holonomic variety, this must be definite by Proposition 1–3, and hence Λ_4 is not a good holonomic variety, i.e., $\Lambda_4 \not\subset W$. Note that the P.V. $(G_{x_4}, \rho_{x_4}, V_{x_4}^*)$ has no relative invariant.

(4) The isotropy subalgebra $\mathfrak{g}_{x_8}$ at $x_8 = (1, e_1e_2e_3e_4)$ is given as follows:

$$(4.5) \qquad \mathfrak{g}_{x_8} = \left\{ \tilde{X} = \left[\begin{array}{c|c|c|c} \varepsilon I_4 + X & \gamma & 0 & 0 \\ \hline 0 & 2\eta & 0 & 0 \\ \hline 0 & \delta & -\varepsilon I_4 - {}^tX & 0 \\ \hline -{}^t\delta & 0 & -{}^t\gamma & -2\eta \end{array} \right] \oplus \left(\begin{array}{cc} \eta + 2\varepsilon & 0 \\ 0 & \eta - 2\varepsilon \end{array} \right); \right.$$

$$\left. X \in \mathfrak{sl}(4), \gamma, \delta \in \boldsymbol{C}^4 \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(4)) \oplus \mathfrak{u}(8).$$

Put $\omega_1 = (e_2e_3e_4e_5, 0)$, $\omega_2 = -(e_1e_3e_4e_5, 0)$, $\omega_3 = (e_1e_2e_4e_5, 0)$, $\omega_4 = -(e_1e_2e_3e_5, 0)$, $\omega_5 = (0, e_1e_5)$, $\omega_6 = (0, e_2e_5)$, $\omega_7 = (0, e_3e_5)$, $\omega_8 = (0, e_4e_5)$. Then the conormal vector space $V_{x_8}^*$ is spanned by $\omega_1, \cdots, \omega_8$, and the action $d\rho_{x_8}$ of $\mathfrak{g}_{x_8}$ on $V_{x_8}^*$ is given as follows.

$$(4.6) \qquad d\rho_{x_8}(\tilde{X})(\omega_1, \cdots, \omega_8) = (\omega_1, \cdots, \omega_8)\left(\begin{array}{c|c} (3\varepsilon-2\eta)I_4+X & 0 \\ \hline 0 & -(3\varepsilon+2\eta)I_4-{}^t X \end{array}\right)$$

where $\tilde{X} \in \mathfrak{g}_{x_8}$ in (4.5).

Any relative invariant of $(G_{x_8}, \rho_{x_8}, V_{x_8}^*)$ is of the form $c \cdot g(x)^m$ $(c \in \mathbf{C}, m \in \mathbf{Z})$ where $g(x) = \sum_{i=1}^4 x_i x_{i+4}$ for $x = \sum_{i=1}^8 x_i \omega_i$. Clearly $y_8 = \omega_1 + \omega_5 = (e_2e_3e_4e_5, e_1e_5)$ is a generic point, and $y_8' = \omega_1 + \omega_6 = (e_2e_3e_4e_5, e_2e_5)$ is a point of the one-codimensional orbit. Hence we have $\Lambda_8 = \Lambda_8^*$ and $\dim \Lambda_1 \cap \Lambda_8 = \dim V - 1$. Since $\Lambda_{13} = \Lambda_1^*$, we have also $\dim \Lambda_8 \cap \Lambda_{13} = \dim V - 1$. Note that $(G_{x_8}, \rho_{x_8}, V_{x_8}^*)$ is a regular P.V. since $\rho_{x_8}(G_{x_8})$ and its generic isotropy subgroup are reductive (See [1]). By Corollary 1–7, Λ_8 is a good holonomic variety. Let $\tilde{X}_0$ be an element of $\mathfrak{g}_{x_8}$ with $\eta = -\frac{1}{2}$, all remaining parts zero in (4.5). Then $d\rho(\tilde{X}_0)x_8 = 0$ and $d\rho^*(\tilde{X}_0)y_8 = y_8$. Since $\delta\chi(\tilde{X}_0) = 4\eta = -2$, $\mathrm{tr}_{V_{x_8}^*}\tilde{X}_0 = -16\eta = 8$ and $\dim V_{x_8}^* = 8$, we have $\mathrm{ord}_{\Lambda_8}f^s = -2s - \frac{8}{2}$. Since $m_{\Lambda_8} - m_{\Lambda_1} = 1$, they intersect regularly. By Corollary 1–2, we have $b_{\Lambda_8}(s)/b_{\Lambda_1}(s) = (s+4)$.

(5) We shall calculate the isotropy subalgebra at $x_9' = (1, e_1e_3 + e_2e_4)$ instead of $x_9 = (1, e_1e_2 + e_3e_4)$. It is given as follows.

$$(4.7) \qquad \mathfrak{g}_{x_9'} = \left\{ \tilde{A} = \left[\begin{array}{c|c} \begin{array}{c|c} \varepsilon I_4 + A & B \\ \hline 0 & 2\eta \end{array} & 0 \\ \hline C & \begin{array}{c|c} -\varepsilon I_4 - {}^t A & 0 \\ \hline -{}^t B & -2\eta \end{array} \end{array}\right] + \left(\begin{array}{c|c} 2\varepsilon + \eta & 0 \\ \hline c_{13} + c_{24} & \eta \end{array}\right); \right.$$

$$\left. A \in \mathfrak{sp}(2),\ B \in \mathbf{C}^4,\ C = -{}^t C = (c_{ij}) \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sp}(2)) \oplus \mathfrak{u}(4).$$

Put $\omega_1 = (e_2e_3e_4e_5, 0)$, $\omega_2 = -(e_1e_3e_4e_5, 0)$, $\omega_3 = (e_1e_2e_4e_5, 0)$, $\omega_4 = -(e_1e_2e_3e_5, 0)$, $\omega_5 = (e_1e_2e_3e_4, 0)$, $\omega_6 = -(e_3e_5, e_2e_3e_4e_5)$, $\omega_7 = (-e_4e_5, e_1e_3e_4e_5)$, $\omega_8 = (e_1e_5, -e_1e_2e_4e_5)$, $\omega_9 = (e_2e_5, e_1e_2e_3e_5)$. Then the conormal vector space $V_{x_9'}^*$ is spanned by these $\omega_1, \cdots, \omega_9$ and the action $d\rho_{x_9'}$ of $\mathfrak{g}_{x_9'}$ on $V_{x_9'}^*$ is given as follows:

$$(4.8) \quad d\rho_{x_{9'}}(\tilde{A})(\omega_1, \cdots, \omega_9) = (\omega_1, \cdots, \omega_9)\left(\begin{array}{c|c|c} A-(3\varepsilon+2\eta)I_4 & B & C' \\ \hline 0 & -4\varepsilon & 0 \\ \hline 0 & 0 & A-(\varepsilon+2\eta)I_4 \end{array}\right)$$

$$\text{with} \quad C' = \left[\begin{array}{c|c|c|c} c_{13}+2c_{24} & -c_{23} & 0 & c_{34} \\ \hline -c_{14} & 2c_{13}+c_{24} & -c_{34} & 0 \\ \hline 0 & -c_{12} & c_{13}+2c_{24} & -c_{14} \\ \hline c_{12} & 0 & -c_{23} & 2c_{13}+c_{24} \end{array}\right].$$

Clearly, $y_9 = \omega_5 + \omega_6$ is its generic point and $y_9' = \omega_1 + \omega_8$ is a point of the one-codimensional orbit. Note that $(G_{x_{9'}}, \rho_{x_{9'}}, V_{x_{9'}}^*)$ has only one orbit of codimension one. Since $y_9, y_9' \in S_9^*$, we have $\Lambda_9 = \Lambda_9^*$, and Λ_9 has no one-codimensional intersection with other conormal bundles. Let $\tilde{A}_0$ be an element of $\mathfrak{g}_{x_{9'}}$ with $\varepsilon = -\frac{1}{4}$, $\eta = -\frac{3}{8}$, all remaining parts zero in (4.7). Then $d\rho(\tilde{A}_0)x_9' = 0$ and $d\rho^*(\tilde{A}_0)y_9 = y_9$. We have $\delta\chi(\tilde{A}_0) = 2\{(2\varepsilon + \eta) + \eta\} = -\frac{5}{2}$, we have $m_{\Lambda_9} = \frac{5}{2}$. This implies that the conormal bundle Λ_9 is not a good holonomic variety, i.e., $\Lambda_9 \not\subset W$ since otherwise m_{Λ_9} must be a non-negative integer (See §1 or [1]).

(6) The isotropy subalgebra $\mathfrak{g}_{x_{13}}$ at $x_{13} = (1, e_1 e_2)$ is given as follows.

$$(4.9) \quad \mathfrak{g}_{x_{13}} = \left\{ \tilde{X} = \left[\begin{array}{c|c|cc|c} \varepsilon_1 I_2 + X & Z & 0 & -b & 0 \\ & & b & 0 & \\ \hline 0 & 2\varepsilon I_3 + Y & 0 & & 0 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{C}} & -\varepsilon_1 I_2 - {}^t X & 0 \\ & & -{}^t Z & -2\varepsilon I_3 - {}^t Y \end{array}\right] \right.$$

$$\left. \oplus \left(\begin{array}{c|c} 3\varepsilon + \varepsilon_1 & b \\ \hline c_{12} & 3\varepsilon - \varepsilon_1 \end{array}\right); \; X \in \mathfrak{sl}(2), \; Y \in \mathfrak{sl}(3), \; {}^t C = -C \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(3)) \oplus \mathfrak{u}(15).$$

Since $\Lambda_{13} = \Lambda_1^*$, $\Lambda_8 = \Lambda_8^*$, and $\dim \Lambda_1^* \cap \Lambda_8^* = \dim V - 1$, the conormal bundle Λ_{13} is a good holonomic variety and $\dim \Lambda_8 \cap \Lambda_{13} = \dim V - 1$. They intersect regularly. Put $\omega_1 = (0, e_2 e_3 e_4 e_5)$, $\omega_2 = (e_2 e_3 e_4 e_5, 0)$, $\omega_3 = (0, e_1 e_3 e_4 e_5)$, $\omega_4 = (e_1 e_3 e_4 e_5, 0)$, $\omega_5 = (e_4 e_5, -e_1 e_2 e_4 e_5)$, $\omega_6 = (e_3 e_5, -e_1 e_2 e_3 e_5)$, $\omega_7 = (e_3 e_4, -e_1 e_2 e_3 e_4)$, $\omega_8 = (0, e_4 e_5)$, $\omega_9 = (0, e_3 e_5)$, $\omega_{10} = (0, e_3 e_4)$, $\omega_{11} = (e_1 e_2 e_4 e_5, 0)$, $\omega_{12} = (e_1 e_2 e_3 e_5, 0)$, $\omega_{13} = (e_1 e_2 e_3 e_4, 0)$. Then the conormal vector space $V_{x_{13}}^*$ is spanned by these $\omega_1, \cdots, \omega_{13}$ and the action $d\rho_{x_{13}}$ of $\mathfrak{g}_{x_{13}}$ on $V_{x_{13}}^*$ is given as follows:

$$d\rho_{x_{13}}(\tilde{X})(\omega_1, \cdots, \omega_{13})$$

$$(4.10) \qquad = (\omega_1, \cdots, \omega_{13})\left(\begin{array}{c|c} -6\varepsilon I_4 + d\rho_1(X \oplus W) & * \\ \hline 0 & -4\varepsilon I_9 + d\rho_2(W \oplus Y) \end{array}\right)$$

where $\rho_1 = \Lambda_1 \otimes \Lambda_1$ for $SL(2) \times SL(2)$, $\rho_2 = (2\Lambda_1) \otimes \Lambda_1$ for $SL(2) \times SL(3)$ and $W = \begin{pmatrix} \varepsilon_1 & b \\ c_{12} & -\varepsilon_1 \end{pmatrix} \in \mathfrak{sl}(2)$.

As a generic point, we may take $y_{13} = \omega_5 + \omega_9 + \omega_{13} = (e_4e_5 + e_1e_2e_3e_4, e_3e_5 - e_1e_2e_4e_5)$. Let $\tilde{X}_0$ be an element of $\mathfrak{g}_{x_{13}}$ with $\varepsilon = -\frac{1}{4}$, all remaining parts zero. Then $d\rho(\tilde{X}_0)x_{13} = 0$ and $d\rho^*(\tilde{X}_0)y_{13} = y_{13}$. Since $\delta\chi(\tilde{X}_0) = 12\varepsilon = -3$, $\mathrm{tr}_{V_{x_{13}}^*}\tilde{X}_0 = -60\varepsilon = 15$ and $\dim V_{x_{13}}^* = 13$, we have $\mathrm{ord}_{\Lambda_{13}}f^s = -3s - \frac{17}{2}$. By Corollary 1-2, we have $b_{\Lambda_{13}}(s)/b_{\Lambda_8}(s) = (s + 5)$. By (4.10), we can see that $(G_{x_{13}}, \rho_{x_{13}}, V_{x_{13}}^*)$ has the unique relative invariant (See Lemma 4 and Proposition 5 in §4 in [1]), i.e., it has the unique one-codimensional orbit.

(7) The isotropy subalgebra $\mathfrak{g}_{x_{15}}$ at $x_{15} = (1 + e_1e_2e_3e_4, 0)$ is given as follows.

$$(4.11) \qquad \mathfrak{g}_{x_{15}} = \left\{ \tilde{X} = \left[\begin{array}{c|c|c|c} X & Y & C'' & 0 \\ \hline 0 & 2\varepsilon & 0 & 0 \\ \hline C & C' & -{}^tX & 0 \\ \hline -{}^tC' & 0 & -{}^tY & -2\varepsilon \end{array}\right] \oplus \begin{pmatrix} \varepsilon & b \\ 0 & \eta \end{pmatrix}; \right.$$

$$X \in \mathfrak{sl}(4),\ CC'' = -\operatorname{Pf} C \cdot I_4,\ {}^tC = -C \in M(4) \left.\vphantom{\begin{array}{c}X\\X\\X\end{array}}\right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{o}(7)) \oplus \mathfrak{u}(9).$$

Note that, in (4.11), $\tilde{X}_0 = \left(\begin{array}{c|c} X & C'' \\ \hline C & -{}^tX \end{array}\right)$ is the spin representation of X_0 in $\mathfrak{o}(7)$. Put $\omega_1 = (0, e_1e_5)$, $\omega_2 = (0, e_2e_5)$, $\omega_3 = (0, e_3e_5)$, $\omega_4 = (0, e_4e_5)$, $\omega_5 = (0, e_2e_3e_4e_5)$, $\omega_6 = (0, -e_1e_3e_4e_5)$, $\omega_7 = (0, e_1e_2e_4e_5)$, $\omega_8 = (0, -e_1e_2e_3e_5)$, $\omega_9 = (0, \frac{1}{2}(1 - e_1e_2e_3e_4))$, $\omega_{10} = (0, e_2e_3)$, $\omega_{11} = (0, -e_2e_4)$, $\omega_{12} = (0, e_3e_4)$, $\omega_{13} = (0, e_1e_4)$, $\omega_{14} = (0, e_1e_3)$, $\omega_{15} = (0, e_1e_2)$. The conormal vector space $V_{x_{15}}^*$ is spanned by $\omega_1, \cdots, \omega_{15}$. Then $y_{15} = \omega_9$ is its generic point and $y_{15}' = \omega_{10} + \omega_{14}$ is a point of the unique one-codimensional orbit. Since $y_{15}, y_{15}' \in S_{15}^*$, we have $\Lambda_{15} = \Lambda_{15}^*$, and Λ_{15} has no one-codimensional intersection with any other conormal bundle. Let $\tilde{X}_1$ be an element of $\mathfrak{g}_{x_{15}}$ with $\varepsilon = \beta + 1$, $\eta = \beta$, all remaining parts zero in (4.11). Then $d\rho(\tilde{X}_1)x_{15} = 0$ and $d\rho^*(\tilde{X}_1)y_{15} = y_{15}$. Since $\delta\chi(\tilde{X}_1) = 2(\varepsilon + \eta) = 2(2\beta + 1)$ is not definite, the conormal bundle Λ_{15} is not a good holonomic variety, i.e., $\Lambda_{15} \not\subset W$.

(8)　Since $\Lambda_{20} = \Lambda_4^*$ and $\Lambda_4 \not\subset W$, the conormal bundle Λ_{20} is not a good holonomic variety. Note that $W \subset V \times V^*$ is symmetric with respect to V and V^*.

(9)　Since $\Lambda_{32} = \{0\} \times V^*$, the conormal bundle Λ_{32} is a good holonomic variety. Put $A_0 = (0) \oplus (-I_2)$. Then $d\rho(A_0)x_{32} = 0$ and $d\rho^*(A_0)y_{32} = y_{32}$ where y_{32} is a generic point of (G, ρ^*, V^*). Since $\delta\chi(A_0) = -4$, $\mathrm{tr}_{V^*_{x_{32}}} A_0 = 32$ and $\dim V^*_{x_{32}} = 32$, we have $\mathrm{ord}_{\Lambda_{32}} f^s = -4s - \frac{32}{2}$ and hence by Corollary 1–2, we have $b_{\Lambda_{32}}(s)/b_{\Lambda_{13}}(s) = s + 8$. Note that $\Lambda_{32} = \Lambda_0^*$ and $\Lambda_{13} = \Lambda_1^*$. Since $b_{\Lambda_0}(s) = 1$ and $b_{\Lambda_{32}}(s) = b(s)$, we have the b-function $b(s) = (s + 1)(s + 4)(s + 5)(s + 8)$, and the holonomy diagram (Figure 4–1). We denote Λ_m by $\textcircled{m}$.
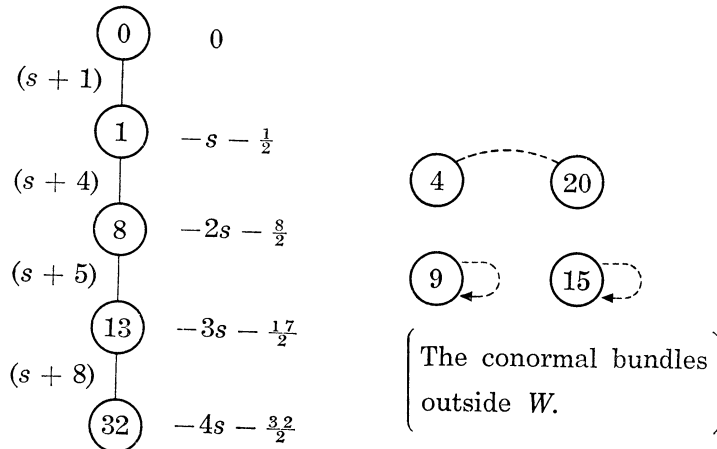
Figure 4–1.　Holonomy diagram of (Spin(10) $\times$ GL(2), half-spin rep. $\otimes \Lambda_1$, $V(16) \otimes V(2)$)

§5.　$(GL(1) \times \mathrm{Spin}(12),\ \square \otimes$ half-spin rep., $V(1) \otimes V(32))$

The representation space $V = V(1) \otimes V(32)$ is spanned by 1, $e_i e_j$, $e_r e_s e_t e_u$, $e_1 e_2 e_3 e_4 e_5 e_6$ $(1 \leq i < j \leq 6, 1 \leq r < s < t < u \leq 6)$ (See [1], [4]). J-I. Igusa has completed the orbital decomposition of this space (See [3]). There exist five G-orbits $S_m = \rho(G)x_m$ $(m = 0, 1, 7, 16, 32)$ where S_m denotes the m-codimensional orbit and $x_0 = 1 + e_1 e_2 e_3 e_4 e_5 e_6$, $x_1 = 1 + e_2 e_3 e_5 e_6 + e_1 e_3 e_4 e_6$, $x_7 = 1 + e_2 e_3 e_5 e_6$, $x_{16} = 1$, $x_{32} = 0$. We identify V^* with V by taking $\{1, e_i e_j, e_r e_s e_t e_u, e_1 e_2 e_3 e_4 e_5 e_6\}$ as a dual basis. Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, there exist also five orbits $S_m^*(m = 0, 1, 7, 16, 32)$ in V^*. We denote by Λ_m (resp. Λ_m^*) the conormal bundle of S_m (resp. S_m^*). Clearly, we have $\Lambda_0 = V \times \{0\} = \Lambda_{32}^*$ and $\Lambda_{32} = \{0\} \times V^* = \Lambda_0^*$. The Lie algebra $\mathfrak{g}$ of $GL(1) \times \mathrm{Spin}(12)$ is given as follows:

(5.1) $\mathfrak{g} = \left\{ (d) \oplus \left(\begin{array}{c|c} A & B \\ \hline C & -{}^t A \end{array} \right); A, B, C \in M(6), \; {}^t B = -B, \; {}^t C = -C \right\}.$

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at x_0 is given as follows (See [1]).

(5.2) $\mathfrak{g}_{x_0} = \left\{ (0) \oplus \left(\begin{array}{c|c} A & 0 \\ \hline 0 & -{}^t A \end{array} \right); A \in \mathfrak{sl}(6) \right\} \cong \mathfrak{sl}(6).$

Since $\Lambda_0 = V \times \{0\}$, we have $\mathrm{ord}_{\Lambda_0} f^s = 0$, where f denotes the relative invariant of degree four (See [1], [3]).

(2) By using (5.29) in [1], we can calculate the isotropy subalgebra $\mathfrak{g}_{x_1}$.

$\mathfrak{g}_{x_1} = \left\{ \tilde{A} = (d) \oplus \left\{ \begin{array}{c|c} A & B \\ \hline C & -{}^t A \end{array} \right\}; \; a_1 + a_4 = a_2 + a_5 = -a_3 - a_6 = 2d, \; c_{36} = 0 \right\}$

(5.3)
$\cong \left\{ (d) \oplus \left(\begin{array}{c|c} dI_6 + A_0 & B_0 \\ \hline 0 & -dI_6 - {}^t A_0 \end{array} \right); A_0 \in \mathfrak{sp}(3), \; {}^t B_0 = -B_0, \; \mathrm{tr}\, B_0 J = 0 \right\}$

with $J = \left(\begin{array}{c|c} 0 & I_3 \\ \hline -I_3 & 0 \end{array} \right),$

$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sp}(3)) \oplus V(14)$ where

$$A = \begin{bmatrix} a_1 & a_{12} & 0 & a_{14} & a_{15} & 0 \\ a_{21} & a_2 & 0 & a_{15} & a_{25} & 0 \\ a_{31} & a_{32} & a_3 & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & 0 & a_4 & -a_{21} & 0 \\ a_{42} & a_{52} & 0 & -a_{12} & a_5 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & c_{46} & 0 & 0 & c_{34} \\ 0 & 0 & c_{56} & 0 & 0 & c_{35} \\ -c_{46} & -c_{56} & 0 & c_{16} & c_{26} & b_{36} \\ 0 & 0 & -c_{16} & 0 & 0 & c_{13} \\ 0 & 0 & -c_{26} & 0 & 0 & c_{23} \\ -c_{34} & -c_{35} & -b_{36} & -c_{13} & -c_{23} & 0 \end{bmatrix}$$

with $b_{36} + c_{14} + c_{25} = 0$.

The conormal vector space $V_{x_1}^*$ is spanned by $e_1 e_2 e_4 e_5$ on which $\mathfrak{g}_{x_1}$ acts as $d\rho_{x_1}(\tilde{A}) e_1 e_2 e_4 e_5 = -4 d e_1 e_2 e_4 e_5$ for $\tilde{A} \in \mathfrak{g}_{x_1}$. This implies that $\Lambda_1 = \overline{G(x_1, e_1 e_2 e_4 e_5)}$ $= \Lambda_{16}^*$. Since 0 is the point of the one-codimensional orbit, we have $\dim \Lambda_0 \cap \Lambda_1 = \dim V - 1$ and $\Lambda_0 \cap \Lambda_1$ is G_0-prehomogeneous, i.e., Λ_1 is a good holonomic variety by Proposition 1–5. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $-4d = 1$. Then $d\rho(A_0) x_1 = 0$ and $d\rho^*(A_0) y_1 = y_1$ where $y_1 = e_1 e_2 e_4 e_5$. Since $\delta\chi(A_0) = 4d = -1$, $\mathrm{tr}_{V_{x_1}^*} A_0 = \dim V_{x_1}^* = 1$, we have $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$. By Proposition 1–4, Λ_0 and Λ_1 intersect regularly and hence $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s + 1)$ by Corollary 1–2.

(3) By using (5.29) in [1], we can calculate the isotropy subalgebra $\mathfrak{g}_{x_7}$.

$\mathfrak{g}_{x_7} = \left\{ \tilde{A} = (d) \oplus \left(\begin{array}{c|c} A & C' \\ \hline C & -{}^t A \end{array} \right); d = \frac{a_1 + a_4}{2}, \; a_2 + a_3 + a_5 + a_6 = 0, \right.$

$$'C = C, \ A \text{ and } C' \text{ are given as follows} \Big\}$$

$$(5.4)$$
$$\cong \left\{ (d) \oplus \left(\begin{array}{c|c|c} dI_2 + A_1 & 0 & 0 \\ \hline U & d\rho_1(V) & 0 \\ \hline W & -{}^tUS & -dI_2 - {}^tA_1 \end{array} \right); \begin{array}{l} A_1 \in \mathfrak{sl}(2), \ W \in \mathfrak{o}(2), \\ V \in \mathfrak{o}(7), \ U \in M(8.2) \end{array} \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{o}(7) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(17)$$

where ρ_1 is the spin-representation of Spin(7), $S = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}$, and

$$A = \begin{bmatrix} a_1 & 0 & 0 & a_{14} & 0 & 0 \\ a_{21} & a_2 & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_3 & a_{34} & a_{35} & a_{36} \\ a_{41} & 0 & 0 & a_4 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_5 & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_6 \end{bmatrix}, \ C' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{56} & 0 & -c_{36} & c_{35} \\ 0 & -c_{56} & 0 & 0 & c_{26} & -c_{25} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{36} & -c_{26} & 0 & 0 & c_{23} \\ 0 & -c_{35} & c_{25} & 0 & -c_{23} & 0 \end{bmatrix}.$$

Put $\omega_1 = e_1e_4 - e_1e_2e_3e_4e_5e_6$, $\omega_2 = e_1e_2e_4e_5$, $\omega_3 = e_1e_2e_3e_4$, $\omega_4 = e_1e_2e_4e_6$, $\omega_5 = e_1e_3e_4e_5$, $\omega_6 = e_1e_3e_4e_6$, $\omega_7 = e_1e_4e_5e_6$. The conormal vector space $V_{x_7}^*$ is spanned by these $\omega_1, \cdots, \omega_7$, and $(G_{x_7}, \rho_{x_7}, V_{x_7}^*) \cong (GL(1) \times SO(7), \square \otimes \Lambda_1, V(1) \otimes V(7))$. Then ω_1 is its generic point and $\omega_2 = e_1e_2e_4e_5$ is a point of the one-codimensional orbit. Since $\Lambda_1 = \overline{G(x_1, \omega_2)}$, we have codim $\Lambda_1 \cap \Lambda_7 = 1$. Since $\Lambda_1 \cap \Lambda_7$ is G_0-prehomogeneous, Λ_7 is a good holonomic variety. We have $\Lambda_7 = \Lambda_7^*$.

Let A_β' be an element of $\mathfrak{g}_{x_7}$ with $4d = 2(a_1 + a_4) = -\beta - 1$, $2(a_3 + a_6) = -2(a_2 + a_5) = 1 - \beta$, all remaining parts zero in (5.4). Then we have $d\rho(A_\beta')x_7 = 0$, $d\rho^*(A_\beta') \ \omega_2 = \omega_2$ and $\text{tr}_{\tilde{V}} A_\beta' = \beta$ where $\tilde{V} = V_{x_7}^* \bmod d\rho_{x_7}(\mathfrak{g}_{x_7})\omega_2$. This implies that Λ_1 and Λ_7 intersect regularly by Proposition 1–4. Let A_0 be an element of $\mathfrak{g}_{x_7}$ with $d = -\frac{1}{2}$, all remaining parts zero in (5.4). Then $d\rho(A_0)x_7 = 0$, $d\rho^*(A_0)\omega_1 = \omega_1$ Since $\delta\chi(A_0) = 4d = -2$, $\text{tr}_{V_{x_7}^*} A_0 = -14d = 7$, $\dim V_{x_7}^* = 7$, we have $\text{ord}_{\Lambda_7} f^s = -2s - \frac{7}{2}$. By Corollary 1–2, we have $b_{\Lambda_7}(s)/b_{\Lambda_1}(s) = (s + \frac{7}{2})$.

(4)   Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, $\Lambda_7^* = \Lambda_7$ and $\Lambda_1^* = \Lambda_{16}$, we have $\dim \Lambda_7 \cap \Lambda_{16} = \dim V - 1$ and they intersect regularly. Since $\Lambda_7 \cap \Lambda_{16} = \Lambda_1^* \cap \Lambda_7^*$ is G_0-prehomogeneous, Λ_{16} is a good holonomic variety. Since $d\rho(\tilde{A}) \cdot 1 = d - (\text{tr} A/2) + \sum_{i<j} b_{ij}e_ie_j$, the isotropy subalgebra $\mathfrak{g}_{x_{16}}$ at $x_{16} = 1$ is

$$(5.5) \qquad \mathfrak{g}_{x_{16}} = \left\{ \tilde{A} = \left(\frac{\text{tr} A}{2} \right) \oplus \left(\begin{array}{cc} A & 0 \\ C & -{}^tA \end{array} \right); {}^tC = -C, A, C \in M(3) \right\}$$

$$\cong \mathfrak{gl}(6) \oplus V(15).$$

The conormal vector space $V_{x_{16}}^*$ is spanned by $e_1 e_2 e_3 e_4 e_5 e_6$ and $e_i e_j e_k e_\ell$ $(1 \leq i < j < k < \ell \leq 6)$. Then the action $d\rho_{x_{16}}$ of $\mathfrak{g}_{x_{16}}$ on $V_{x_{16}}^*$ is given by

$$(5.6) \qquad d\rho_{x_{16}}(\tilde{A})(\omega_1, \cdots, \omega_{16}) = (\omega_1, \cdots, \omega_{16}) \begin{pmatrix} -\operatorname{tr} A & c' \\ 0 & d\rho_1^*(A) \end{pmatrix}$$

where $\omega_1 = e_1 e_2 e_3 e_4 e_5 e_6$, $\{\omega_2, \cdots, \omega_{16}\} = \{e_i e_j e_k e_\ell, 1 \leq i < j < k < \ell \leq 6\}$, ${}^t c' \in \mathbf{C}^{15}$, $\rho_1 = \Lambda_2$ for $GL(6)$.

Then $y_{16} = e_1 e_2 e_4 e_5 + e_1 e_3 e_4 e_6 + e_2 e_3 e_5 e_6$ is its generic point. Let A_0 be an element of $\mathfrak{g}_{x_{16}}$ with $A = -\frac{1}{4} I_6$, $C = 0$ in (5.5). Then $d\rho(A_0) x_{16} = 0$ and $d\rho^*(A_0) y_{16} = y_{16}$. Since $\delta\chi(A_0) = 4d = 2\operatorname{tr}(-\frac{1}{4} I_6) = -3$, $\operatorname{tr}_{V_{x_{16}}^*} A_0 = -11 \operatorname{tr}(-\frac{1}{4} I_6) = \frac{33}{2}$ and $\dim V_{x_{16}}^* = 16$, we have $\operatorname{ord}_{\Lambda_{16}} f^s = s\delta\chi(A_0) - \operatorname{tr}_{V_{x_{16}}^*} A_0 + \frac{1}{2} \dim V_{x_{16}}^* = -3s - \frac{17}{2}$. By Corollary 1–2, we have $b_{\Lambda_{16}}(s)/b_{\Lambda_7}(s) = (s + \frac{11}{2})$. By (5.6), the character group of $\rho_{x_{16}}(G_{x_{16}})$ is one-dimensional and hence $(G_{x_{16}}, \rho_{x_{16}}, V_{x_{16}}^*)$ has (at most) the unique one-codimensional orbit.

(5) Since $\Lambda_{32} = \Lambda_0^*$ and $\Lambda_{16} = \Lambda_1^*$, they intersect regularly with codimension one. We shall calculate the order $\operatorname{ord}_{\Lambda_{32}} f^s$. Since $(G_{x_{32}}, \rho_{x_{32}}, V_{x_{32}}) \cong (G, \rho^*, V^*)$, $y_{32} = 1 + e_1 e_2 e_3 e_4 e_5 e_6$ is its generic point. Let A_0 be an element of $\mathfrak{g}$ with $d = -1$, all remaining parts zero in (5.1). Then $d\rho(A_0) x_{32} = 0$, $d\rho^*(A_0) y_{32} = y_{32}$. Since $\delta\chi(A_0) = -4$, $\operatorname{tr}_{V_{32}^*} A_0 = -32d = 32$, $\dim V_{x_{32}}^* = 32$, we have $\operatorname{ord}_{\Lambda_{32}} f^s = -4s - \frac{32}{2}$. By Corollary 1–2, we have $b_{\Lambda_{32}}(s)/b_{\Lambda_{16}}(s) = s + 8$. Since $b_{\Lambda_0}(s) = 1$ and $b_{\Lambda_{32}}(s) = b(s)$, we obtain the b-function $b(s) = (s + 1)(s + \frac{7}{2})(s + \frac{11}{2})(s + 8)$ and the holonomy diagram (Figure 5–1). We denote $\textcircled{$\Lambda_m$}$ by $\textcircled{$m$}$.

$$
\begin{array}{cl}
\textcircled{0} & \quad 0 \\[4pt]
(s+1) \qquad | & \\[4pt]
\textcircled{1} & \quad -s - \tfrac{1}{2} \\[4pt]
(s+\tfrac{7}{2}) \qquad | & \\[4pt]
\textcircled{7} & \quad -2s - \tfrac{7}{2} \\[4pt]
(s+\tfrac{11}{2}) \qquad | & \\[4pt]
\textcircled{16} & \quad -3s - \tfrac{17}{2} \\[4pt]
(s+8) \qquad | & \\[4pt]
\textcircled{32} & \quad -4s - \tfrac{32}{2}
\end{array}
$$
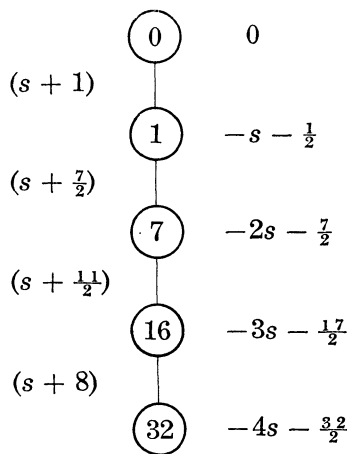
Figure 5–1. Holonomy diagram of $(GL(1) \times \operatorname{Spin}(12)$,
$\square \otimes$ half-spin rep., $V(1) \otimes V(32))$.

Remark. (1) $(GL(1) \times \mathrm{Spin}(7), \ \square \otimes \mathrm{spin \ rep.}, \ V(1) \otimes V(8))$

(2) $(\mathrm{Spin}(7) \times GL(2), \ \mathrm{spin \ rep.} \otimes \varLambda_1, \ V(8) \otimes V(2))$

(3) $(\mathrm{Spin}(7) \times GL(3), \ \mathrm{spin \ rep.} \otimes \varLambda_1, \ V(8) \otimes V(3))$

(4) $(GL(1) \times \mathrm{Spin}(9), \ \square \otimes \mathrm{spin \ rep.}, \ V(1) \otimes V(16))$

(5) $(GL(1) \times (G_2), \ \square \otimes \varLambda_2, \ V(1) \otimes V(7))$

(6) $((G_2) \times GL(2), \ \varLambda_2 \otimes \varLambda_1, \ V(7) \otimes V(2))$

(7) $(GL(1) \times \mathrm{Spin}(11), \ \square \otimes \mathrm{spin \ rep.}, \ V(1) \otimes V(32))$

Since $\mathrm{Spin}\,(7) \hookrightarrow SO(8)$ by the spin representation, the first three triplets (1), (2), (3) are reduced to the triplet $(SO(8) \times GL(n), \ \varLambda_1 \otimes \varLambda_1, \ V(8) \otimes V(n))$ $(n = 1, 2, 3)$ (See [1]). Since $\mathrm{Spin}(9) \hookrightarrow SO(16)$ by the spin representation, (4) is reduced to $(SO(16) \times GL(1), \ \varLambda_1 \otimes \varLambda_1, \ V(16) \otimes V(1))$ (See [1]). Since $(G_2) \hookrightarrow SO(7)$ by $\varLambda_2$, (5) and (6) are reduced to $(SO(7) \times GL(n), \ \varLambda_1 \otimes \varLambda_1,$ $V(7) \otimes V(n))$ $(n = 1, 2)$ (See [1]).

Since the spin representation of $\mathrm{Spin}(11)$ is obtained by the restriction of the half-spin representation of $\mathrm{Spin}(12)$ to $\mathrm{Spin}(11)$, (7) is reduced to $\mathrm{Spin}\,(12)$ in §5. Note that the b-function depends essentially on the relative invariant itself, not on the group.

§6. $(GL(1) \times E_6, \ \square \otimes \varLambda_1, \ V(1) \otimes V(27))$

The Lie algebra $\mathfrak{g}$ of $G = GL(1) \times E_6$ can be written as $\mathfrak{g} = \mathscr{D}_0 \oplus \mathscr{T}_1$ $\oplus \mathscr{T}_2 \oplus \mathscr{T}_3 \oplus \{R_y\}_{y \in \mathscr{I}}$ (See Proposition 37 and Example 39 of §1 in [1]). The representation space is identified with the simple Jordan algebra $\mathscr{I}$.

$$(6.1) \qquad \mathscr{I} = \left\{ X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}; \xi_1, \xi_2, \xi_3 \in C; x_1, x_2, x_3 \in \mathscr{L} \right\}$$

where $\mathscr{L}$ denotes the complex Cayley algebra.

DEFINITION 6–1. For $a \in \mathscr{L}$, we define elements $T_i(a)$ and $T_i'(a)$ $(i = 1, 2, 3)$ of $\mathfrak{g}$ as follows:

$$T_1(a) \cdot X = [R_{A_1(a)} + \mathscr{T}_1(2a)]X = \begin{pmatrix} 0 & 0 & x_3 a \\ 0 & 0 & a\xi_2 \\ \overline{x_3 a} & \overline{a\xi_2} & \mathrm{tr}(\bar{x}_1 a) \end{pmatrix}$$

$$T_1'(a) \cdot X = [R_{A_1(a)} - \mathscr{T}_1(2a)]X = \begin{pmatrix} 0 & \overline{a x_2} & 0 \\ a x_2 & \mathrm{tr}(a \bar{x}_1) & a\xi_3 \\ 0 & \overline{a\xi_3} & 0 \end{pmatrix}$$

$$T_2(a) \cdot X = [R_{A_2(\bar{a})} + \mathscr{T}_2(2\bar{a})]X = \begin{pmatrix} 0 & 0 & a\xi_1 \\ 0 & 0 & \bar{x}_3 a \\ \overline{a\xi_1} & \overline{a x_3} & \mathrm{tr}(x_2 a) \end{pmatrix}$$

$$T_2'(a) \cdot X = [R_{A_2(\bar a)} - \mathscr{T}_2(2\bar a)]X = \begin{pmatrix} \operatorname{tr}(ax_2) & a\bar x_1 & a\xi_3 \\ \overline{x_1\bar a} & 0 & 0 \\ \overline{a\xi_3} & 0 & 0 \end{pmatrix}$$

$$T_3(a) \cdot X = [R_{A_3(a)} + \mathscr{T}_3(2a)]X = \begin{pmatrix} 0 & a\xi_1 & 0 \\ \overline{a\xi_1} & \operatorname{tr}(\bar a x_3) & \overline{x_2 a} \\ 0 & x_2 a & 0 \end{pmatrix}$$

$$T_3'(a) \cdot X = [R_{A_3(a)} - \mathscr{T}_3(2a)]X = \begin{pmatrix} \operatorname{tr}(a\bar x_3) & a\xi_2 & ax_1 \\ \overline{a\xi_2} & 0 & 0 \\ \overline{ax_1} & 0 & 0 \end{pmatrix}$$

where $A_i(a)$ denotes the element of $\mathscr{J}$ with $x_i = a$, all remaining terms zero in (6.1) for $i = 1, 2, 3$, and $\operatorname{tr} b = b + \bar b$ for $b \in \mathscr{L}$. Thus we have $\mathfrak{g} = \mathscr{D}_0 \oplus T_1 \oplus T_2 \oplus T_3 \oplus T_1' \oplus T_2' \oplus T_3' \oplus \{R_{\begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}}\}$. For $a \in \mathscr{L}$, we put $t_i(a) = \exp T_i(a)$ and $t_i'(a) = \exp T_i'(a)$ for $i = 1, 2, 3$. They are elements of G. For $\xi \in C$, let $B_i(\xi)$ be the element of $\mathscr{J}$ with $\xi_i = \xi$, all remaining terms zero in (6.1) for $i = 1, 2, 3$ and put $c = \exp \xi$. We define an element $S_i(c)$ of G by $S_i(c) = \exp R_{B_i(\xi)}$ for $i = 1, 2, 3$. The following proposition is well-known.

PROPOSITION 6–2. *There exist four orbits* $S_m = \rho(G)x_m$ ($m = 0, 1, 10, 27$) *where* S_m *denotes the m-codimensional orbit, and* x_m *is given as follows*:

$$x_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad x_{10} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad x_{27} = (0).$$

Proof. Let X be a non-zero element of $\mathscr{J}$. Then we may assume that $\xi_1 = 1$ by t_i, t_i' and S_1. By $t_2(-\bar x_2)$ and $t_3(-x_3)$, we have $x_2 = x_3 = 0$. Unless $\xi_2 = \xi_3 = x_1 = 0$, we have $\xi_2 = 1$ by t_1' and S_2. Then by $t_1(-x_1)$, we have $x_1 = 0$. If $\xi_3 \neq 0$, we have $\xi_3 = 1$ by S_3. Thus we obtain four orbits. We shall calculate their codimension later.   Q.E.D.

DEFINITION 6–3. We identify the dual vector space V^* of $V = \mathscr{J}$ with V by $\langle X, Y \rangle = \operatorname{tr} X \circ Y$. Then the dual actions are given as follows: (i) $D^*Y = DY$ for $D \in \mathscr{D}_0$. (ii) $T_i^*(a)Y = -T_i'(a)Y$ for $a \in \mathscr{L}$ and $i = 1, 2, 3$. (iii) $T_i'^*(a)Y = -T_i(a)Y$ for $i = 1, 2, 3$ and $a \in \mathscr{L}$. (iv) $R_z^* Y = -R_z Y$ for $z \in \mathscr{J}$.

DEFINITION 6–4. Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, the dual space has also four orbits S_m^* ($m = 0, 1, 10, 27$). We denote by $\varLambda_m$ (resp. $\varLambda_m^*$) the conormal

bundle of S_m (resp. S_m^*). Clearly we have $\Lambda_0 = V \times \{0\} = \Lambda_{27}^*$ and $\Lambda_{27} = \{0\} \times V^* = \Lambda_0^*$.

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at x_0 is $\mathscr{D}_0$ which is the Lie algebra of F_4 (See [1]). Since $\Lambda_0 = V \times \{0\}$, we have $\mathrm{ord}_{\Lambda_0} f^s = 0$.

(2) For $\tilde{A} = D \oplus \sum_{i=1}^{3} (T_i(a_i) \oplus T_i'(a_i')) \oplus R_{\left(\begin{smallmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ 0 & & \alpha_3 \end{smallmatrix}\right)}$, we have $d\rho(\tilde{A})x_1 =$

$\begin{pmatrix} \alpha_1 & a_3+a_3' & a_2 \\ \overline{a_3+a_3'} & \alpha_2 & a_1 \\ \bar{a}_2 & \bar{a}_1 & 0 \end{pmatrix}$, and hence $\tilde{A}$ is an element of the isotropy subalgebra

$\mathfrak{g}_{x_1}$ at x_1 if and only if $\tilde{A} = D \oplus T_1'(a_1') \oplus T_2'(a_2') \oplus [T_3(a_3) \oplus T_3'(-a_3)] \oplus R_{\left(\begin{smallmatrix} 0 & & \\ & 0 & \\ & & \alpha_3 \end{smallmatrix}\right)}$.

The conormal vector space $V_{x_1}^*$ is given by $V_{x_1}^* = \left\{ \begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & \eta \end{pmatrix}; \eta \in C \right\} \cong \{\eta\}$

and $d\rho_{x_{10}}(\tilde{A})\eta = -\alpha_3\eta$. For $\tilde{A}_0 = R_{\left(\begin{smallmatrix} 0 & & \\ & 0 & \\ & & -1 \end{smallmatrix}\right)}$, we have $d\rho(\tilde{A}_0)x_1 = 0$ and $d\rho^*(\tilde{A}_0)y_1$

$= y_1$ where $y_1 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$. Since $\delta\chi(\tilde{A}_0) = -1$, $\mathrm{tr}_{V_{x_1}^*} \tilde{A}_0 = \dim V_{x_1}^* = 1$, we

have $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$. By Corollary 1–2, we have $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$.

(3) For $\tilde{A} = D \oplus \sum_{i=1}^{3} (T_i(a_i) \oplus T_i'(a_i')) \oplus R_{\left(\begin{smallmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{smallmatrix}\right)}$, we have $d\rho(\tilde{A})x_{10} =$

$\begin{pmatrix} \alpha_1 & a_3 & a_2 \\ \bar{a}_3 & 0 & 0 \\ \bar{a}_2 & 0 & 0 \end{pmatrix}$ and hence $\tilde{A} \in \mathfrak{g}_{x_{10}}$ if and only if $\tilde{A} = D \oplus T_1(a_1) \oplus T_1'(a_1') \oplus T_2'(a_2')$

$\oplus T_3'(a_3') \oplus R_{\left(\begin{smallmatrix} 0 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{smallmatrix}\right)}$. In this case, $\tilde{A}$ acts on the conormal vector space as

follows:

$$d\rho_{x_{10}}(\tilde{A}) \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & y_1 \\ 0 & \bar{y}_1 & \eta_3 \end{pmatrix} = \left[\begin{array}{c|c|c} 0 & & \\ \hline \begin{array}{c} -\alpha_2\eta_2 \\ -\mathrm{tr}(y_1\bar{a}_1) \end{array} & \begin{array}{c} Uy_1 - a_1\eta_3 - a_1'\eta_2 \\ -\frac{1}{2}(\alpha_2+\alpha_3)y_1 \end{array} \\ \hline \begin{array}{c} \overline{Uy_1} - a_1\eta_3 - a_1'\eta_2 \\ -\frac{1}{2}(\alpha_2+\alpha_3)\bar{y}_1 \end{array} & \begin{array}{c} -\alpha_3\eta_3 - \mathrm{tr}(\bar{y}_1a_1') \end{array} \end{array} \right].$$

Let y_{10} (resp. y_{10}') be the element of $\mathscr{J}$ with $x_1 = 1$ (resp. $\xi_3 = 1$), all remaining parts zero in (6.1). Then y_{10} is a generic point of $(G_{x_{10}}, \rho_{x_{10}}, V_{x_{10}}^*)$ and y_{10}' is a point of the one-codimensional orbit. Thus we have $\Lambda_{10} = \Lambda_1^*$, $\Lambda_1 = \Lambda_{10}^*$ and $\dim \Lambda_1 \cap \Lambda_{10} = \dim V - 1$. Put $\tilde{A}_0 = R_{\left(\begin{smallmatrix} 0 & & \\ & -1 & \\ & & -1 \end{smallmatrix}\right)}$. Then we have

$d\rho(\tilde{A}_0)x_{10} = 0$ and $d\rho^*(\tilde{A}_0)y_{10} = y_{10}$. Since $\delta\chi(\tilde{A}_0) = -2$, $\mathrm{tr}_{V_{x_{10}}^*} \tilde{A}_0 = 10 = \dim V_{x_{10}}^*$, we have $\mathrm{ord}_{\Lambda_{10}} f^s = -2s - \frac{10}{2}$. By Corollary 1–2, we have $b_{\Lambda_{10}}(s)/b_{\Lambda_1}(s) = (s+5)$.

(4) The isotropy subalgebra $\mathfrak{g}_{x_{27}}$ is $\mathfrak{g}$. Put $y_{27} = x_0$ and $y_{27}' = x_1'$. Then $d\rho(\tilde{A}_0)x_{27} = 0$ and $d\rho^*(\tilde{A}_0)y_{27} = y_{27}$ for $\tilde{A}_0 = R_{-I_3}$. Since $\delta\chi(\tilde{A}_0) = -3$, $\mathrm{tr}_{V_{x_{27}}^*} \tilde{A}_0$

$= \dim V^*_{x_{27}} = 27$, we have $\mathrm{ord}_{A_{27}} f^s = -3s - \frac{27}{2}$. Since $\Lambda_{10} = \Lambda_1^*$, $\Lambda_{27} = \Lambda_0^*$, we have codim $\Lambda_{10} \cap \Lambda_{27} = 1$ and $b_{A_{27}}(s)/b_{A_{10}}(s) = (s+9)$. Thus we obtain the b-function $b(s) = (s+1)(s+5)(s+9)$ and the holonomy diagram (Figure 6–1). Note that the relative invariant $f(X)$ is given by the determinant $\det X$ of X in $\mathscr{J}$.
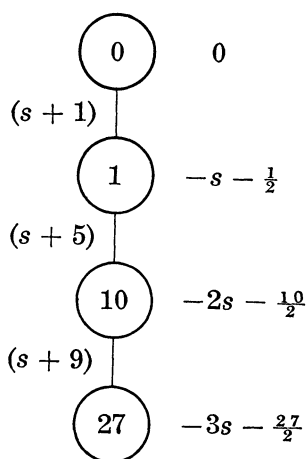

Figure 6–1. Holonomy diagram of $(GL(1) \times E_6,$
$\square \otimes \Lambda_1,\ V(1) \otimes V(27))$

§ 7. $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$

The representation space $V(1) \otimes V(56)$ is identified with

$$(7.1) \qquad V = \{X = (x, x');\ x, x' \in M(8),\ {}^t x = -x,\ {}^t x' = -x'\} .$$

Then the infinitesimal action $d\rho$ of $\mathfrak{g} = \mathfrak{gl}(1) \oplus E_7$ is given by

$$(1) \quad (x, x') \xmapsto{p} (px + x{}^t p,\ -{}^t py - yp) \quad \text{for} \quad p \in SL(8, C)$$

$$(7.2) \qquad (2) \quad (x, x') \xmapsto{c} (cx, cx') \quad \text{for} \quad c \in \mathfrak{gl}(1)$$

$$(3) \quad ((x_{ij}), (x'_{ij})) \xmapsto{\theta} \left(\left(\sum_{m,n=1}^{8} \vartheta^{ijmn} y_{mn} \right),\ \left(-\sum_{m,n=1}^{8} \vartheta_{ijmn} x_{mn} \right) \right)$$

where ϑ denotes a tensor, antisymmetric in its indices, and upper, lower indices satisfy the relation

$$\vartheta_{i_1,\dots,i_4} = \frac{1}{4!} \sum_{j_1,\dots,j_4} I^{1,\dots,8}_{i_1,\dots,i_4,j_1,\dots,j_4} \vartheta^{j_1,\dots,j_4} .$$

Here $I^{1,\cdots,8}_{i_1,\cdots,i_4,j_1,\cdots,j_4}$ denotes the signature of the permutation $\begin{pmatrix} 1, & \cdots & \\ i_1, & \cdots, & i_4, j_1, \\ & & \cdots, 8 \\ & & \cdots, j_4 \end{pmatrix}$ when $\{i_1, \cdots, i_4, j_1, \cdots, j_4\} = \{1, \cdots, 8\}$, and 0 otherwise. The product as a Lie algebra is given as follows:

(7.3)
(1) $[p, p'] = pp' - p'p$ where pp' denotes the matrix multiplication

(2) $[p, \vartheta] = \vartheta'$ where $(\vartheta')^{ijkl} = \sum_m (\vartheta^{mjkl}p_{im} + \vartheta^{imkl}p_{jm} + \vartheta^{ijml}p_{km} + \vartheta^{ijkm}p_{lm})$

(3) $[\vartheta, \vartheta'] = p$ where $p_{ij} = \frac{2}{3}\left(\sum_{\ell,m,n} \left(\vartheta^{\ell mni}(\vartheta')_{\ell mnj} - \frac{1}{8}\left(\sum_r \vartheta^{\ell mnr}(\vartheta')_{\ell mnr} \right)\delta_{ij} \right)\right)$

PROPOSITION 7–1 (Stephen J. Haris). *There exist five orbits* $S_m = \rho(G)x_m$ $(m = 0, 1, 11, 28, 56)$ *where* S_m *denotes the m-codimensional G-orbit and* x_m *is given as follows.*

$$x_0 = \left\{ \left(\begin{bmatrix} 1 & & & & & \\ -1 & & & & & \\ & & 1 & & & \\ & & -1 & & & \\ \hline & & & 1 & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & -1 \end{bmatrix}, 0 \right) \right\}, \quad x_1 = \left\{ \left(\begin{bmatrix} 1 & & & & & \\ -1 & & & & & \\ & & 1 & & & \\ & & -1 & & & \\ \hline & & & 1 & & \\ & & & -1 & & \\ & & & & 0 & \end{bmatrix}, 0 \right) \right\}$$

$$x_{11} = \left\{ \left(\begin{bmatrix} 1 & & & \\ -1 & & & \\ & 1 & & \\ & -1 & & \\ \hline & & 0 & \end{bmatrix}, 0 \right) \right\}, \quad x_{28} = \left\{ \left(\begin{bmatrix} 1 & & \\ -1 & & \\ \hline & 0 & \end{bmatrix}, 0 \right) \right\}$$

and $x_{56} = (0, 0)$.

Proof. See [5].   Q.E.D.

We identify the dual vector space V^* with V by $\langle X, Y \rangle = \operatorname{tr} xx' + \operatorname{tr} yy'$ for $X = (x, x')$, $Y = (y, y') \in V$.

Then the dual action $d\rho^*$ is given as follows:

(1) $(y, y') \overset{p^*}{\mapsto} (-{}^t py - yp, py' + y'{}^t p)$

(7.4)
(2) $(y, y') \overset{c^*}{\mapsto} (- cy, - cy')$

(3) $((y_{ij}), (y'_{ij})) \overset{\theta^*}{\mapsto} \left(\left(\sum_{m,n=1}^{8} \vartheta_{ijmn} y'_{mn} \right), \left(- \sum_{m,n=1}^{8} \vartheta^{ijmn} y_{mn} \right) \right)$

Since $G = GL(1) \times E_7$ is reductive, the dual triplet (G, ρ^*, V^*) has also five orbits S_m^* ($m = 0, 1, 11, 28, 56$). We denote by Λ_m (resp. Λ_m^*) the conormal bundle of S_m (resp. S_m^*).

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at x_0 is the Lie algebra of E_6 (See [5], [1]). Since $\Lambda_0 = V \times \{0\} = \Lambda_{56}^*$, we have $\operatorname{ord}_{\Lambda_0} f^s = 0$ where $f(X) = \operatorname{Pf}(x) + \operatorname{Pf}(x') - \frac{1}{4} \operatorname{tr}(xx'xx') + \frac{1}{16} \operatorname{tr}(xx')^2$ for $X = (x, x') \in V$.

(2) The isotropy subalgebra $\mathfrak{g}_{x_1}$ at x_1 is the set $\{c \oplus p \oplus \theta\}$ satisfying the following conditions:

$$(7.4) \quad p = \begin{bmatrix} -\dfrac{c}{2} I_6 & \\ & \dfrac{3}{2} c I_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_{12} & p_{13} & p_{14} \\ p_{12}' & p_2 & p_{23} & p_{24} \\ p_{13}' & p_{23}' & p_3 & p_{34} \\ 0 & 0 & 0 & p_4 \end{bmatrix} \quad \text{where } \operatorname{tr} p_i = 0$$

for $i = 1, \cdots, 4$, and $p_{ij} p_{ij}' = -\det p_{ij} \cdot I_2$ for $1 \le i < j \le 3$. $\vartheta = (\vartheta_{ijk\ell})$ satisfies $\vartheta_{12ij} + \vartheta_{34ij} + \vartheta_{56ij} = 0$.

In fact, the isotropy subgroup is connected, and is isomorphic to $(GL(1) \times F_4) \cdot U$ where U is unipotent of dimension 26 (See [5]). The conormal vector space $V_{x_1}^*$ is given by

$$(7.5) \qquad V_{x_1}^* = \left\{ x = \left(\left(\begin{array}{c|cc} 0 & & 0 \\ \hline 0 & 0 & x \\ & -x & 0 \end{array} \right), 0 \right); \ x \in C \right\}.$$

Let y_1 be the element of $V_{x_1}^*$ with $x = 1$ in (7.5). Then it is a generic point, and $y_1' = 0$ is the point of the one-codimensional orbit. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $c = -\frac{1}{4}$, all remaining parts zero in (7.4). Then $d\rho(A_0)x_1 = 0$ and $d\rho^*(A_0)y_1 = y_1$. Since $\delta\chi(A_0) = -1$, $\operatorname{tr}_{V_{x_1}^*} A_0 = \dim V_{x_1}^* = 1$, we have $\operatorname{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$ and $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s + 1)$.

(3) The isotropy subalgebra $\mathfrak{g}_{x_{11}}$ at x_{11} is the set $\{c \oplus p \oplus \theta\}$ satisfying the following conditions

$$(7.6) \quad \begin{aligned} p &= \left[\begin{array}{c|c} -\dfrac{c}{2} I_4 + p_1 & p_2 \\ \hline 0 & \dfrac{c}{2} I_4 + p_4 \end{array} \right] \quad \text{where } K^{-1} p_1 K \in \hat{\mathfrak{sp}}(2) \text{ with} \\[2em] K &= \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix}, \ \operatorname{tr} p_4 = 0, \ (\vartheta) \text{ with } \vartheta_{12ij} + \vartheta_{34ij} = 0 \text{ for all } i, j. \end{aligned}$$

The conormal vector space $V_{x_{11}}^*$ is given by

$$(7.7) \qquad V_{x_{11}}^* = \left\{ \tilde{Y} = \left(\begin{pmatrix} 0 & 0 \\ 0 & Y_4 \end{pmatrix}, \begin{pmatrix} Y_1' & 0 \\ 0 & 0 \end{pmatrix} \right); y_{12}' + y_{34}' = 0 \right\}.$$

Then, for $A = c \oplus p \oplus \theta$ in $\mathfrak{g}_{x_{11}}$, we have $d\rho^*(A)\tilde{Y} = \left(\begin{pmatrix} 0 & 0 \\ 0 & Z_4 \end{pmatrix}, \begin{pmatrix} Z_1' & 0 \\ 0 & 0 \end{pmatrix} \right)$ where

$$Z_4 = -2cY_4 - {}^t P_4 Y_4 - Y_4 P_4 + \left(\sum_{m,n=1}^4 \vartheta_{ijmn} y_{mn}' \right)$$

and

$$Z_1' = -2cY_1' + P_1 Y_1' + Y_1' {}^t P_1 - \left(\sum_{m,n=5}^8 \vartheta^{ijmn} y_{mn} \right).$$

Put $y_{11} = \left\{ \begin{bmatrix} 0 & 0 \\ \hline & 1 \\ 0 & -1 \\ & & 1 \\ & & & -1 \end{bmatrix}, 0 \right\}$ and $y_{11}' = \left\{ \begin{bmatrix} 0 & 0 \\ \hline 0 & \\ & & 1 \\ & & -1 \end{bmatrix}, 0 \right\}.$

Then y_{11} is a generic point and y_{11}' is a point of the unique one-codimensional orbit. Thus we have $\Lambda_{11} = \Lambda_{11}^*$ and $\dim \Lambda_1 \cap \Lambda_{11} = \dim V - 1$. Let A_c be an element of $\mathfrak{g}_{x_{11}}$ with $c = -\frac{1}{2}$, all remaining parts zero in (7.7). Then $d\rho(A_0)x_{11} = 0$ and $d\rho^*(A_0)y_{11} = y_{11}$. Since $\delta\chi(A_0) = 4c = -2$, $\mathrm{tr}_{V_{x_{11}}^*} A_0 = -22c = 11$ and $\dim V_{x_{11}}^* = 11$, we have $\mathrm{ord}_{\Lambda_{11}} f^s = -2s - \frac{11}{2}$ and hence $b_{\Lambda_{11}}(s)/b_{\Lambda_1}(s) = (s + \frac{11}{2})$.

(4) The isotropy subalgebra $\mathfrak{g}_{x_{28}}$ at x_{28} is the set $\{c \oplus p \oplus \theta\}$ satisfying the following conditions:

$$(7.8) \qquad p = \left[\begin{array}{c|c} -\dfrac{c}{2} I_2 + p_1 & p_2 \\ \hline 0 & \dfrac{c}{6} I_2 + p_4 \end{array} \right] \quad \text{with} \quad \mathrm{tr}\, p_1 = \mathrm{tr}\, p_4 = 0$$

$$\vartheta = (\vartheta^{ijk\ell}) \quad \text{with} \quad \vartheta_{12ij} = 0 \quad \text{for all } i, j.$$

The conormal vector space $V_{x_{28}}^*$ is given by

$$(7.9) \qquad V_{x_{28}}^* = \left\{ \tilde{Y} = \left(\begin{pmatrix} 0 & 0 \\ \underbrace{0}_{2} & \underbrace{Y_4}_{6} \end{pmatrix}, \begin{pmatrix} \underbrace{Y_1'}_{2} & \underbrace{Y_3'}_{6} \\ -{}^t Y_3' & 0 \end{pmatrix} \right) \in V^* \right\}.$$

Then for $A = c \oplus p \oplus \theta$ in (7.8), we have

$$d\rho^*(A)\tilde{Y} = \left(\begin{pmatrix} 0 & 0 \\ 0 & Z_4 \end{pmatrix}, \begin{pmatrix} Z_1' & Z_3' \\ -{}^t Z_3' & 0 \end{pmatrix} \right)$$

where

$$Z_4 = -\frac{4}{3} c Y_4 - {}^t p_4 Y_4 - Y_4 p_4 + \left(\sum_{m,n} \vartheta_{ijmn} y'_{mn} \right)$$

$$Z_1' = -2 c Y_1' + p_1 Y_1' + Y_1'{}^t p_1 - p_2{}^t Y_3' + Y_3'{}^t p_2 - \left(\sum_{m,n=3}^{8} \vartheta^{ijmn} y_{mn} \right)$$

$$Z_3' = -\frac{4}{3} c Y_3' + p_1 Y_3' + Y_3'{}^t p_4 - \left(\sum_{m,n=3}^{8} \vartheta^{ijmn} y_{mn} \right).$$

Therefore, one can see that the colocalization at x_{28} has at most unique one-codimensional orbit.

Since $\Lambda_{28} = \Lambda_1^*$ and $\Lambda_{11} = \Lambda_{11}^*$, Λ_{28} is a good holonomic variety and $\dim \Lambda_{28} \cap \Lambda_{11} = \dim V - 1$.
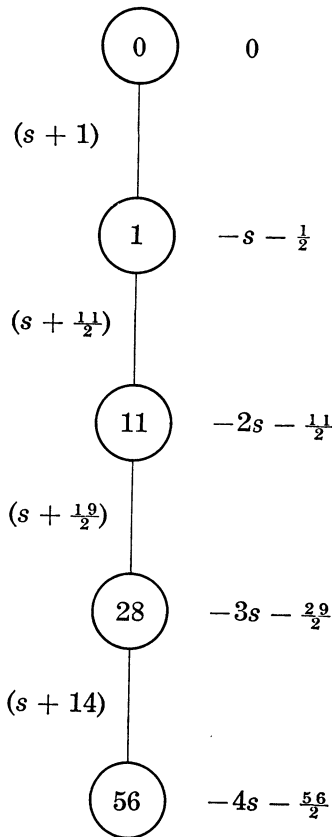
Figure 7–1. Holonomy diagram of $(GL(1) \times E_7,$
$\square \otimes \Lambda_1, \; V(1) \otimes V(56))$

$$\text{Put } y_{28} = \left\{ \left[\begin{array}{c|ccccc} 0 & & & & & \\ \hline & 1 & & & & \\ -1 & & & & & \\ & & & 1 & & \\ & & -1 & & & \\ & & & & & 1 \\ & & & & -1 & \end{array} \right], 0 \right\}. \quad \text{Then } y_{28} \text{ is a generic point.}$$

Let A_0 be an element of $\mathfrak{g}_{x_{28}}$ with $c = -\frac{3}{4}$, all remaining parts zero in (7.8). Then we have $d\rho(A_0)x_{28} = 0$ and $d\rho^*(A_0)y_{28} = y_{28}$. Since $\delta\chi(A_0) = 4c = -3$, $\operatorname{tr}_{V_{x_{28}}^*} A_0 = -38c = \frac{57}{2}$ and $\dim V_{x_{28}}^* = 28$, we have $\operatorname{ord}_{A_{28}} f^s = -3s - \frac{29}{2}$ and hence $b_{A_{28}}(s)/b_{A_{11}}(s) = (s + \frac{19}{2})$ by Corollary 1-2.

(5) Since $x_{56} = 0$, we have $(G_{x_{56}}, \rho_{x_{56}}, V_{x_{56}}^*) = (G, \rho^*, V^*)$. Since $\Lambda_{56} = \{0\} \times V^* = \Lambda_0^*$ and $\Lambda_{28} = \Lambda_1^*$, we have $\dim \Lambda_{56} \cap \Lambda_{28} = \dim V - 1$ and they intersect regularly. Let A_0 be an element of $\mathfrak{g}_{x_{28}}$ with $c = -1$, all remaining parts zero in (7.2). Then $d\rho(A_0)x_{56} = 0$ and $d\rho^*(A_0)y_{56} = y_{56}$ where $y_{56} = x_0$. Since $\delta\chi(A_0) = -4$, $\operatorname{tr}_{V_{x_{56}}^*} A_0 = \dim V_{x_{56}}^m = 56$, we have $\operatorname{ord}_{A_{56}} f^s = -4s - \frac{56}{2}$, and hence $b_{A_{56}}(s)/b_{A_{28}}(s) = s + 14$. Thus we obtain the b-function $b(s) = (s+1)(s+\frac{11}{2})(s+\frac{19}{2})(s+14)$ and the holonomy diagram (Figure 7-1).

§8. $(GL(6), \Lambda_3, V(20))$

Let V_1 be a 6-dimensional vector space spanned by $u_1, \cdots, u_6$. Then $G = GL(6)$ acts on V_1 by $\rho_1(g)(u_1, \cdots, u_6) = (u_1, \cdots, u_6)g$ for $g \in G$. The representation space $V = V(20)$ is spanned by skew-tensors $u_i \wedge u_j \wedge u_k$ $(1 \le i < j < k \le 6)$, and $\rho = \Lambda_3$ is given by $\rho(g)(u_i \wedge u_j \wedge u_k) = \rho_1(g)u_i \wedge \rho_1(g)u_j \wedge \rho_1(g)u_k$ for $1 \le i < j < k \le 6$, and $g \in G$. Then it is well-known (and also one can easily check) that there exist five G-orbits $S_m = \rho(G)x_m$ $(m = 0, 1, 5, 10, 20)$ where S_m denotes the m-codimensional orbit, and $x_0 = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6$, $x_1 = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_4 \wedge u_5 + u_2 \wedge u_4 \wedge u_6$, $x_5 = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_4 \wedge u_5$, $x_{10} = u_1 \wedge u_2 \wedge u_3$, and $x_{20} = 0$. We identify the dual space V^* with V by $(\sum a_{ijk} u_i \wedge u_j \wedge u_k, \sum b_{rst} u_r \wedge u_s \wedge u_t) = \sum_{1 \le i < j < k \le 6} a_{ijk} b_{ijk}$. Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, there exist also five orbits S_m^* $(m = 0, 1, 5, 10, 20)$ in V^*. We denote by Λ_m the conormal bundle of S_m. The isotropy subalgebra $\mathfrak{g}_x$ at $x \in V(20)$ is, by definition, $\mathfrak{g}_x = \{A \in \mathfrak{gl}(6); d\rho(A)x = 0\}$ where $d\rho(A)(u_i \wedge u_j \wedge u_k) = d\rho_1(A)u_i \wedge u_j \wedge u_k + u_i \wedge d\rho_1(A)u_j \wedge u_k + u_i \wedge u_j \wedge d\rho_1(A)u_k$.

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ is, by simple calculation, given as follows:

$$(8.1) \qquad \mathfrak{g}_{x_0} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{gl}(6); A, B \in \tilde{\mathfrak{sl}}(3) \right\}.$$

We have $\Lambda_0 = V \times \{0\}$, and hence $\operatorname{ord}_{\Lambda_0} f^s = 0$ where f denotes the relatively invariant irreducible polynomial of degree four (See [1], [14]).

(2) Put $x_1' = u_1 \wedge u_2 \wedge u_6 + u_2 \wedge u_3 \wedge u_4 - u_1 \wedge u_3 \wedge u_5$. Then $x_1' \in S_1$. By simple calculation, the isotropy subalgebra $\mathfrak{g}_{x_1'}$ at x_1' is given by

$$(8.2) \qquad \mathfrak{g}_{x_1'} = \left\{ \tilde{A} = \left(\begin{array}{c|c} A & B \\ \hline 0 & A - (\operatorname{tr} A) \cdot I_3 \end{array} \right) \in \mathfrak{gl}(6); A, B \in M(3), \operatorname{tr} B = 0 \right\}.$$

Therefore we have $G_{x_1'} \sim GL(3) \cdot (G_a)^8$ where $\cdot$ denotes a semi-direct product and $G_a \cong C$. The conormal vector space $V_{x_1'}^*$ is of one-dimension with a basis $u_4 \wedge u_5 \wedge u_6$. The action $d\rho_{x_1'}$ of $\mathfrak{g}_{x_1'}$ on $V_{x_1'}^*$ is $d\rho_{x_1'}(\tilde{A}) u_4 \wedge u_5 \wedge u_6 = -2 \operatorname{tr} \tilde{A} \cdot u_4 \wedge u_5 \wedge u_6$. Take $A_0 \in \mathfrak{g}_{x_1'}$ with $\operatorname{tr} A_0 = -\frac{1}{2}$. Then we have $d\rho(A_0) x_1' = 0$ and $d\rho^*(A_0) y_1 = y_1$ where $y_1 = u_4 \wedge u_5 \wedge u_6$. Since $\delta\chi(A_0) = (\deg f / \dim V) \operatorname{tr} d\rho(A_0) = \frac{4}{20} \times (10 \operatorname{tr} A_0) = -1$ and $\operatorname{tr}_{V_{x_1'}^*} A_0 = \dim V_{x_1'}^* = 1$, we have $\operatorname{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$ by Proposition 1–3. Since 0 is the point of the one-codimensional orbit, we have $\dim \Lambda_0 \cap \Lambda_1 = \dim V - 1$ and $\Lambda_0 \cap \Lambda_1$ is G_0-prehomogeneous, i.e., Λ_1 is a good holonomic variety by Proposition 1–5. Also we have $\mu = 1$ and $\nu = 0$ by Proposition 1–4, i.e., Λ_0 and Λ_1 intersect regularly. By Corollary 1–2, we have $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s + 1)$.

(3) Put $x_5' = u_1 \wedge (u_2 \wedge u_4 + u_3 \wedge u_5) \in S_5$. Then the isotropy subalgebra $\mathfrak{g}_{x_5'}$ is given as follows:

$$(8.3) \qquad \mathfrak{g}_{x_5'} = \left\{ \left(\begin{array}{c|c|c} -2\varepsilon & B & C \\ \hline 0 & A + \varepsilon I_4 & D \\ \hline 0 & 0 & \eta \end{array} \right) \in \mathfrak{gl}(6); A \in \tilde{\mathfrak{sp}}(2), {}^t B, D \in C^4, C \in C \right\}$$

$$= (\tilde{\mathfrak{sp}}(2) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(9)$$

where $\mathfrak{u}(9)$ denotes the Lie algebra of 9-dimensional unipotent group. Put $\omega_1 = (u_2 \wedge u_4 - u_3 \wedge u_5) \wedge u_6$, $\omega_2 = u_4 \wedge u_5 \wedge u_6$, $\omega_3 = u_3 \wedge u_4 \wedge u_6$, $\omega_4 = u_2 \wedge u_5 \wedge u_6$ and $\omega_5 = u_2 \wedge u_3 \wedge u_6$. Then the conormal vector space $V_{x_5'}^*$ is spanned by $\omega_1, \cdots, \omega_5$ and $(G_{x_5'}, \rho_{x_5'}, V_{x_5'}^*) \cong (GL(1) \times Sp(2), \Lambda_1 \otimes \Lambda_2, V(1) \otimes V(5)) \cong (GL(1) \times SO(5), \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(5))$, where ω_1 is a generic point and $\omega_2 = u_4 \wedge u_5 \wedge u_6$ is a point of the one-codimensional orbit. Therefore we have $\dim \Lambda_1 \cap \Lambda_5 = \dim V - 1$. Since the $(G_{x_5'} \cap G_0)$-orbit of ω_2 is one-codimensional in $V_{x_5'}^*$, i.e., $\Lambda_1 \cap \Lambda_5$ is G_0-prehomogeneous, Λ_5 is a good holonomic variety by (2) and Proposition 1–5. Let A_0 be an element of $\mathfrak{g}_{x_5'}$ with $\eta = -1$ and all remaining parts zero in (8.3). Then we have

$d\rho(A_0)x_5' = 0$ and $d\rho^*(A_0)\omega_1 = \omega_1$. Since $\delta\chi(A_0) = 2\mathrm{tr}\,A_0 = -2$, $\mathrm{tr}_{V\cdot x_5'}A_0 = -5(2\varepsilon + \eta) = 5$, and $\dim V_{x_5'}^* = 5$, we have $\mathrm{ord}_{A_5}f^s = -2s - \frac{5}{2}$. Put $A_\beta = \beta(E_{22} - E_{44}) + (\beta + 1)E_{66}$ for $\beta \in C$ where E_{ij} denotes the matrix unit. Then $d\rho(A_\beta)x_5' = 0$ and $d\rho^*(A_\beta)\omega_2 = \omega_2$. Since $\tilde{V} = V_{x_5'}^* \bmod d\rho_{x_5'}\,(\mathfrak{g}_{x_5'})\omega_2$ is spanned by $u_2 \wedge u_3 \wedge u_6$, we have $\mathrm{tr}_{\tilde{V}}A_\beta = 2\beta + 1$. Hence we have $\mu = 1$ and $\nu = 0$ by Proposition 1–4, i.e., $\varLambda_1$ and $\varLambda_5$ intersect regularly. One can also get this from the fact $m_{\varLambda_5} - m_{\varLambda_1} = 1$. By Corollary 1–2, we have $b_{\varLambda_5}(s)/b_{\varLambda_1}(s) = s + \frac{5}{2}$.

(4) Put $x_{10} = u_1 \wedge u_2 \wedge u_3 \in S_{10}$. Then the isotropy subalgebra $\mathfrak{g}_{x_{10}}$ is given as follows:

$$(8.4) \quad \mathfrak{g}_{x_{10}} = \left\{ \tilde{A} = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) \in \mathfrak{gl}(6);\ A,\ B,\ C \in M(3),\ \mathrm{tr}\,A = 0 \right\}$$

$$\cong (\mathfrak{sl}(3) \oplus \mathfrak{gl}(3)) \oplus V(9),\ \text{i.e.,}\ G_{x_{10}} \sim (SL(3) \times GL(3))\cdot(G_a)^9.$$

In general, we write $G_1 \sim G_2$ when two groups G_1 and G_2 are locally isomorphic to each other. Put $\omega_1 = u_1 \wedge u_4 \wedge u_5$, $\omega_2 = u_1 \wedge u_4 \wedge u_6$, $\omega_3 = u_1 \wedge u_5 \wedge u_6$, $\omega_4 = u_2 \wedge u_4 \wedge u_5$, $\omega_5 = u_2 \wedge u_4 \wedge u_6$, $\omega_6 = u_2 \wedge u_5 \wedge u_6$, $\omega_7 = u_3 \wedge u_4 \wedge u_5$, $\omega_8 = u_3 \wedge u_4 \wedge u_6$, $\omega_9 = u_3 \wedge u_5 \wedge u_6$ and $\omega_{10} = u_4 \wedge u_5 \wedge u_6$. Then the conormal vector space $V_{x_{10}}^*$ is spanned by $\omega_1, \cdots, \omega_{10}$. The action $d\rho_{x_{10}}$ of $\mathfrak{g}_{x_{10}}$ on $V_{x_{10}}^*$ is given by

$$(8.5) \quad d\rho_{x_{10}}(\tilde{A})(\omega_1, \cdots, \omega_{10}) = (\omega_1, \cdots, \omega_{10}) \left(\begin{array}{c|c} d\tilde{\rho}(A \oplus C) & 0 \\ \hline B' & -\mathrm{tr}\,A \end{array} \right) ({}^t B' \in C^9)$$

where $\tilde{\rho} = \varLambda_2 \otimes \varLambda_2^*$, i.e., the action of G_{x_0} induced on the subspace spanned by $\omega_1, \cdots, \omega_9$ is isomorphic to a triplet $(SL(3) \times GL(3), \varLambda_1 \otimes \varLambda_1, V(3) \otimes V(3))$ as a triplet (See [1]). Then $\omega_1 + \omega_5 + \omega_9 \in S_1^*$ is a generic point and $\omega_1 + \omega_5 \in S_5^*$ is a point of the one-codimensional orbit. This implies that $\dim \varLambda_5 \cap \varLambda_{10} = \dim V - 1$. Since $\varLambda_5 \cap \varLambda_{10}$ is G_0-prehomogeneous, $\varLambda_{10}$ is a good holonomic variety by Proposition 1–5. Let $\tilde{A}$ be an element of $\mathfrak{g}_{x_{10}}$ with $A = B = 0$ and $C = -\frac{1}{2}I_3$ in (8.4). Then $d\rho(\tilde{A})x_{10} = 0$ and $d\rho^*(\tilde{A})(\omega_1 + \omega_5 + \omega_9) = (\omega_1 + \omega_5 + \omega_9)$. Since $\delta\chi(\tilde{A}) = 2\cdot\mathrm{tr}\,\tilde{A} = -3$, $\mathrm{tr}_{V_{x_{10}}^*}\tilde{A} = -7\mathrm{tr}\,C = \frac{21}{2}$ and $\dim V_{x_{10}}^* = 10$, we have $\mathrm{ord}_{A_{10}}f^s = -3s - \frac{11}{2}$ by Proposition 1–3. Let A_β be an element of $\mathfrak{g}_{x_{10}}$ with $A = B = 0$ and $C = \begin{pmatrix} 1 - (\beta/2) & 0 \\ 0 & (\beta/2)I_2 \end{pmatrix}$ in (8.4). Then $d\rho(A_\beta)x_{10} = 0$ and $d\rho(A_\beta)(\omega_1 + \omega_5) = (\omega_1 + \omega_5)$. Since $\tilde{V} = V_{x_{10}}^* \bmod d\rho_{x_{10}}(\mathfrak{g}_{x_{10}})(\omega_1 + \omega_5)$ is spanned by $u_3 \wedge u_5 \wedge u_6$, we have $\mathrm{tr}_{\tilde{V}}A_\beta = \beta$. This implies that $\mu = 1$ and $\nu = 0$, i.e., $\varLambda_5$ and $\varLambda_{10}$ intersect regularly by Proposition 1–4. One can also get this from the fact $m_{\varLambda_{10}} - m_{\varLambda_5} = 1$. By

Corollary 1–2, we have $b_{A_{10}}(s)/b_{A_5}(s) = s + \frac{7}{2}$.

(5)   Put $x_{20} = 0 \in S_{20}$. In this case, $(G_{x_{20}}, \rho_{x_{20}}, V_{x_{20}}^*) \cong (GL(6), \Lambda_3, V(20))$. $\Lambda_{x_{20}} = \{0\} \times V^*$ is a good holonomic variety. Put $\tilde{A} = -\frac{1}{3}I_6$. Then $d\rho(\tilde{A})x_{20} = 0$ and $d\rho^*(\tilde{A})x_0^* = x_0^*$ where $x_0^* = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6 \in S_0^*$. Since $\delta\chi(\tilde{A}) = 2\operatorname{tr}\tilde{A} = -4$, $\operatorname{tr}_{V_{x_{20}}^*}\tilde{A} = 20$ and $\dim V_{x_{20}}^* = 20$, we have $\operatorname{ord}_{A_{20}}f^s$

$= -4s - \frac{20}{2}$ by Proposition 1–3. Put $A_\beta = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_6 \end{pmatrix}$ with $a_1 = a_2 = a_4$

$= 1/2 - \beta/6$, $a_3 = a_5 = a_6 = \beta/3$. Then $d\rho(A_\beta)x_{20} = 0$ and $d\rho^*(A_\beta)x_1^* = x_1^*$ where $x_1^* = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_4 \wedge u_5 + u_2 \wedge u_4 \wedge u_6$. Since $\tilde{V} = V_{x_{20}}^*$ mod $d\rho_{x_{20}}(\mathfrak{g}_{x_{20}})x_1^*$ is spanned by $u_3 \wedge u_5 \wedge u_6$, we have $\operatorname{tr}_{\tilde{V}}A_\beta = \beta$. This implies that $\mu = 1$ and $\nu = 0$, i.e., Λ_{20} and Λ_{10} intersect regularly. One can also get this from $m_{A_{20}} - m_{A_{10}} = 1$. By Corollary 1–2, we have $b_{A_{20}}(s)/b_{A_{10}}(s) = s + 5$. Thus we obtain the b-function $b(s) = (s+1)(s+\frac{5}{2})(s+\frac{7}{2})(s+5)$ and the holonomy diagram (Figure 8–1). We denote $\boxed{\Lambda_m}$ by $\boxed{m}$.
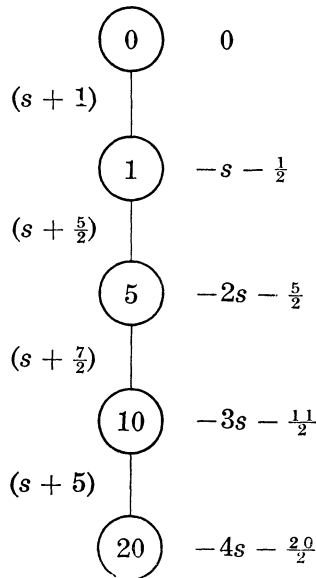
Figure 8–1.   Holonomy diagram of $(GL(6), \Lambda_3, V(20))$.

§ 9. $(GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$

Put $\omega_1 = u_1 \wedge u_2 \wedge u_3$, $\omega_2 = u_4 \wedge u_5 \wedge u_6$, $\omega_3 = u_2 \wedge u_3 \wedge u_4$, $\omega_4 = u_1 \wedge u_5 \wedge u_6$, $\omega_5 = u_1 \wedge u_3 \wedge u_5$, $\omega_6 = u_2 \wedge u_4 \wedge u_6$, $\omega_7 = u_1 \wedge u_2 \wedge u_6$, $\omega_8 = u_3 \wedge u_4 \wedge u_5$, $\omega_9 = u_1 \wedge u_2 \wedge u_5 - u_1 \wedge u_3 \wedge u_6$, $\omega_{10} = u_2 \wedge u_4 \wedge u_5 - u_3 \wedge u_4 \wedge u_6$,

$\omega_{11} = u_1 \wedge u_2 \wedge u_4 + u_2 \wedge u_3 \wedge u_6$, $\omega_{12} = u_1 \wedge u_4 \wedge u_5 + u_3 \wedge u_5 \wedge u_6$, $\omega_{13} = u_1 \wedge u_3 \wedge u_4 - u_2 \wedge u_3 \wedge u_5$, $\omega_{14} = u_1 \wedge u_4 \wedge u_6 - u_2 \wedge u_5 \wedge u_6$. Then the representation space V is identified with the subspace of $V(20)$ in §8 generated by $\omega_1, \cdots, \omega_{14}$. Then the representation $\rho = \square \otimes \Lambda_3$ is the restriction of Λ_3 for $GL(6)$ to $G = GL(1) \times Sp(3)$. The orbital decomposition of this space has been completed by J-I. Igusa (See [3]). There exist five G-orbits $S_m = \rho(G)x_m$ ($m = 0, 1, 4, 7, 14$) where S_m denotes the m-codimensional orbit, and $x_0 = \omega_1 + \omega_2$, $x_1 = \omega_7 + \omega_{13}$, $x_4 = \omega_{13}$, $x_7 = \omega_1$, $x_{14} = 0$. We identify the dual space V^* with V by $(\sum_{i=1}^{14} a_i \omega_i, \sum_{j=1}^{14} b_j \omega_j) = \sum_{k=1}^{14} a_k b_k$. Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, there exist also five G-orbits S_m^* ($m = 0, 1, 4, 7, 14$) in V^*. We denote by Λ_m the conormal bundle of S_m. The Lie algebra $\mathfrak{g}$ of $G = GL(1) \times Sp(3)$ is given as follows:

$$(9.1) \quad \mathfrak{g} = \left\{ \tilde{A} = (d) \oplus \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix}; A, B, C \in M(3), {}^t B = B, {}^t C = C \right\}.$$

(1) Since $d\rho(\tilde{A})x_0 = (d + \operatorname{tr} A)\omega_1 + (d - \operatorname{tr} A)\omega_2 + c_1\omega_3 + b_1\omega_4 - c_2\omega_5 - b_2\omega_6 + c_3\omega_7 + b_3\omega_8 + c_{23}\omega_9 + b_{23}\omega_{10} + c_{13}\omega_{11} + b_{13}\omega_{12} - c_{12}\omega_{13} - b_{12}\omega_{14}$ where $c_i = c_{ii}$ and $b_i = b_{ii}$ for $i = 1, 2, 3$, we have

$$(9.2) \quad \mathfrak{g}_{x_0} = \left\{ \tilde{A} = (0) \oplus \begin{pmatrix} A & 0 \\ 0 & -{}^t A \end{pmatrix}; A \in \mathfrak{sl}(3) \right\} \cong \mathfrak{sl}(3).$$

We have $\Lambda_0 = V \times \{0\}$, and hence $\operatorname{ord}_{\Lambda_0} f^s = 0$ where f denotes the relatively invariant irreducible polynomial of degree four (See [1], [3]).

(2) Since $d\rho(\tilde{A})x_1 = (b_3 - 2b_{12})\omega_1 + 2a_{21}\omega_3 + c_2\omega_4 - 2a_{12}\omega_5 - c_1\omega_6 + (d + a_1 + a_2 - a_3)\omega_7 + 2c_{12}\omega_8 + (a_{13} - a_{32})\omega_9 - c_{13}\omega_{10} + (a_{23} - a_{31})\omega_{11} - c_{23}\omega_{12} + (d + a_3)\omega_{13} + (c_{12} - c_3)\omega_{14}$ where $a_i = a_{ii}$ for $i = 1, 2, 3$, we have

$$(9.3) \quad \mathfrak{g}_{x_1} = \left\{ (d) \oplus \left[\begin{array}{ccc|ccc} -d+\alpha & 0 & \beta & b_1 & b_{12} & b_{13} \\ 0 & -d-\alpha & \gamma & b_{12} & b_2 & b_{23} \\ \gamma & \beta & -d & b_{13} & b_{23} & 2b_{12} \\ \hline & & & d-\alpha & 0 & -\gamma \\ & 0 & & 0 & d+\alpha & -\beta \\ & & & -\beta & -\gamma & d \end{array} \right] \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{o}(3)) \oplus V(5)$$

where $V(5)$ denotes the Lie algebra of $(G_a)^5$.

The conormal vector space $V_{x_1}^*$ is one-dimensional with a basis ω_2. The action $d\rho_{x_1}$ of $\mathfrak{g}_{x_1}$ on $V_{x_1}^*$ is given by $d\rho_{x_1}(\tilde{A})\omega_2 = (-d + a_1 + a_2 + a_3)\omega_2 = -4d\omega_2$. Therefore we have $\Lambda_1 = \overline{G(x_1, y_1)}$ where $y_1 = \omega_2$. Let A_0 be an

element of $\mathfrak{g}_{x_1}$ with $d = -\frac{1}{4}$, all remaining parts zero in (9.3). Then we have $d\rho(A_0)x_1 = 0$ and $d\rho^*(A_0)y_1 = y_1$. Since $\delta\chi(A_0) = 4d = -1$, $\operatorname{tr}_{V_{x_1}^*}\tilde{A} = \dim V_{x_1}^* = 1$, we have $\operatorname{ord}_{A_1}f^s = -s - \frac{1}{2}$ by Proposition 1–3. Since 0 is the point of the one-codimensional orbit, we have $\dim \Lambda_0 \cap \Lambda_1 = \dim V - 1$ and $\Lambda_0 \cap \Lambda_1$ is G_0-prehomogeneous, i.e., Λ_1 is a good holonomic variety by Proposition 1–5. Also we have $\mu = 1$ and $\nu = 0$ by Proposition 1–4, i.e., Λ_0 and Λ_1 intersect regularly. By Corollary 1–2, we have $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s + 1)$.

(3) Since $d\rho(\tilde{A})x_4 = -2b_{12}\omega_1 + 2a_{21}\omega_3 - 2a_{12}\omega_5 + 2c_{12}\omega_8 + a_{13}\omega_9 - c_{13}\omega_{10} + a_{23}\omega_{11} - c_{23}\omega_{12} + (d + a_3)\omega_{13} - c_3\omega_{14}$, we have

$$
\mathfrak{g}_{x_4} = \left\{ (d) \oplus \begin{pmatrix} a_1 & 0 & 0 & b_1 & 0 & \beta \\ 0 & a_2 & 0 & 0 & b_2 & \delta \\ \alpha & \gamma & -d & \beta & \delta & \varepsilon \\ c_1 & 0 & 0 & -a_1 & 0 & -\alpha \\ 0 & c_2 & 0 & 0 & -a_2 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & d \end{pmatrix} \right\}
$$

(9.4)
$$
\cong \left\{ (d) \oplus \begin{pmatrix} -d & \alpha & \beta & \gamma & \delta & \varepsilon \\ 0 & a_1 & b_1 & & & \beta \\ & c_1 & -a_1 & & 0 & -\alpha \\ 0 & & & a_2 & b_2 & \delta \\ & & 0 & c_2 & -a_2 & -\gamma \\ 0 & & 0 & & 0 & d \end{pmatrix} \right\}
$$

$$
\simeq (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(5) .
$$

The conormal vector space $V_{x_4}^*$ is spanned by ω_2, ω_4, ω_6, ω_7 on which $\mathfrak{g}_{x_4}$ acts as follows:

$$
(\omega_2, \omega_4, \omega_6, \omega_7) \mapsto (\omega_2, \omega_4, \omega_6, \omega_7) \begin{pmatrix} A_1 & -b_1 & b_2 & 0 \\ -c_1 & A_2 & 0 & -b_2 \\ c_2 & 0 & A_3 & b_1 \\ 0 & -c_2 & c_1 & A_4 \end{pmatrix}
$$

where $A_1 = a_1 + a_2 - 2d$, $A_2 = -a_1 + a_2 - 2d$, $A_3 = a_1 - a_2 - 2d$, $A_4 = -a_1 - a_2 - 2d$.

Hence we have $(G_{x_4}, \rho_{x_4}, V_{x_4}^*) \cong (SL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2)) \cong (GL(1) \times SO(4), \square \otimes \Lambda_1, V(1) \otimes V(4))$.

Clearly, $y_4 = \omega_4 + \omega_6$ is its generic point, and ω_2 is a point of the one-codimensional orbit. Since $\Lambda_1 = \overline{G(x_1, \omega_2)}$, we have $\dim \Lambda_1 \cap \Lambda_4 = \dim V - 1$. Since $\Lambda_1 \cap \Lambda_4$ is G_0-prehomogeneous, Λ_4 is a good holonomic variety by (2) and Proposition 1–5. Let A_0 be an element of $\mathfrak{g}_{x_4}$ with d

$= -\frac{1}{2}$ and all remaining parts zero in (9.4). Then $d\rho(A_0)x_4 = 0$ and $d\rho^*(A_0)y_4 = y_4$. Since $\delta\chi(A_0) = 4d = -2$, $\mathrm{tr}_{V_{x_4}^*}A_0 = -8d = 4$ and $\dim V_{x_4}^* = 4$, we have $\mathrm{ord}_{A_4}f^s = -2s - \frac{4}{2}$. Let A_β be an element of $\mathfrak{g}_{x_4}$ with $a_1 = \frac{1}{2}(1-\beta)$, $d = -\frac{1}{4}(\beta+1)$, all remaining parts zero in (9.4). Then we have $d\rho(A_\beta)x_4 = 0$, $d\rho^*(A_\beta)\omega_2 = \omega_2$ and $\mathrm{tr}_{\tilde V}A_\beta = \beta$ where $\tilde V = V_{x_4}^* \bmod d\rho_{x_4}(\mathfrak{g}_{x_4})\omega_2$. This implies that Λ_4 and Λ_1 intersect regularly by Proposition 1–4. By Corollary 1–2, we have $b_{A_4}(s)/b_{A_1}(s) = (s+2)$.

(4) Since $d\rho(\tilde A)x_7 = (d + a_1 + a_2 + a_3)\omega_1 + c_1\omega_3 - c_2\omega_5 + c_3\omega_7 + c_{23}\omega_9 + c_{13}\omega_{11} - c_{12}\omega_{13}$, we have

$$(9.5) \qquad \mathfrak{g}_{x_7} = \left\{ \tilde A = (-\mathrm{tr}\,A) \oplus \left(\begin{array}{c|c} A & B \\ \hline 0 & -{}^tA \end{array} \right);\ {}^tB = B \right\} \simeq \mathfrak{gl}(3) \oplus V(6).$$

The conormal vector space $V_{x_7}^*$ is spanned by ω_2, ω_4, ω_6, ω_8, ω_{10}, ω_{12}, ω_{14}, and $\mathfrak{g}_{x_7}$ acts on $V_{x_7}^*$ as follows:

$$(9.6) \quad d\rho_{x_7}(\tilde A)(\omega_2, \omega_4, \cdots, \omega_{14}) = (\omega_2, \omega_4, \cdots, \omega_{14}) \left(\begin{array}{c|c} 2\,\mathrm{tr}\,A & B \\ \hline 0 & 2\,\mathrm{tr}\,A \cdot I_6 \oplus d\rho_1^*(A) \end{array} \right)$$

where ${}^tB \in \mathbf{C}^6$ and $\rho_1 = 2\Lambda_1$.

Then $y_7 = \omega_4 + \omega_{10}$ is its generic point, and $\omega_4 + \omega_6$ is a point of the one-codimensional orbit. Since $\Lambda_4 = \overline{G(x_4, \omega_4 + \omega_6)}$, we have $\dim \Lambda_4 \cap \Lambda_7 = \dim V - 1$. Since $\Lambda_4 \cap \Lambda_7$ is G_0-prehomogeneous, Λ_7 is a good holonomic variety by (3) and Proposition 1–5. Let A_0 be an element of $\mathfrak{g}_{x_7}$ with $A = \frac{1}{4}I_3$ and $B = 0$ in (9.5). Then $d\rho(A_0)x_7 = 0$ and $d\rho^*(A_0)y_7 = y_7$. Since $\delta\chi(A_0) = -4\,\mathrm{tr}\,A = -3$, $\mathrm{tr}_{V_{x_7}^*}A_0 = 10\,\mathrm{tr}\,A = \frac{15}{2}$ and $\dim V_{x_7}^* = 7$, we have $\mathrm{ord}_{A_7}f^s = -3s - \frac{8}{2}$. Let A_β be an element of $\mathfrak{g}_{x_7}$ with $a_1 = a_2 = \frac{\beta}{4}$, $a_3 = \frac{1}{2} - \frac{\beta}{4}$, all remaining parts zero in (9.5). Then $d\rho(A_\beta)x_7 = 0$, $d\rho(A_\beta)(\omega_4 + \omega_6) = (\omega_4 + \omega_6)$ and $\mathrm{tr}_{\tilde V}A_\beta = \beta$ where $\tilde V = V_{x_7}^* \bmod d\rho_{x_7}(\mathfrak{g}_{x_7})(\omega_4 + \omega_6) = C\omega_8$. This implies that Λ_4 and Λ_7 intersect regularly by Proposition 1–4. By Corollary 1–2, we have $b_{A_7}(s)/b_{A_4}(s) = (s + \frac{5}{2})$.

(5) Since $x_{14} = 0$, we have $(G_{x_{14}}, \rho_{x_{14}}, V_{x_{14}}^*) \cong (GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$ and $\Lambda_{14} = \{0\} \times V^*$ is a good holonomic variety. Take $\tilde A = (-1) \oplus (0) \in \mathfrak{g} = \mathfrak{gl}(1) \oplus \mathfrak{sp}(3)$. Then $d\rho(\tilde A)x_{14} = 0$, $d\rho^*(\tilde A)(\omega_1 + \omega_2) = (\omega_1 + \omega_2)$. Since $\delta\chi(\tilde A) = -4$, $\mathrm{tr}_{V_{x_{14}}^*}\tilde A = 14$ and $\dim V_{x_{14}}^* = 14$, we have $\mathrm{ord}_{A_{14}}f^s = -4s - \frac{14}{2}$. Since $\Lambda_{14} = \Lambda_0^*$, $\Lambda_7 = \Lambda_1^*$ where Λ_m^* denotes the conormal bundle of $S_m^*(\subset V^*)$, they intersect regularly by (2). Note that $(G, \rho, V) \cong (G, \rho^*, V^*)$ since $G = GL(1) \times Sp(3)$ is reductive. By Corollary 1–2, we have $b_{A_{14}}(s)/b_{A_7}(s) = s + \frac{7}{2}$. Since $b_{A_{14}}(s) = b(s)$ and $b_{A_0}(s) = 1$, we obtain the b-function
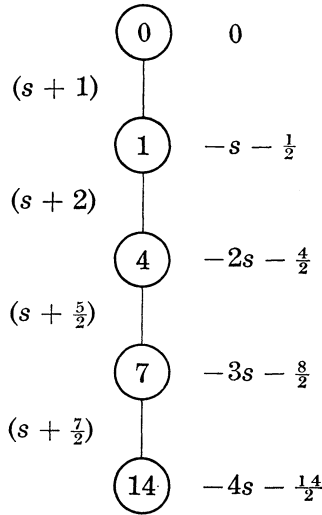
Figure 9–1. Holonomy diagram of $(GL(1) \times Sp(3)$,
$\square \otimes \Lambda_3,\ V(1) \otimes V(14))$.

$b(s) = (s + 1)(s + 2)(s + \frac{5}{2})(s + \frac{7}{2})$ and the holonomy diagram (Figure 9–1).
We denote Λ_m by $\boxed{m}$.

§ 10. $(GL(7),\ \Lambda_3,\ V(35))$

The representation space $V = V(35)$ is spanned by the skew-tensors $u_i \wedge u_j \wedge u_k$ $(1 \leq i < j < k \leq 7)$ of degree three, on which $G = GL(7)$ acts as in § 8. Then it is known (See [6], [7]) that there exist ten orbits $S_m = \rho(G)x_m$ $(m = 0, 1, 4, 7, 9, 10, 14, 15, 22, 35)$, where S_m denotes the m-codimensional orbit, and $x_0 = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$, $x_1 = u_2 \wedge u_3 \wedge u_5 + u_3 \wedge u_4 \wedge u_6 + u_1 \wedge (u_2 \wedge u_7 - u_4 \wedge u_5)$, $x_4 = u_1 \wedge u_3 \wedge u_4 + u_2 \wedge u_5 \wedge u_6 + u_1 \wedge u_2 \wedge u_7$, $x_7 = u_2 \wedge u_3 \wedge u_4 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$, $x_9 = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6$, $x_{10} = u_1 \wedge u_2 \wedge u_6 - u_1 \wedge u_3 \wedge u_5 + u_2 \wedge u_3 \wedge u_4$, $x_{14} = u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$, $x_{15} = u_1 \wedge (u_2 \wedge u_4 + u_3 \wedge u_5)$, $x_{22} = u_1 \wedge u_2 \wedge u_3$ and $x_{35} = 0$. Note that we chose these representative points x_m so that the isotropy subalgebra $\mathfrak{g}_{x_m}$ at x_m will be the standard form. The relative invariant $f(x)$ of this space is of degree seven (See [1], [14]). Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, there exist also ten G-orbits S_m^* $(m = 0, 1, 4, 7, 9, 10, 14, 15, 22, 35)$ in V^*. We denote by Λ_m (resp. Λ_m^*) the conormal bundle of S_m (resp. S_m^*). Clearly we have $\Lambda_0 = V \times \{0\} = \Lambda_{35}^*$ and $\Lambda_{35} = \{0\} \times V^* = \Lambda_0^*$.

(1)　The isotropy subalgebra $\mathfrak{g}_{x_0}$ at x_0 is given as follows (See [1]).

$$(10.1) \quad \mathfrak{g}_{x0} = \left\{ \left[\begin{array}{ccc|ccc} & 2d & 2e & 2f & 2a & 2b & 2c \\ \hline a & & & & 0 & f & -e \\ b & & X & & -f & 0 & d \\ c & & & & e & -d & 0 \\ \hline d & 0 & -c & b & & & \\ e & c & 0 & -a & & -{}^tX & \\ f & -b & a & 0 & & & \end{array} \right]; X \in \mathfrak{sl}(3) \right\} \cong \mathfrak{g}_2 .$$

Since $\Lambda_0 = V \times \{0\}$, we have $\mathrm{ord}_{\Lambda_0} f^s = 0$.

(2)　The isotropy subalgebra $\mathfrak{g}_{x_1}$ at x_1 is given as follows.

$$(10.2) \quad \mathfrak{g}_{x1} = \left\{ \left[\begin{array}{cc|cc|ccc} \frac{d}{2}+\alpha+\beta & a_{12} & & & \gamma_1 & \gamma_2 & \gamma_3 \\ a_{21} & \frac{d}{2}-\alpha+\beta & & b_{12}I_2 & \gamma_4 & \gamma_5 & \gamma_6 \\ \hline & & \frac{d}{2}+\alpha-\beta & a_{12} & -\gamma_2 & \gamma_7 & -\gamma_1 \\ b_{21}I_2 & & a_{21} & \frac{d}{2}-\alpha-\beta & -\gamma_5 & \gamma_8 & -\gamma_4 \\ \hline & & & & -d & 2b_{21} & 2b_{12} \\ & 0 & & & b_{12} & -d+2\beta & 0 \\ & & & & b_{21} & 0 & -d-2\beta \end{array} \right] \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(8).$$

The conormal vector space $V_{x_1}^*$ is spanned by $u_5 \wedge u_6 \wedge u_7$. Then $d\rho_{x_1}(A)u_5 \wedge u_6 \wedge u_7 = 3d\, u_5 \wedge u_6 \wedge u_7$ for $A \in \mathfrak{g}_{x_1}$. Since 0 is the point of the one-codimensional G-orbit, Λ_0 and Λ_1 intersect regularly with codimension one. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $d = \frac{1}{3}$, all remaining parts zero in (10.2). Then $d\rho(A_0)x_1 = 0$ and $d\rho^*(A_0)y_1 = y_1$ where $y_1 = u_5 \wedge u_6 \wedge u_7$. Since $\delta\chi(A_0) = (\deg f/\dim V)\cdot \mathrm{tr}_V A_0 = 3\,\mathrm{tr}\,A_0 = -3d = -1$ (See Proposition 1–9), $\mathrm{tr}_{V_{x_1}^*} A_0 = \dim V_{x_1}^* = 1$, we have $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$ and hence $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$. We have also $\Lambda_1 = \Lambda_{22}^*$, and hence $\Lambda_{22} = \Lambda_1^*$.

(3)　The isotropy subalgebra $\mathfrak{g}_{x_4}$ at x_4 is given as follows.

$$(10.3) \quad \mathfrak{g}_{x4} = \left\{ \left[\begin{array}{cc|cc|c} -\mathrm{tr}\,X & 0 & \alpha_1 & \alpha_2 & \alpha_3 \ \ \alpha_4 & \alpha_5 \\ 0 & -\mathrm{tr}\,Y & \beta_1 & \beta_2 & \beta_3 \ \ \beta_4 & \beta_5 \\ \hline & & & & \beta_2 \\ & 0 & & X & 0 \quad -\beta_1 \\ \hline & & & & -\alpha_4 \\ & 0 & & 0 & Y \quad \alpha_3 \\ \hline & 0 & & 0 & 0 \quad \mathrm{tr}(X+Y) \end{array} \right] \right\}$$

$$\cong (\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)) \cdot \mathfrak{u}(10).$$

The conormal vector space $V_{x_4}^*$ is spanned by $\omega_1 = u_3 \wedge u_5 \wedge u_7$, $\omega_2 = u_3$

$\wedge u_6 \wedge u_7$, $\omega_3 = u_4 \wedge u_5 \wedge u_7$, $\omega_4 = u_4 \wedge u_6 \wedge u_7$. Then we have $(G_{x_4}, \rho_{x_4}, V_{x_4}^*) \cong (GL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2))$, and $y_4 = \omega_1 + \omega_4$ is its generic point, $y_4' = \omega_1$ is a point of the one-codimensional orbit. Since the colocalization $(G_{x_4}, \rho_{x_4}, V_{x_4}^*)$ is an irreducible regular P.V., Λ_4 is a good holonomic variety by Corollary 1–8. Let A_0 be an element of $\mathfrak{g}_{x_4}$ with $X = -\frac{1}{3}I_2$, all remaining parts zero in (10.3). Then $d\rho(A_0)x_4 = 0$ and $d\rho^*(A_0)y_4 = y_4$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = -2$, $\mathrm{tr}_{V_{x_4}^*}A_0 = 4\dim V_{x_4}^*$, we have $\mathrm{ord}_{\Lambda_4}f^s = -2s - \frac{4}{2}$ by Proposition 1–3. We have also $d\rho^*(A_0)y_4' = y_4'$ and $\mathrm{tr}_{\tilde{V}}A_0 = 1$ where $\tilde{V} = V_{x_4}^* \bmod d\rho_{x_4}(\mathfrak{g}_{x_4})y_4' = \boldsymbol{C}\omega_4$. This implies that Λ_1 and Λ_4 intersect regularly with codimension one by Proposition 1–4. By Corollary 1–2, we have $b_{\Lambda_4}(s)/b_{\Lambda_1}(s) = (s+2)$. We have also $\Lambda_4 = \Lambda_{15}^*$ and hence $\Lambda_{15} = \Lambda_4^*$.

(4) The isotropy subalgebra $\mathfrak{g}_{x_7}$ at x_7 is given as follows.

$$(10.4) \quad \mathfrak{g}_{x_7} = \left\{ \left(\begin{array}{c|ccc|ccc} \varepsilon & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \hline & & & & \delta_4 & (\gamma_3+\delta_3) & \delta_2 \\ 0 & & X & & \delta_3 & \delta_5 & (\gamma_1+\delta_1) \\ & & & & (\gamma_2+\delta_2) & \delta_1 & \delta_6 \\ \hline 0 & & 0 & & & -\varepsilon I_3 - {}^t X \end{array} \right) ; X \in \mathfrak{sl}(3) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(3)) \oplus \mathfrak{u}(12).$$

Put $\omega_1 = u_5 \wedge u_6 \wedge u_7$, $\omega_2 = u_2 \wedge u_6 \wedge u_7$, $\omega_3 = u_4 \wedge u_5 \wedge u_6$, $\omega_4 = u_3 \wedge u_5 \wedge u_7$, $\omega_5 = u_2 \wedge u_5 \wedge u_7 - u_3 \wedge u_6 \wedge u_7$, $\omega_6 = u_4 \wedge u_5 \wedge u_7 - u_3 \wedge u_5 \wedge u_6$, $\omega_7 = u_2 \wedge u_5 \wedge u_6 + u_4 \wedge u_6 \wedge u_7$. Then the conormal vector space $V_{x_7}^*$ is spanned by these $\omega_1, \cdots, \omega_7$. The action $d\rho_{x_7}$ of $\mathfrak{g}_{x_7}$ on $V_{x_7}^*$ is given by

$$(10.5) \qquad d\rho_{x_7}(A)(\omega_1, \cdots, \omega_7) = (\omega_1, \cdots, \omega_7) \left(\begin{array}{c|cccccc} 3\varepsilon & * & * & * & * & * & * \\ \hline 0 & & 2\varepsilon I_6 \oplus d\rho_1^*(X) \end{array} \right)$$

where $\rho_1 = 2\Lambda_1$ for $SL(3)$.

Here $y_7 = \omega_2 + \omega_3 + \omega_4$ is its generic point, and $y_7' = \omega_2 + \omega_3$ is a point of the one-codimensional orbit. This implies that $\Lambda_7 = \Lambda_{10}^*$, $\Lambda_{10} = \Lambda_7^*$ and $\dim \Lambda_4 \cap \Lambda_7 = \dim V - 1$. Since $\Lambda_4 \cap \Lambda_7$ is G_0-prehomogeneous, Λ_7 is a good holonomic variety. Let A_0 be an element of $\mathfrak{g}_{x_7}$ with $\varepsilon = \frac{1}{2}$, all remaining parts zero in (10.4). Then $d\rho(A_0)x_7 = 0$ and $d\rho^*(A_0)y_7 = y_7$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = -6\varepsilon = -3$, $\mathrm{tr}_{V_{x_7}^*}A_0 = \frac{15}{2}$ and $\dim V_{x_7}^* = 7$, we have $\mathrm{ord}_{\Lambda_7}f^s = -3s - \frac{8}{2}$ by Proposition 1–3. Let A_β be an element of $\mathfrak{g}_{x_7}$ with $\varepsilon = \frac{\beta}{6} + \frac{1}{3}$, $X = \begin{pmatrix} \eta & & \\ & \eta & \\ & & -2\eta \end{pmatrix}$ with $\eta = \frac{\beta}{6} - \frac{1}{6}$, all remaining parts zero in (10.4). Then we have $d\rho(A_\beta)x_7 = 0$, $d\rho(A_\beta)y_7' = y_7'$ and $\mathrm{tr}_{\tilde{V}}A_\beta = \beta$ where $\tilde{V} = V_{x_7}^* \bmod d\rho_{x_7}(\mathfrak{g}_{x_7})y_7' = \boldsymbol{C}\omega_4$. This implies that Λ_4 and Λ_7 intersect regularly

by Proposition 1–4. By Corollary 1–2, we have $b_{\Lambda_7}(s)/b_{\Lambda_4}(s) = (s + \frac{5}{2})$.

(5) The isotropy subalgebra $\mathfrak{g}_{x_9}$ at x_9 is given as follows.

(10.6) $$\mathfrak{g}_{x_9} = \left\{ \begin{pmatrix} X & 0 & Z \\ 0 & Y & W \\ 0 & 0 & \varepsilon \end{pmatrix}; X, Y \in \mathfrak{sl}(3), Z, W \in \boldsymbol{C}^3 \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(3) \oplus \mathfrak{sl}(3)) \oplus V(6).$$

The conormal vector space $V_{x_9}^*$ is spanned by $u_i \wedge u_j \wedge u_7 (1 \leq i \leq 3; 4 \leq j \leq 6)$. By seeing the weights, we have $(G_{x_9}, \rho_{x_9}, V_{x_9}^*) \cong (SL(3) \times GL(3), \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3))$. Since this is an irreducible regular P.V., Λ_9 is a good holonomic variety by Corollary 1–8. As a generic point, we may take $y_9 = (u_1 \wedge u_4 + u_2 \wedge u_5 + u_3 \wedge u_6) \wedge u_7$, and $y_9' = (u_1 \wedge u_4 + u_2 \wedge u_5) \wedge u_7$ is a point of the one-codimensional orbit. This implies that $\Lambda_9 = \Lambda_{14}^*$, $\Lambda_{14} = \Lambda_9^*$ and dim $\Lambda_4 \cap \Lambda_9 = \dim V - 1$. Let A_0 be an element of $\mathfrak{g}_{x_9}$ with $\varepsilon = -1$, all remaining parts zero in (10.6). Then $d\rho(A_0)x_9 = 0$, $d\rho^*(A_0)y_9 = y_9$. Since $\delta\chi(A_0) = 3\operatorname{tr} A_0 = -3$, $\operatorname{tr}_{V_{x_9}^*} A_0 = 9\varepsilon = -9$, $\dim V_{x_9}^* = 9$, we have $\operatorname{ord}_{\Lambda_9} f^s = -3s - \frac{9}{2}$. Let A_β be an element of $\mathfrak{g}_{x_9}$ with $\varepsilon = ((\beta + 2)/3)$, $X = \begin{pmatrix} \eta & & \\ & \eta & \\ & & -2\eta \end{pmatrix}$ with $\eta = ((1 - \beta)/3)$, all remaining parts zero in (10.6). Then we have $d\rho(A_0)x_9 = 0$, $d\rho^*(A_0)y_9' = y_9'$ and $\operatorname{tr}_{\tilde{V}} A_0 = \beta$ where $\tilde{V} = V_{x_9}^* \operatorname{mod} d\rho_{x_9}(\mathfrak{g}_{x_9})y_9' = \boldsymbol{C} u_3 \wedge u_6 \wedge u_7$. This implies that Λ_4 and Λ_9 intersect regularly. By Corollary 1–2, we have $b_{\Lambda_9}(s)/b_{\Lambda_4}(s) = (s + 3)$.

(6) The isotropy subalgebra $\mathfrak{g}_{x_{10}}$ at x_{10} is given as follows.

(10.7) $$\mathfrak{g}_{x_{10}} = \left\{ A = \left(\begin{array}{c|c|c} \varepsilon I_3 + X & B & C \\ \hline 0 & -2\varepsilon I_3 + X & F \\ \hline 0 & 0 & \eta \end{array} \right); X \in \mathfrak{sl}(3), \operatorname{tr} B = 0, C, D \in \boldsymbol{C}^3 \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(3)) \oplus \mathfrak{u}(14).$$

Put $\omega_1 = u_5 \wedge u_6 \wedge u_7$, $\omega_2 = u_4 \wedge u_6 \wedge u_7$, $\omega_3 = u_4 \wedge u_5 \wedge u_7$, $\omega_4 = u_1 \wedge u_4 \wedge u_7$, $\omega_5 = u_2 \wedge u_5 \wedge u_7$, $\omega_6 = u_3 \wedge u_6 \wedge u_7$, $\omega_7 = (u_1 \wedge u_5 + u_2 \wedge u_4) \wedge u_7$, $\omega_8 = (u_2 \wedge u_6 + u_3 \wedge u_5) \wedge u_7$, $\omega_9 = (u_1 \wedge u_6 + u_3 \wedge u_4) \wedge u_7$, $\omega_{10} = u_4 \wedge u_5 \wedge u_6$. Then the conormal vector space $V_{x_{10}}^*$ is spanned by $\omega_1, \cdots, \omega_{10}$, and the action $d\rho_{x_{10}}$ of $\mathfrak{g}_{x_{10}}$ on $V_{x_{10}}^*$ is given as follows.

(10.8) $$d\rho_{x_{10}}(A)(\omega_1, \cdots, \omega_{10}) = (\omega_1, \cdots, \omega_{10}) \left(\begin{array}{c|c|c} (4\varepsilon - \eta)I_3 + X & B' & F \\ \hline 0 & (\varepsilon - \eta)I_6 + d\rho_1^*(X) & 0 \\ \hline 0 & 0 & 6\varepsilon \end{array} \right)$$

where $\rho_1 = 2\Lambda_1$ for $SL(3)$.

Then $y_{10} = \omega_4 + \omega_8 + \omega_{10}$ is a generic point. There exist two one-codimensional orbits. As a representative point, we may take $y'_{10} = \omega_8 + \omega_{10}$ and $y''_{10} = \omega_4 + \omega_5 + \omega_6$ respectively. This implies that $\dim \Lambda_7 \cap \Lambda_{10} = \dim \Lambda_9 \cap \Lambda_{10} = \dim V - 1$. Since $\Lambda_{10} = \Lambda_7^*$, Λ_{10} is a good holonomic variety. Let A_0 be an element of $\mathfrak{g}_{x_{10}}$ with $\varepsilon = \frac{1}{6}$, $\eta = -\frac{5}{6}$, all remaining parts zero in (10.7). Then $d\rho(A_0)x_{10} = 0$ and $d\rho^*(A_0)y_{10} = y_{10}$. Since $\delta\chi(A_0) = -9\varepsilon + 3\eta = -4$, $\operatorname{tr}_{V_{x_{10}}^*} A_0 = 24\varepsilon - 9\eta = \frac{23}{2}$, and $\dim V_{x_{10}}^* = 10$, we have $\operatorname{ord}_{\Lambda_{10}} f^s = -4s - \frac{13}{2}$ by Proposition 1–3.

Since $d\rho^*(A_0)y'_{10} = y'_{10}$ and $\operatorname{tr}_{\tilde{V}} A_0 = 1$ where $\tilde{V} = V_{x_{10}}^* \bmod d\rho_{x_{10}}(\mathfrak{g}_{x_{10}})y'_{10} = C\omega_4$, Λ_7 and Λ_{10} intersect regularly by Proposition 1–4. Let A_β be an element of $\mathfrak{g}_{x_{10}}$ with $\varepsilon = \frac{\beta}{6}$, $\eta = \frac{\beta}{6} - 1$, all remaining parts zero in (10.7). Then we have $d\rho(A_\beta)x_{10} = 0$, $d\rho^*(A_\beta)y''_{10} = y''_{10}$, and $\operatorname{tr}_{\tilde{V}} A_\beta = \beta$ where $\tilde{V} = V_{x_{10}}^* \bmod d\rho_{x_{10}}(\mathfrak{g}_{x_{10}})y''_{10} = C\omega_{10}$. This implies that Λ_9 and Λ_{10} intersect regularly by Proposition 1–4. By Corollary 1–2, we have $b_{\Lambda_{10}}(s)/b_{\Lambda_7}(s) = (s + 3)$ and $b_{\Lambda_{10}}(s)/b_{\Lambda_9}(s) = (s + \frac{5}{2})$.

(7) The isotropy subalgebra $\mathfrak{g}_{x_{14}}$ at x_{14} is given as follows.

$$(10.9) \quad \mathfrak{g}_{x_{14}} = \left\{ \left(\begin{array}{c|c} -2\varepsilon & Y \\ \hline 0 & \varepsilon I_6 + X \end{array} \right); X \in \mathfrak{sp}(3), {}^t Y \in C^6 \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{sp}(3)) \oplus V(6) .$$

Put $\omega_1 = u_2 \wedge u_3 \wedge u_4$, $\omega_2 = u_5 \wedge u_6 \wedge u_7$, $\omega_3 = u_3 \wedge u_4 \wedge u_5$, $\omega_4 = u_2 \wedge u_6 \wedge u_7$, $\omega_5 = u_2 \wedge u_4 \wedge u_6$, $\omega_6 = u_3 \wedge u_5 \wedge u_7$, $\omega_7 = u_2 \wedge u_3 \wedge u_7$, $\omega_8 = u_4 \wedge u_5 \wedge u_6$, $\omega_9 = u_2 \wedge u_3 \wedge u_6 - u_2 \wedge u_4 \wedge u_7$, $\omega_{10} = u_3 \wedge u_5 \wedge u_6 - u_4 \wedge u_5 \wedge u_7$, $\omega_{11} = u_2 \wedge u_3 \wedge u_5 + u_3 \wedge u_4 \wedge u_7$, $\omega_{12} = u_2 \wedge u_5 \wedge u_6 + u_4 \wedge u_6 \wedge u_7$, $\omega_{13} = u_2 \wedge u_4 \wedge u_5 - u_3 \wedge u_4 \wedge u_6$, $\omega_{14} = u_2 \wedge u_5 \wedge u_7 - u_3 \wedge u_6 \wedge u_7$. The conormal vector space $V_{x_{14}}^*$ is spanned by these $\omega_1, \cdots, \omega_{14}$. By seeing the weights, we have $(G_{x_{14}}, \rho_{x_{14}}, V_{x_{14}}^*) \cong (GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$. Since this is an irreducible regular P.V., Λ_{14} is a good holonomic variety by Corollary 1–8. As we have seen in § 9, $y_{14} = \omega_1 + \omega_2$ is a generic point. Let A_0 be an element of $\mathfrak{g}_{x_{14}}$ with $\varepsilon = -\frac{1}{3}$, $X = Y = 0$ in (10.9). Then $d\rho(A_0)x_{14} = 0$ and $d\rho^*(A_0)y_{14} = y_{14}$. Since $\delta\chi(A_0) = 3\operatorname{tr} A_0 = -4$, $\operatorname{tr}_{V_{x_{14}}^*} A_0 = 14 \times 3\varepsilon = -14$ and $\dim V_{x_{14}}^* = 14$, we have $\operatorname{ord}_{\Lambda_{14}} f^s = -4s - \frac{14}{2}$. Since $\Lambda_{14} = \Lambda_9^*$, $\Lambda_7 = \Lambda_{10}^*$, and Λ_9 and Λ_{10} intersect regularly with codimension one, so do Λ_{14} and Λ_7. By Corollary 1–2, we have $b_{\Lambda_{14}}(s)/b_{\Lambda_7}(s) = (s + \frac{7}{2})$.

(8) The isotropy subalgebra $\mathfrak{g}_{x_{15}}$ at x_{15} is given as follows.

$$(10.10) \quad \mathfrak{g}_{x_{15}} = \left\{ \left(\begin{array}{c|c|c} -2\varepsilon & W & U \\ \hline 0 & \varepsilon I_4 + X & Z \\ \hline 0 & 0 & \eta I_2 + Y \end{array} \right); \begin{array}{l} X \in \mathfrak{sp}(2),\ Y \in \mathfrak{sl}(2),\ Z \in M(4, 2) \\ {}^t W \in C^2,\ {}^t U \in C^4,\ \varepsilon, \eta \in C \end{array} \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sp}(2) \oplus \mathfrak{gl}(2)) \oplus \mathfrak{u}(14).$$

Put $\omega_i = u_{i+1} \wedge u_6 \wedge u_7$ $(1 \leq i \leq 4)$, $\omega_5 = u_1 \wedge u_6 \wedge u_7$, $\omega_{6+j} = (u_2 \wedge u_4 - u_3 \wedge u_5) \wedge u_{6+j}$, $\omega_{8+j} = u_2 \wedge u_3 \wedge u_{6+j}$, $\omega_{10+j} = u_2 \wedge u_5 \wedge u_{6+j}$, $\omega_{12+j} = u_3 \wedge u_4 \wedge u_{6+j}$, $\omega_{14+j} = u_4 \wedge u_5 \wedge u_{6+j}$ $(j = 0, 1)$. Then the conormal vector space $V_{x_{15}}^*$ is spanned by $\omega_1, \cdots, \omega_{15}$. The action $d\rho_{x_{15}}$ of $\mathfrak{g}_{x_{15}}$ on $V_{x_{15}}^*$ is as follows.

$$
d\rho_{x_{15}}(A)(\omega_1, \cdots, \omega_{15}) = (\omega_1, \cdots, \omega_{15})
$$

(10.11)

$$
\times \left(\begin{array}{c|c|c} -(\varepsilon + 2\eta)I_4 + d\rho_1(X) & * & * \\ \hline 0 & 2\varepsilon - 2\eta & * \\ \hline 0 & 0 & -(2\varepsilon + \eta)I_{10} + d\rho_2^*(X \oplus Y) \end{array} \right)
$$

where $\rho_1 = \Lambda_1$ for $Sp(2)$ and $\rho_2 = \Lambda_2 \otimes \Lambda_1$ for $Sp(2) \times SL(2)$. Since $\Lambda_{15} = \Lambda_4^*$ and Λ_4 is a good holonomic variety, Λ_{15} is also a good holonomic variety. $y_{15} = \omega_5 + \omega_{11} + \omega_{12}$ is a generic point. Let A_0 be an element of $\mathfrak{g}_{x_{15}}$ with $\varepsilon = -\frac{1}{6}$, $\eta = -\frac{2}{3}$, all remaining parts zero in (10.10). Then $d\rho(A_0)x_{15} = 0$ and $d\rho^*(A_0)y_{15} = y_{15}$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = 6(\varepsilon + \eta) = -5$, $\mathrm{tr}_{V_{x_{15}}^*}A_0 = -22\varepsilon - 20\eta = \frac{102}{6}$, and $\dim V_{x_{15}}^* = 15$, we have $\mathrm{ord}_{\Lambda_{15}}f^s = -5s - \frac{19}{2}$. Since $\Lambda_{15} = \Lambda_4^*$, $\Lambda_{14} = \Lambda_9^*$, $\Lambda_{10} = \Lambda_7^*$, we have $\dim \Lambda_{15} \cap \Lambda_{14} = \dim \Lambda_{15} \cap \Lambda_{10} = \dim V - 1$ and they intersect regularly. By Corollary 1–2, we have $b_{\Lambda_{15}}(s)/b_{\Lambda_{14}}(s) = (s + 3)$ and $b_{\Lambda_{15}}(s)/b_{\Lambda_{10}}(s) = (s + \frac{7}{2})$.

(9) The isotropy subalgebra $\mathfrak{g}_{x_{22}}$ at x_{22} is given as follows.

(10.12) $\mathfrak{g}_{x_{22}} = \left\{ \tilde{X} = \left(\begin{array}{c|c} X & Z \\ \hline 0 & \varepsilon I_4 + Y \end{array} \right); X \in \mathfrak{sl}(3),\ Y \in \mathfrak{sl}(4),\ Z \in M(3, 4) \right\}$

$$\cong (\mathfrak{sl}(3) \oplus \mathfrak{gl}(4)) \oplus V(12).$$

The conormal vector space $V_{x_{22}}^*$ is spanned by $u_i \wedge u_j \wedge u_k$ $(4 \leq i < j < k \leq 7)$ and $u_i \wedge u_j \wedge u_k$ $(1 \leq i \leq 3, 4 \leq j < k \leq 7)$. The action $d\rho_{x_{22}}$ of $\mathfrak{g}_{x_{22}}$ is given by

$$
d\rho(\tilde{X})(u_5 \wedge u_6 \wedge u_7, \cdots)
$$

(10.13)

$$
= (u_5 \wedge u_6 \wedge u_7, \cdots)\left(\begin{array}{c|c} Y - 3\varepsilon I_4 & * \\ \hline 0 & -2\varepsilon I_{18} + d\rho_1^*(X \oplus Y) \end{array} \right)
$$

where $\rho_1 = \Lambda_1 \otimes \Lambda_2$ for $SL(3) \times SL(4)$. For example, $y_{22} = u_1 \wedge (u_4 \wedge u_5 + u_6 \wedge u_7) + u_2 \wedge u_4 \wedge u_6 + u_3 \wedge u_5 \wedge u_7$ is a generic point. Since $\Lambda_{22} = \Lambda_1^*$, Λ_{22} is a good holonomic variety. Let A_0 be an element of $\mathfrak{g}_{x_{22}}$ with $\varepsilon = -\frac{1}{2}$, $X = Y = Z = 0$ in (10.12). Then $d\rho(A_0)x_{22} = 0$ and $d\rho^*(A_0)y_{22} = y_{22}$. Since $\delta\chi(A_0) = 12\varepsilon = -6$, $\mathrm{tr}_{V_{x_{22}}^*}A_0 = -48\varepsilon = 24$ and $\dim V_{x_{22}}^* = 22$, we have $\mathrm{ord}_{\Lambda_{22}}f^s = -6s - \frac{26}{2}(= -6s - 13)$. Since $\Lambda_{22} = \Lambda_1^*$ and $\Lambda_{15} = \Lambda_4^*$, we have

$\dim \Lambda_{22} \cap \Lambda_{15} = \dim V - 1$ and they intersect regularly.

By Corollary 1–2, we have $b_{\Lambda_{22}}(s)/b_{\Lambda_{15}}(s) = (s + 4)$.

(10)   The isotropy subalgebra $\mathfrak{g}_{x_{35}}$ at $x_{35} = 0$ is $\mathfrak{g}$ itself and we have $(G_{x_{35}}, \rho_{x_{35}}, V^*_{x_{35}}) = (G, \rho^*, V^*) \cong (GL(7), \Lambda_3, V(35))$. Then $y_{35} = x_0 = u_2 \wedge u_3 \wedge \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$ is its generic point. Put $A_0 = -\frac{1}{3}I_7$. Then $d\rho(A_0)x_{35} = 0$ and $d\rho^*(A_0)y_{35} = y_{35}$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = -7$, $\mathrm{tr}_{V^*_{x_{35}}}A_0 = -35$ and $\dim V^*_{x_{35}} = 35$, we have $\mathrm{ord}_{\Lambda_{35}}f^s = -7s - \frac{35}{2}$. Since $\Lambda_{22} = \Lambda_1^*$ and $\Lambda_{35} = \Lambda_0^*$, they intersect regularly with codimension one. By Corollary 1–2, we have $b_{\Lambda_{35}}(s)/b_{\Lambda_{22}}(s) = (s + 5)$. Since $b_{\Lambda_0}(s) = 1$ and $b_{\Lambda_{35}}(s) = b(s)$, we obtain the b-function $b(s) = (s + 1)(s + 2)(s + \frac{5}{2})(s + \frac{7}{2})(s + 3)(s + 4)(s + 5)$, and the holonomy diagram (Figure 10–1). We denote Λ_m by $\boxed{m}$.

Note that the colocalization at x_1, x_4, x_7, x_9, x_{14}, x_{22} and x_{35} has the unique one-codimensional orbit respectively, and the colocalization at x_{10} and x_{15} has the two one-codimensional orbits respectively. Therefore we have obtained all one-codimensional intersections among the conormal bundles.
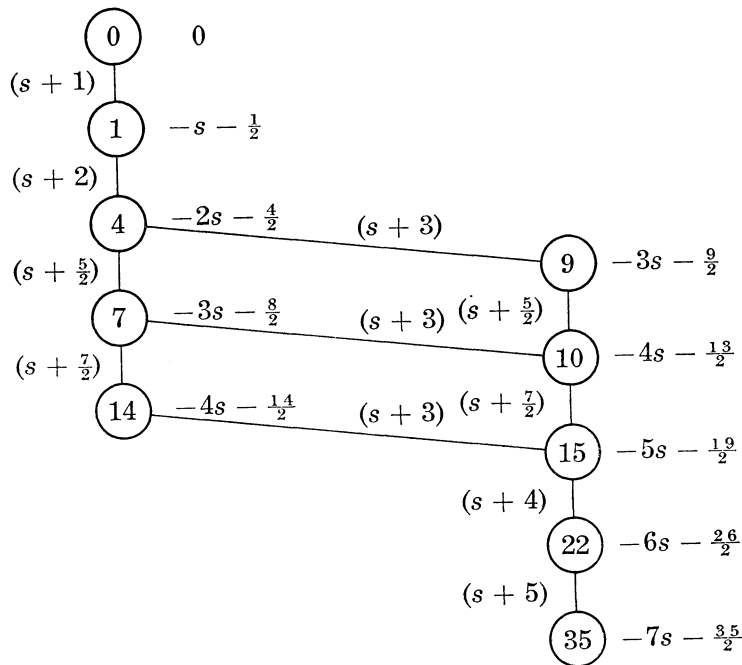
Figure 10–1.   Holonomy diagram of $(GL(7), \Lambda_3, V(35))$.

§ 11. $(SL(5) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3))$

Let $V(10)$ be a vector space spanned by 2-forms $u_i \wedge u_j$ $(1 \leq i < j \leq 5)$. Then the representation space is identified with $V = V(10) \oplus V(10) \oplus V(10)$ (See [1]). Let Λ be the conormal bundle of an orbit S in V and Λ^* that of an orbit S^* in V^*. When $\Lambda = \Lambda^*$, we say that S and S^* are the dual orbits of each other. We denote by $S_{i,j}^{(k)}$ the i-codimensional orbit whose dual orbit is j-codimensional, where k denotes the dimension of the central torus of the isotropy subgroup of this orbit. When there is no confusion, denote this by S_i or S_{ij}. We denote by $\Lambda_{i,j}^{(k)}$ (resp. $\Lambda_{i,j}$, Λ_i) the conormal bundle of $S_{i,j}^{(k)}$ (resp. $S_{i,j}$, S_i). We identify V and its dual V^* by taking $(u_i \wedge u_{i'}, u_j \wedge u_{j'}, u_k \wedge u_{k'})$ $(i < i', j < j', k < k')$ as a dual basis.

PROPOSITION 11–1. *This space has following twenty five orbits* $S_{i,j}^{(k)}$.

(1) $S_{0,30}^{(0)}$: $(u_1 \wedge u_2 + u_3 \wedge u_4,\ u_2 \wedge u_3 + u_4 \wedge u_5,\ u_1 \wedge u_3 + u_2 \wedge u_5)$ $(= x_0)$

(2) $S_{1,21}^{(2)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_5 + u_3 \wedge u_4,\ u_2 \wedge u_3 + u_4 \wedge u_5)$ $(= x_1)$

(3) $S_{2,16}^{(2)}$: $(u_1 \wedge u_2,\ u_2 \wedge u_3 + u_3 \wedge u_4,\ u_1 \wedge u_3 + u_4 \wedge u_5)$ $(= x_2)$

(4) $S_{3,15}^{(3)}$: $(u_1 \wedge u_2,\ u_3 \wedge u_4,\ u_1 \wedge u_5 + u_4 \wedge u_5)$ $(= x_3)$

(5) $S_{3,13}^{(2)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3 + u_2 \wedge u_4,\ u_2 \wedge u_3 + u_4 \wedge u_5)$ $(= x_3')$

(6) $S_{4,11}^{(3)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3 + u_2 \wedge u_4,\ u_4 \wedge u_5)$ $(= x_4)$

(7) $S_{5,8}^{(2)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3 + u_2 \wedge u_4,\ u_1 \wedge u_5 + u_3 \wedge u_4)$ $(= x_5)$

(8) $S_{6,12}^{(1)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_2 \wedge u_3 + u_4 \wedge u_5)$ $(= x_6)$

(9) $S_{7,9}^{(2)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_4 \wedge u_5)$ $(= x_7)$

(10) $S_{7,7}^{(1)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3 + u_2 \wedge u_4,\ u_2 \wedge u_3 + u_1 \wedge u_5)$ $(= x_7')$

(11) $S_{7,7}^{(2)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_2 \wedge u_4 + u_3 \wedge u_5)$ $(= x_7'')$

(12) $S_{8,18}^{(2)}$: $(u_1 \wedge u_2 + u_3 \wedge u_4,\ u_2 \wedge u_3 + u_4 \wedge u_5,\ 0)$ $(= x_8)$

(13) $S_{8,14}^{(1)}$: $(u_1 \wedge u_2,\ u_3 \wedge u_4,\ u_1 \wedge u_3 + u_2 \wedge u_4)$ $(= x_8')$

(14) $S_{8,5}^{(3)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_1 \wedge u_5 + u_2 \wedge u_4)$ $(= x_8'')$

(15) $S_{9,7}^{(4)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_2 \wedge u_4)$ $(= x_9)$

(16) $S_{10,10}^{(3)}$: $(u_1 \wedge u_2,\ u_3 \wedge u_4 + u_1 \wedge u_5,\ 0)$ $(= x_{10})$

(17) $S_{11,4}^{(2)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_1 \wedge u_4 + u_2 \wedge u_3)$ $(= x_{11})$

(18) $S_{12,6}^{(3)}$: $(u_1 \wedge u_2,\ u_3 \wedge u_4,\ 0)$ $(= x_{12})$

(19) $S_{13,3}^{(3)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3 + u_2 \wedge u_4,\ 0)$ $(= x_{13})$

(20) $S_{14,8}^{(2)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_1 \wedge u_4)$ $(= x_{14})$

(21) $S_{15,3}^{(1)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_2 \wedge u_3)$ $(= x_{15})$

(22) $S_{16,2}^{(3)}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ 0)$ $(= x_{16})$

(23) $S_{18,8}^{(2)}$: $(u_1 \wedge u_2 + u_3 \wedge u_4,\ 0,\ 0)$ $(= x_{18})$

(24) $S_{21,1}^{(2)}$: $(u_1 \wedge u_2,\ 0,\ 0)$ $(= x_{21})$

(25) $S_{30,0}^{(1)}$: $(0, 0, 0)$ $(= x_{30})$.

Proof. It is easy to check that the non-regular P.V. $(SL(5) \times GL(2),$ $\Lambda_2 \otimes \Lambda_1, V(10) \otimes V(2))$ has eight orbits which are represented by the following points; [1] $(0, 0)$, [2] $(u_1 \wedge u_2, 0)$, [3] $(u_1 \wedge u_2 + u_3 \wedge u_4, 0)$, [4] $(u_1 \wedge u_2, u_1 \wedge u_3)$, [5] $(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4)$, [6] $(u_1 \wedge u_2, u_3 \wedge u_4)$, [7] $(u_1 \wedge u_2, u_3 \wedge u_4 + u_1 \wedge u_5)$, [8] $(u_1 \wedge u_2 + u_3 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5)$. Therefore, for a point $x = (x_1, x_2, x_3)$ of V, we may assume that (x_1, x_2) is one of these points. In the first three cases, repeating the same argument, we obtain (12), (16), (18), (19), (22), (23), (24) and (25). For $\lambda \in C$, we define $S_{ij}(\lambda)$ by $S_{ij}(\lambda)u_k = u_k$ for $k \neq i$ and $S_{ij}(\lambda)u_i = u_i + \lambda u_j$. Then $S_{ij}(\lambda)$ is an element of $\rho(G)$. Put $x_3 = \sum_{i<j} a_{ij} u_i \wedge u_j$. First we consider the case [4], i.e., $(x_1, x_2) = (u_1 \wedge u_2, u_1 \wedge u_3)$. Assume that $a_{45} \neq 0$. Then we may assume that $x_3 = a_{23} u_2 \wedge u_3 + u_4 \wedge u_5$. In fact, we have $a_{35} = 0$ by $S_{43}(-a_{35}/a_{45})$ and so on. If $a_{23} = 0$, then we have (9). If $a_{23} \neq 0$, then we have (8). Next assume that $a_{45} = 0$. If one of a_{ij} $(i = 2, 3; j = 4, 5)$ is not zero, we may assume that $x_3 = u_2 \wedge u_4 + a_{15}u_1 \wedge u_5 + a_{35}u_3 \wedge u_5$. If $a_{35} \neq 0$ (resp. $a_{35} = 0$ and $a_{15} \neq 0$, $a_{35} = a_{15} = 0$), then we have (11) (resp. (14), (15)). If any $a_{ij} = 0$ $(i = 2, 3; j = 4, 5)$, then we may assume that $x_3 = a_{14}u_1 \wedge u_4 + a_{23}u_2 \wedge u_3$. If $a_{14} \neq 0$ and $a_{23} \neq 0$ (resp. $a_{14} \neq 0$ and $a_{23} = 0$, $a_{14} = 0$ and $a_{23} \neq 0$, $a_{14} = a_{23} = 0$), then we have (17) (resp. (20), (21), (22)). Next we consider the case [5], i.e., $(x_1, x_2) = (u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4)$. If $a_{35} \neq 0$ or $a_{45} \neq 0$, we may assume that $x_3 = a_{23}u_2 \wedge u_3 + u_4 \wedge u_5$ and hence we have (5) (resp. (6)) for $a_{23} \neq 0$ (resp. $a_{23} = 0$). If $a_{35} = a_{45} = 0$ and one of a_{k5} $(k = 1, 2)$ is not zero, then we may assume that $x_3 = a_{23}u_2 \wedge u_3 + a_{34}u_3 \wedge u_4 + u_1 \wedge u_5$ and hence we have (7) (resp. (10), (14)) for $a_{34} \neq 0$ (resp. $a_{34} = 0$ and $a_{23} \neq 0$, $a_{34} = a_{23} = 0$). If $a_{k5} = 0$ for $1 \leq k \leq 4$, we may assume that $x_3 = a_{13}u_1 \wedge u_3 + a_{14}u_1 \wedge u_4 + a_{23}u_2 \wedge u_3 + a_{34}u_3 \wedge u_4$. Then we have (13) (resp. we have (19); it is reduced to the case [4]) for $a_{34} \neq 0$ (resp. $x_3 = 0$; $a_{34} = 0$ and $x_3 \neq 0$). Now we consider the case [6], i.e., $(x_1, x_2) = (u_1 \wedge u_2, u_3 \wedge u_4)$. (i) If $a_{35} \neq 0$ or $a_{45} \neq 0$, we may assume that $x_3 = a_{12}u_1 \wedge u_3 + a_{15}u_1 \wedge u_5 + a_{25}u_2 \wedge u_5 + u_4 u_5$. Moreover if $a_{25} \neq 0$, then we have (3) (resp. (4)) for $a_{13} \neq 0$ (resp. $a_{13} = 0$). If $a_{25} = 0$, then we have (4) (resp. it is reduced to the case [4] or [5]) for $a_{15} \neq 0$ (resp. $a_{15} = 0$). (ii) If $a_{35} = a_{45} = 0$, it is reduced to the previous cases. Next we shall consider the case [7], i.e., $(x_1, x_2) = (u_1 \wedge u_2, u_3 \wedge u_4 + u_1 \wedge u_5)$. (i) If $a_{35} \neq 0$ or $a_{45} \neq 0$, then we may assume that $a_{45} = 1$ and $a_{35} = a_{24} = a_{14} = a_{12} = 0$. By $S_{53}(\lambda)$, $S_{41}(\mu)$, $S_{21}(\nu)$ and $GL(3)$, we have $x_3 = (a_{13} + (a_{15} - a_{34})\lambda + (a_{23} + a_{25})\nu + \lambda^2)u_1 \wedge u_3 + (a_{23} + \lambda a_{25})u_2 \wedge u_3$

$+ a_{25}u_2 \wedge u_5 + (a_{15} + \nu a_{25} + \mu + \lambda)u_1 \wedge u_5 + (a_{34} + \mu - \lambda)u_3 \wedge u_4 + u_4 \wedge u_5$. If $a_{25} \neq 0$, we may take λ, μ, ν so that $a_{13} + (a_{15} - a_{34})\lambda + \nu(a_{23} + \lambda a_{25}) + \lambda^2 = a_{15} + \nu a_{25} + \mu + \lambda = a_{34} + \mu - \lambda = 0$ and hence we have $x_3 = \alpha u_2 \wedge u_3 + u_2 \wedge u_5 + u_4 \wedge u_5$. If $\alpha \neq 0$ (resp. $\alpha = 0$), then we have (2) (resp. (3)) by $S_{35}(-1/2\alpha)$, $S_{42}(-1/2)$, $S_{12}(1/4\alpha)$, $\{u_3 \mapsto (1/\sqrt{\alpha})u_3, u_4 \mapsto \sqrt{\alpha}\,u_4, u_j \mapsto u_j \ (j \neq 3, 4)\}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2\alpha \\ 0 & 0 & 1/\sqrt{\alpha} \end{pmatrix}$ (resp. by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $\{u_5 \mapsto u_3, u_3 \mapsto u_5, u_4 \mapsto -u_4, u_j \mapsto u_j \ (j = 1, 2)\}$). If $a_{25} = 0$, taking λ and μ satisfying $a_{15} + \mu + \lambda = a_{34} + \mu - \lambda = 0$, we have $x_3 = a'_{13}u_1 \wedge u_3 + a'_{23}u_2 \wedge u_3 + u_4 \wedge u_5$. If $a'_{23} \neq 0$ (resp. $a'_{23} = a'_{13} = 0$), then we have (2) (resp. (6)). If $a'_{23} = 0$ and $a'_{13} \neq 0$, then we have (4) by $S_{41}(\lambda)$, $\begin{pmatrix} 1 & & \\ & 1 & -1/\gamma \\ & 0 & 1 \end{pmatrix}$, $S_{35}\left(-\dfrac{1}{\gamma}\right)$, $S_{53}\left(\dfrac{\gamma}{2}\right)$, $\begin{pmatrix} 1 & & \\ & 1 & \\ & \gamma/2 & 1 \end{pmatrix}$ and $\{u_1 \mapsto (1/2\gamma)u_1, u_2 \mapsto 2\gamma u_2, u_j \mapsto u_j \ (j \neq 1, 2)\}$ where $\gamma = \sqrt{-a'_{13}}$. (ii) If $a_{35} = a_{45} = 0$, it is reduced to the previous cases. Finally we shall consider the case [8], i.e., $(x_1, x_2) = (u_1 \wedge u_2 + u_3 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5)$. The isotropy subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(5) \oplus \mathfrak{gl}(2)$ at this point (x_1, x_2) is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & \gamma_1 & & \\ \alpha_1 & a_2 & \beta_1 & \gamma_2 & \beta_2 \\ -2\gamma_2 & & a_3 & & 2\gamma_1 \\ \beta_1 & -\gamma_2 & \beta_2 & a_4 & \alpha_2 \\ & & -\gamma_2 & & a_5 \end{pmatrix} \oplus \begin{pmatrix} -(a_1+a_2) & -\gamma_2 \\ \gamma_1 & -(a_4+a_5) \end{pmatrix} \right\}$$

with $a_1 + a_2 = a_3 + a_4$, $a_2 + a_3 = a_4 + a_5$ and $\sum_{i=1}^{5} a_i = 0$.

Taking one-parameter subgroups from $\mathfrak{h}$, we obtain the following actions which fix (x_1, x_2). (i) $\alpha_1(\lambda)$: $u_1 \mapsto u_1 + \lambda u_2, u_j \mapsto u_j \ (j \neq 1)$, (ii) $\alpha_2(\lambda)$: $u_5 \mapsto u_5 + \lambda u_4, u_j \mapsto u_j \ (j \neq 5)$, (iii) $\beta_1(\lambda)$: $u_1 \mapsto u_1 + \lambda u_4, u_3 \mapsto u_3 + \lambda u_2, u_j \mapsto u_j \ (j \neq 1, 3)$ (iv) $\beta_2(\lambda)$: $u_3 \mapsto u_3 + \lambda u_4, u_5 \mapsto u_5 + \lambda u_2, u_j \mapsto u_j \ (j \neq 3, 5)$ (v) $\gamma_1(\lambda)$: $u_2 \mapsto u_2 - \lambda u_4, u_3 \mapsto u_3 + \lambda u_1, u_5 \mapsto u_5 + 2\lambda u_3 + \lambda^2 u_1, u_j \mapsto u_j \ (j \neq 2, 3, 5)$ and $(x_1, x_2, x_3) \mapsto (x_1, \lambda x_1 + x_2, x_3)$, (vi) $\gamma_2(\lambda)$: $u_1 \mapsto u_1 - 2\lambda u_3 + \lambda^2 u_5, u_3 \mapsto u_3 - \lambda u_5, u_4 \mapsto u_4 + \lambda u_2, u_j \mapsto u_j \ (j \neq 1, 3, 4)$ and $(x_1, x_2, x_3) \mapsto (x_1 - \lambda x_2, x_2, x_3)$. We have also $\xi_1(\lambda)$ (resp. $\xi_2(\lambda)$): $(x_1, x_2, x_3) \mapsto (x_1, x_2, \lambda x_1 + x_3)$ (resp. $(x_1, x_2, \lambda x_2 + x_3)$) and $\eta(\mu)$: $(x_1, x_2, x_3) \to (x_1, x_2, \mu x_3)$ with $\mu \neq 0$. By using these actions, we shall do the orbital decomposition leaving (x_1, x_2) fixed. If at least one of a_{13}, a_{15} and a_{35} is not zero, then by γ_1, γ_2, ξ_1, ξ_2 and η, we may assume that $a_{13} = 1$, $a_{34} = a_{35} = a_{45} = 0$. (i) If $a_{15} \neq 0$, by α_1, β_1, ξ_1, ξ_2, β_2, α_2, γ_1 and η, we may assume that $x_3 = a_{24}u_2 \wedge u_4 + u_1 \wedge u_5$. If $a_{24} \neq 0$ (resp. $a_{24} = 0$), we have (1) (resp. (2)). (ii) If $a_{15} = 0$, by ξ_1, ξ_2, β_1, β_2 and α_1, we may assume that $x_3 = u_1 \wedge u_3 + a_{24}u_2 \wedge u_4 + a_{25}u_2 \wedge u_5$. If $a_{25} \neq 0$, then we have (1).

If $a_{25} = 0$, then it is reduced to previous cases. Finally, if $a_{13} = a_{15} = a_{35} = 0$, we may assume that $a_{15} = a_{25} = a_{35} = a_{45} = 0$ by the action of ξ_1, ξ_2 and γ_2. By considering (x_1, x_3) instead of (x_1, x_2), it is reduced to the previous cases. We shall see later, by calculating the isotropy subalgebras, that these orbits are different from each other.   Q.E.D.

(1) Put $x_0' = (3u_3 \wedge u_4 - u_2 \wedge u_5, u_1 \wedge u_5 - 2u_2 \wedge u_4, 3u_2 \wedge u_3 - u_1 \wedge u_4)$. Then the isotropy subalgebra $\mathfrak{g}_{x_0'}$ is the following standard form.

$$(11.1) \qquad \mathfrak{g}_{x_0'} = \left\{ \begin{bmatrix} 4\alpha & \beta & & & \\ 4\gamma & 2\alpha & 2\beta & & \\ & 3\gamma & 0 & 3\beta & \\ & & 2\gamma & -2\alpha & 4\beta \\ & & & \gamma & -4\alpha \end{bmatrix} \oplus \begin{pmatrix} 2\alpha & \beta & \\ 2\gamma & & 2\beta \\ & \gamma & -2\alpha \end{pmatrix} \right\}$$

$$\cong \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \right\} = \mathfrak{sl}(2).$$

Since $\Lambda_0 = V \times \{0\}$, we have $\mathrm{ord}_{V_{0,30}^{(0)}} f^s = 0$ where f denotes the relative invariant of degree 15 (See [1]).

(2) The isotropy subgroup at x_1 is locally isomorphic to $(GL(1) \times GL(1)) \cdot U(2)$ where $U(2)$ denotes a 2-dimensional unipotent group (See [1]). The conormal vector space $V_{x_1}^*$ is spanned by $(u_3 \wedge u_5, 0, 0) \in S_{21,1}^*$. We have $\dim \Lambda_0 \cap \Lambda_1 = \dim V - 1$; $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$ and $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$.

(3) The isotropy subalgebra $\mathfrak{g}_{x_2}$ at x_2 is given as follows.

$$(11.2) \qquad \mathfrak{g}_{x_2} = \left\{ A = \begin{bmatrix} \varepsilon & 0 & 0 & 0 & -\beta \\ \alpha & -2(\varepsilon+\eta) & \beta & 0 & 0 \\ 0 & 0 & \eta & 0 & -\alpha \\ 0 & 0 & -\beta & -2(\varepsilon+\eta) & \gamma \\ 0 & 0 & 0 & 0 & 3(\varepsilon+\eta) \end{bmatrix} \right.$$

$$\left. \oplus \begin{pmatrix} \varepsilon+2\eta & & \\ & 2\varepsilon+\eta & \\ -\beta & -\alpha & -(\varepsilon+\eta) \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(3) .$$

The conormal vector space $V_{x_2}^*$ is spanned by $v_1 = (u_3 \wedge u_5, 0, 0)$, and $v_2 = (0, u_1 \wedge u_5, 0)$, and the action $d\rho_{x_2}$ is given by

$$d\rho_{x_2}(A)(v_1, v_2) = (v_1, v_2) \begin{pmatrix} -(4\varepsilon+6\eta) & 0 \\ 0 & -(6\varepsilon+4\eta) \end{pmatrix}$$

i) $V_{x_2}^* - S_{x_2}^* \leftrightarrow v_1 + v_2 = (u_3 \wedge u_5, u_1 \wedge u_5, 0)$, i.e., $v_1 + v_2$ is a generic point of $V_{x_2}^*$, where $S_{x_2}^*$ is the singular set of the P.V. $(G_{x_2}, \rho_{x_2}, V_{x_2}^*)$. We use this notation from now in § 11. Put $y = y_1 v_1 + y_2 v_2$.

ii) $(S_{x_2}^*)_1 \leftrightarrow d\rho_1(A) = -(4\varepsilon + 6\eta) \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 = (0, u_1 \wedge u_5, 0) \in S_{21,1}^*$, i.e., $(S_{x_2}^*)_1 = \{y \in V_{x_2}^*; f_1^*(y) = 0\} = \overline{\rho_{v_2}(G_{x_2}) \cdot v_2}$ and $f_1^*(\rho_{x_2}(g)y) = \rho_1(g) f_1^*(y)$ for $y \in V_{x_2}^*$, $g \in G_{x_2}$. From now on, we use this notation in § 11.

iii) $(S_{x_2}^*)_2 \leftrightarrow d\rho_2(A) = -(6\varepsilon + 4\eta) \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 = (u_3 \wedge u_5, 0, 0) \in S_{21,1}^*$

iv) $-\delta\chi = d\rho_1 + d\rho_2$, $\mathrm{tr}_{V_{x_2}^*} = d\rho_1 + d\rho_2$

v) $\mathrm{ord}_{\Lambda_2} f^s = -2s - 2/2$.

Since the Hessian of the localization $f_{x_2}(z) = z_1 z_2$ $(z = z_1 v_1 + z_2 v_2 \in V_{x_2})$ of $f(x)$ is not identically zero, $\Lambda_2 = \Lambda_{2,16}^{(2)}$ is a good holonomic variety. We have $\dim \Lambda_1 \cap \Lambda_2 = \dim V - 1$ and $b_{\Lambda_2}(s)/b_{\Lambda_1}(s) = (s+1)$.

(4) The isotropy subalgebra $\mathfrak{g}_{x_3}$ is given as follows.

(11.3)

$$\mathfrak{g}_{x_3} = \left\{ A = \begin{pmatrix} \varepsilon & \alpha & 0 & 0 & \beta \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 \\ 0 & 0 & \gamma & \varepsilon & \beta \\ 0 & 0 & 0 & 0 & -(2\varepsilon+\eta+\xi) \end{pmatrix} \right.$$

$$\oplus \left. \begin{pmatrix} -(\varepsilon+\eta) & & \\ & -(\varepsilon+\xi) & \\ & & (\varepsilon+\eta+\xi) \end{pmatrix} \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(3) .$$

The conormal vector space $V_{x_3}^*$ is spanned by $v_1 = (u_3 \wedge u_5, 0, 0)$, $v_2 = (0, u_2 \wedge u_5, 0)$ $v_3 = (0, 0, u_2 \wedge u_3)$, and

$$d\rho_{x_3}(A)(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} 3\varepsilon+2\eta & & \\ & 3\varepsilon+2\xi & \\ & & -(\varepsilon+2\eta+2\xi) \end{pmatrix}$$

i) $V_{x_3}^* - S_{x_3}^* \leftrightarrow v_1 + v_2 + v_3 = (u_3 \wedge u_5, u_2 \wedge u_5, u_2 \wedge u_3) \in S_{15,3}^*$

ii) $(S_{x_3}^*)_1 \leftrightarrow d\rho_1(A) = 3\varepsilon + 2\eta \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 + v_3 = (0, u_2 \wedge u_5, u_2 \wedge u_3)$ $\in S_{16,2}^*$

iii) $(S_{x_3}^*)_2 \leftrightarrow d\rho_2(A) = 3\varepsilon + 2\xi \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 + v_3 = (u_3 \wedge u_5, 0, u_2 \wedge u_3)$ $\in S_{16,2}^*$

iv) $(S_{x_3}^*)_3 \leftrightarrow d\rho_3(A) = -(\varepsilon + 2\eta + 2\xi) \leftrightarrow f_3^*(y) = y_3 \leftrightarrow v_1 + v_2 = (u_3 \wedge u_5,$ $u_2 \wedge u_5, 0) \in S_{16,2}^*$

v) $-\delta\chi = d\rho_1 + d\rho_2 + d\rho_3$, $\mathrm{tr}_{V_{x_3}^*} = d\rho_1 + d\rho_2 + d\rho_3$.

Since the localization $f_{x_3}(z) = z_1 z_2 z_3$ $(z = \sum z_i v_i \in V_{x_3})$ is non-degenerate, $\Lambda_{3,15}$ is a good holonomic variety and $\operatorname{ord}_{\Lambda_{3,15}} f^s = -3s - \frac{3}{2}$. We have $\dim \Lambda_2 \cap \Lambda_{3,15} = \dim V - 1$ and $b_{\Lambda_{3,15}}(s)/b_{\Lambda_2}(s) = (s+1)$.

(5) The isotropy subalgebra $\mathfrak{g}_{x_3'}$ at x_3' is given as follows.

(11.4)
$$\mathfrak{g}_{x_3'} = \left\{ A = \begin{bmatrix} -2(\varepsilon+\eta) & \alpha & \beta & 0 & 0 \\ 0 & \varepsilon & \gamma & 0 & -2\alpha \\ 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & -\alpha & -(3\varepsilon+\eta) & \delta \\ 0 & 0 & 0 & 0 & 4\varepsilon+2\eta \end{bmatrix} \right.$$

$$\left. \oplus \begin{pmatrix} \varepsilon+2\eta & 0 & 0 \\ -\gamma & 2\varepsilon+\eta & 0 \\ \beta & -\alpha & -\varepsilon-\eta \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(4) .$$

The conormal vector space $V_{x_3'}^*$ is spanned by $v_1 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_2 = (u_3 \wedge u_4, 0, 0)$, $v_3 = (u_3 \wedge u_5, 0, 0)$, and

$$d\rho_{x_3'}(A)(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} -(6\varepsilon+4\eta) & 0 & 0 \\ 0 & 2(\varepsilon-\eta) & 0 \\ -2\gamma & -\delta & -5(\varepsilon+\eta) \end{pmatrix}$$

i) $V_{x_3'}^* - S_{x_3'}^* \leftrightarrow v_1 + v_2 = (u_2 \wedge u_5 + u_3 \wedge u_4, -u_3 \wedge u_5, 0) \in S_{13,3}^*$

ii) $(S_{x_3'}^*)_1 \leftrightarrow d\rho_1(A) = -(6\varepsilon + 4\eta) \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 \in S_{21,1}^*$

iii) $(S_{x_3'}^*)_2 \leftrightarrow d\rho_2(A) = 2(\varepsilon - \eta) \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 \in S_{16,2}^*$

iv) $-\delta\chi = 2d\rho_1 + d\rho_2 = -10(\varepsilon + \eta)$, $\operatorname{tr}_{V_{x_3}^*} = 2d\rho_1 + \frac{3}{2}d\rho_2 = -9\varepsilon - 11\eta$.

Since $\dim \Lambda_{3,13} \cap \Lambda_2 = \dim \Lambda_{3,13} \cap \Lambda_1 = \dim V - 1$ and they intersect G_0-prehomogeneously, $\Lambda_{3,13}$ is a good holonomic variety and $\operatorname{ord}_{\Lambda_{3,13}} f^s = -3s - \frac{4}{2}$. The intersection exponent of $\Lambda_{3,13}$ and Λ_1 is $(1:0)$. We have $b_{\Lambda_{3,13}}(s)/b_{\Lambda_1}(s) = (s+1)(s+\frac{3}{2})$ and $b_{\Lambda_{3,13}}(s)/b_{\Lambda_2}(s) = (s+\frac{3}{2})$.

(6) The isotropy subalgebra $\mathfrak{g}_{x_4}$ at x_4 is given as follows.

(11.5)
$$\mathfrak{g}_{x_4} = \left\{ A = \begin{bmatrix} \varepsilon & \alpha & \gamma & 0 & 0 \\ 0 & \eta & \beta & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 \\ 0 & 0 & -\alpha & \varepsilon-\eta+\xi & \delta \\ 0 & 0 & 0 & 0 & -2(\varepsilon+\xi) \end{bmatrix} \right.$$

$$\left. \oplus \begin{pmatrix} -(\varepsilon+\eta) & 0 & 0 \\ -\beta & -(\varepsilon+\xi) & 0 \\ 0 & 0 & (\varepsilon+\eta+\xi) \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(4) .$$

The conormal vector space $V_{x_4}^*$ is spanned by $v_1 = (u_3 \wedge u_4, 0, 0)$, $v_2 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_3 = (u_3 \wedge u_5, 0, u_2 \wedge u_3)$, $v_4 = (0, 0, u_2 \wedge u_3)$, and

$$d\rho_{x_4}(A)(v_1, \cdots, v_4) = (v_1, \cdots, v_4) \begin{bmatrix} 2(\eta-\varepsilon) & 0 & 0 & 0 \\ 0 & 3\varepsilon+2\eta & 0 & 0 \\ \alpha & -2\beta & 3\varepsilon+\eta+\xi & 0 \\ 0 & 0 & 0 & -(\varepsilon+2\eta+2\xi) \end{bmatrix}$$

i) $V_{x_4}^* - S_{x_4}^* \leftrightarrow v_1 + v_2 + v_4 = (u_3 \wedge u_4 + u_2 \wedge u_5, -u_3 \wedge u_5, u_2 \wedge u_3) \in S_{11,4}^*$

ii) $(S_{x_4}^*)_1 \leftrightarrow d\rho_1(A) = 2(\eta - \xi) \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 + v_4$
$\qquad = (u_2 \wedge u_5, -u_3 \wedge u_5, u_2 \wedge u_3) \in S_{15,3}^*$

iii) $(S_{x_4}^*)_2 \leftrightarrow d\rho_2(A) = 3\varepsilon + 2\xi \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 + v_4$
$\qquad = (u_3 \wedge u_4, 0, u_2 \wedge u_3) \in S_{16,2}^*$

iv) $(S_{x_4}^*)_3 \leftrightarrow d\rho_3(A) = -(\varepsilon + 2\eta + 2\xi) \leftrightarrow f_3^*(y) = y_4 \leftrightarrow v_1 + v_2$
$\qquad = (u_2 \wedge u_5 + u_3 \wedge u_4, -u_3 \wedge u_5, 0) \in S_{13,3}^*$

v) $-\delta\chi = d\rho_1 + 2d\rho_2 + d\rho_3$, $\mathrm{tr}_{V_{x_4}^*} = \frac{3}{2}d\rho_1 + 2d\rho_2 + d\rho_3$.

The conormal bundle Λ_4 is a good holonomic variety with $\mathrm{ord}_{\Lambda_4} f^s = -4s - \frac{5}{2}$. We have $b_{\Lambda_4}(s)/b_{\Lambda_2}(s) = (s+1)(s+\frac{3}{2})$, $b_{\Lambda_4}(s)/b_{\Lambda_{3,13}}(s) = (s+1)$ and $b_{\Lambda_4}(s)/b_{\Lambda_{3,15}}(s) = (s+\frac{3}{2})$. Note that these intersections are regular and G_0-prehomogeneous.

(7) The isotropy subalgebra $\mathfrak{g}_{x_5}$ at x_5 is given as follows.

$$\text{(11.6)} \quad \mathfrak{g}_{x_5} = \left\{ A = \begin{bmatrix} \varepsilon & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & 2\varepsilon+4\eta & \gamma_5 & 0 & \gamma_6 \\ 0 & 0 & \eta & 0 & \gamma_3+\gamma_5 \\ 0 & 0 & -\gamma_1 & -(\varepsilon+3\eta) & -\gamma_2 \\ 0 & 0 & 0 & 0 & -2(\varepsilon+\eta) \end{bmatrix} \right.$$

$$\left. \oplus \begin{pmatrix} -(3\varepsilon+4\eta) & 0 & 0 \\ \gamma_3-\gamma_5 & -(\varepsilon+\eta) & 0 \\ -\gamma_6 & -\gamma_5 & \varepsilon+2\eta \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(6) .$$

Then $V_{x_5}^*$ is spanned by $v_1 = (u_1 \wedge u_5 - u_3 \wedge u_4, u_4 \wedge u_5, 0)$, $v_2 = (u_2 \wedge u_3, -u_2 \wedge u_5, u_3 \wedge u_5)$, $v_3 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_4 = (u_4 \wedge u_5, 0, 0)$, $v_5 = (u_3 \wedge u_5, 0, 0)$, and

$$\text{(11.7)} \quad d\rho_{x_5}(A)(v_1, \cdots, v_5) = (v_1, \cdots, v_5) \begin{bmatrix} 4\varepsilon+6\eta & 0 & 0 & 0 & 0 \\ 0 & \varepsilon-\eta & 0 & 0 & 0 \\ -\gamma_1 & -\gamma_5 & 3\varepsilon+2\eta & 0 & 0 \\ -3\gamma_3 & 0 & 0 & 6\varepsilon+9\eta & 0 \\ -2\gamma_5 & 2\gamma_6 & \gamma_3-2\gamma_5 & \gamma_1 & 5(\varepsilon+\eta) \end{bmatrix}$$

i) $V_{x_5}^* - S_{x_5}^* \leftrightarrow v_1 + v_2 = (u_1 \wedge u_5 - u_3 \wedge u_4 + u_2 \wedge u_3, u_4 \wedge u_5$

$\qquad - u_2 \wedge u_5, u_3 \wedge u_5) \in S_{8,5}^*$

ii) $(S_{x_5}^*)_1 \leftrightarrow d\rho_1(A) = 4\varepsilon + 6\eta \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 + v_4 \in S_{11,4}^*$

iii) $(S_{x_5}^*)_2 \leftrightarrow d\rho_2(A) = \varepsilon - \eta \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 \in S_{13,3}^*$

iv) $- \delta\chi = 3d\rho_1 + 3d\rho_2 (= 15\varepsilon + 15\eta)$, $\mathrm{tr}_{V_{x_5}^*} = 4d\rho_1 + 3d\rho_2 (= 19\varepsilon + 21\eta)$.

The conormal bundle Λ_5 is a good holonomic variety with $\mathrm{ord}_{\Lambda_5} f^s = - 6s$ $- \frac{9}{2}$. We have $b_{\Lambda_5}(s)/b_{\Lambda_4}(s) = (s + \frac{4}{3})(s + \frac{5}{3})$ and $b_{\Lambda_5}(s)/b_{\Lambda_{3,13}}(s) = (s + 1)(s + \frac{4}{3})$ $(s + \frac{5}{3})$. Note that the intersection exponent of Λ_5 and Λ_4 is $(2 : 1)$. The intersection of Λ_5 and $\Lambda_{3,13}$ is regular and G_0-prehomogeneous.

(8) The isotropy subalgebra $\mathfrak{g}_{x_6}$ at x_6 is given as follows.

$$(11.8) \quad \mathfrak{g}_{x_6} = \left\{ \tilde{A} = \left(\begin{array}{c|cc|c} -4\varepsilon & \gamma_1 & \gamma_2 & 0 \\ \hline 0 & \varepsilon I_2 + A & & 0 \\ \hline 0 & & 0 & \varepsilon I_2 + B \end{array} \right) \oplus \left(\begin{array}{c|c} 3\varepsilon I_2 - {}^t A & 0 \\ \hline \gamma_2 - \gamma_1 & -2\varepsilon \end{array} \right); A, B \in \mathfrak{sl}(2) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(2) \,.$$

Then $V_{x_6}^*$ is spanned by $v_1 = (u_2 \wedge u_4, - u_3 \wedge u_4, 0)$, $v_2 = (0, u_2 \wedge u_4, 0)$, $v_3 = (u_3 \wedge u_4, 0, 0)$, $v_4 = (u_2 \wedge u_5, - u_3 \wedge u_5, 0)$, $v_5 = (0, u_2 \wedge u_5, 0)$, $v_6 = (u_3 \wedge u_5, 0, 0)$, and $(G_{x_6}, \rho_{x_6}, V_{x_6}^*) \cong (GL(1) \times SL(2) \times SL(2), 5\Lambda_1 \otimes 2\Lambda_1 \otimes \Lambda_1, V(1) \otimes V(3) \otimes V(2)) \cong (SO(3) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(2))$ and hence Λ_6 is a good holonomic variety.

i) $V_{x_6}^* - S_{x_6}^* \leftrightarrow v_2 + v_6 = (u_3 \wedge u_5, u_2 \wedge u_4, 0) \in S_{12,6}^*$

ii) $(S_{x_6}^*)_1 \leftrightarrow v_2 + v_4 = (u_2 \wedge u_5, u_2 \wedge u_4 - u_3 \wedge u_5, 0) \in S_{13,3}^*$

iii) $- \delta\chi = d\rho_1$, $\mathrm{tr}_{V_{x_6}^*} = \frac{3}{2}d\rho_1$

We have $\mathrm{ord}_{\Lambda_6} f^s = - 4s - \frac{6}{2}$ and $b_{\Lambda_6}(s)/b_{\Lambda_{3,13}}(s) = (s + \frac{3}{2})$.

(9) The isotropy subalgebra $\mathfrak{g}_{x_7}$ at x_7 is given as follows.

$$(11.9) \quad \mathfrak{g}_{x_7} = \left\{ \tilde{A} = \left(\begin{array}{c|cc|c} -2(\varepsilon+\eta) & F & & 0 \\ \hline 0 & \varepsilon I_2 + A & & 0 \\ \hline 0 & & 0 & \eta I_2 + B \end{array} \right) \oplus \left(\begin{array}{c|c} (\varepsilon+2\eta)I_2 - {}^t A & 0 \\ \hline 0 & -2\eta \end{array} \right); \right.$$

$$\left. A, B \in \mathfrak{sl}(2), {}^t F \in \boldsymbol{C}^2 \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(2) \,.$$

Then $V_{x_7}^*$ is spanned by $v_1 = (u_2 \wedge u_4, - u_3 \wedge u_4, 0)$, $v_2 = (0, u_2 \wedge u_4, 0)$, $v_3 = (u_3 \wedge u_4, 0, 0)$, $v_4 = (u_2 \wedge u_5, - u_3 \wedge u_5, 0)$, $v_5 = (0, u_2 \wedge u_5, 0)$, $v_6 = (u_3 \wedge u_5, 0, 0)$, $v_7 = (0, 0, u_2 \wedge u_3)$, and $(G_{x_7}, \rho_{x_7}, V_{x_7}^*) \cong (GL(1) \times GL(1) \times SL(2) \times SL(2), (2\Lambda_1^* \otimes 3\Lambda_1^* \otimes 2\Lambda_1 \otimes \Lambda_1) \oplus (2\Lambda_1^* \otimes 2\Lambda_1^* \otimes 1 \otimes 1), V(6) \oplus V(1))$.

i) $V_{x_7}^* - S_{x_7}^* \leftrightarrow v_2 + v_6 + v_7 = (u_3 \wedge u_5, u_2 \wedge u_4, u_2 \wedge u_3) \in S_{9,7}^*$

ii) $(S_{x_7}^*)_1 \leftrightarrow d\rho_1(\tilde{A}) = - 8\varepsilon - 12\eta \leftrightarrow f_1^*(y)(\deg f_1^* = 4) \leftrightarrow v_2 + v_4 + v_7 \in S_{11,4}^*$

iii) $(S_{x_7}^*)_2 \leftrightarrow d\rho_2(\tilde{A}) = -2\varepsilon + 2\eta \leftrightarrow f_2^*(y) = y_7 \leftrightarrow v_2 + v_6 \in S_{12,6}^*$

iv) $-\delta\chi = d\rho_1 + d\rho_2, \ \mathrm{tr}_{V_{x_7}^*} = \frac{3}{2}d\rho_1 + d\rho_2.$

Then by Corollary 1–7 conormal bundle $\Lambda_{7,9}$ is a good holonomic variety with $\mathrm{ord}_{\Lambda_{7,9}}f^s = -5s - \frac{7}{2}$. We have $\dim\Lambda_{7,9} \cap \Lambda_4 = \dim\Lambda_{7,9} \cap \Lambda_6 = \dim V - 1$, $b_{\Lambda_{7,9}}(s)/b_{\Lambda_4}(s) = (s + \frac{3}{2})$, and $b_{\Lambda_{7,9}}(s)/b_{\Lambda_6}(s) = (s + 1).$

(10) The isotropy subalgebra $\mathfrak{g}_{x_7'}$ at x_7' is given as follows.

(11.10)
$$\mathfrak{g}_{x_7'} = \left\{ A = \begin{bmatrix} \begin{array}{cc|ccc} 3\varepsilon+\alpha & \beta & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma & 3\varepsilon-\alpha & \gamma_4 & \gamma_5 & \gamma_6 \\ \hline & & -2\varepsilon & -2\gamma & -2\beta \\ \multicolumn{2}{c|}{\mathbf{0}} & -\beta & -2\varepsilon+2\alpha & 0 \\ & & -\gamma & 0 & -2\varepsilon-2\alpha \end{array} \end{bmatrix} \right.$$
$$\left. \oplus \begin{pmatrix} \begin{array}{c|cc} -6\varepsilon & \multicolumn{2}{c}{0} \\ \hline \gamma_2-\gamma_4 & -\varepsilon-\alpha & \gamma \\ \gamma_1-\gamma_6 & \beta & -\varepsilon+\alpha \end{array} \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus V(6) .$$

Then $V_{x_7'}^*$ is spanned by $v_1 = (u_1 \wedge u_4, 0, u_3 \wedge u_4)$, $v_2 = (u_1 \wedge u_3 - u_2 \wedge u_4, -u_3 \wedge u_4, -2u_4 \wedge u_5)$, $v_3 = (u_1 \wedge u_5 - u_2 \wedge u_3, 2u_4 \wedge u_5, -u_3 \wedge u_5)$, $v_4 = (-u_2 \wedge u_5, u_3 \wedge u_5, 0)$, $v_5 = (u_4 \wedge u_5, 0, 0)$, $v_6 = (u_3 \wedge u_4, 0, 0)$, $v_7 = (u_3 \wedge u_5, 0, 0)$ and the action $d\rho_{x_7'}$ of $\mathfrak{g}_{x_7'}$ on $V_{x_7'}^*$ is given by

$$d\rho_{x_7'}(A)(v_1, \cdots, v_7) = (v_1, \cdots, v_7)\left(\begin{array}{c|c} 5\varepsilon I_4 + A_1 & 0 \\ \hline C & 10\varepsilon I_3 + A_2 \end{array} \right)$$

where

$$(C, A_2) = \begin{pmatrix} \gamma_3 & 2\gamma_1-\gamma_6 & 3\gamma_2-2\gamma_4 & -\gamma_5 & 0 & -2\beta & 2\gamma \\ \gamma_6-2\gamma_1 & 2\gamma_2 & \gamma_5 & 0 & -\gamma & -2\alpha & 0 \\ 0 & \gamma_3 & 2\gamma_6 & \gamma_2-2\gamma_4 & \beta & 0 & 2\alpha \end{pmatrix}$$

and

$$A_1 = \begin{bmatrix} -3\alpha & 3\gamma & 0 & 0 \\ \beta & -\alpha & -2\gamma & 0 \\ 0 & -2\beta & \alpha & \gamma \\ 0 & 0 & 3\beta & 3\alpha \end{bmatrix}$$

i) $V_{x_7'}^* - S_{x_7'}^* \leftrightarrow v_1 + v_4 = (u_1 \wedge u_4 - u_2 \wedge u_5, u_3 \wedge u_5, u_3 \wedge u_4) \in S_{7,7}^{(2)*}$

ii) $(S_{x_7'}^*)_1 \leftrightarrow d\rho_1(A) = 20\varepsilon \leftrightarrow f_1^*(y_1, \cdots, y_4)$: the discriminant of binary cubic forms $\leftrightarrow v_2 \in S_{8,5}^*$

iii) $-\delta\chi = 2d\rho_1, \ \mathrm{tr}_{V_{x_7'}^*} = \frac{5}{2}d\rho_1.$

The conormal bundle $\Lambda_{7,7}^{(1)}$ is a good holonomic variety with $\mathrm{ord}_{V_{7,7}^{(1)}} f^s = -8s - \frac{13}{2}$. We have $\dim \Lambda_5 \cap \Lambda_{7,7}^{(1)} = \dim V - 1$ and $b_{\Lambda_{7,7}^{(1)}}(s)/b_{\Lambda_5}(s) = (s + \frac{5}{4})(s + \frac{7}{4})$. The intersection is regular and G_0-prehomogeneous.

(11) The isotropy subalgebra $\mathfrak{g}_{x_7''}$ at x_7'' is given as follows.

$$(11.11) \quad \mathfrak{g}_{x_7''} = \left\{ \tilde{A} = \left(\begin{array}{c|c|c} -2(\varepsilon+\eta) & 0 & C \\ \hline 0 & \varepsilon I_2 + A & D \\ \hline 0 & 0 & \eta I_2 - {}^t A \end{array} \right) \oplus \left(\begin{array}{c|c} (\varepsilon+2\eta)I_2 - {}^t A & 0 \\ \hline C & -(\varepsilon+\eta) \end{array} \right); \right.$$

$$\left. A \in \hat{\mathfrak{sl}}(2) \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \hat{\mathfrak{sl}}(2)) \oplus \mathfrak{u}(5) .$$

Then $V_{x_7''}^*$ is spanned by $v_1 = (u_3 \wedge u_4, 0, 0)$, $v_2 = (u_2 \wedge u_4 - u_3 \wedge u_5, -u_3 \wedge u_4, 0)$, $v_3 = (u_2 \wedge u_5, u_2 \wedge u_4 - u_3 \wedge u_5, 0)$, $v_4 = (0, u_2 \wedge u_5, 0)$, $v_5 = (u_1 \wedge u_5, -u_1 \wedge u_4, u_4 \wedge u_5)$, $v_6 = (u_4 \wedge u_5, 0, 0)$, $v_7 = (0, u_4 \wedge u_5, 0)$, and

$$d\rho_{x_7''}(\tilde{A})(v_1, \cdots, v_7) = (v_1, \cdots, v_7) \left(\begin{array}{c|c|c} -(2\varepsilon+3\eta)I_4 + 3\Lambda_1(A) & 0 & 0 \\ \hline 0 & \varepsilon - \eta & 0 \\ \hline * & * & -(\varepsilon+4\eta)I_2 + A \end{array} \right).$$

i) $V_{x_7''}^* - S_{x_7''}^* \leftrightarrow v_1 + v_4 + v_5 \in S_{7,7}^{(1)*}$

ii) $(S_{x_7''}^*)_1 \leftrightarrow d\rho_1(\tilde{A}) = -8\varepsilon - 12\eta \leftrightarrow f_1^*(y_1, \cdots, y_4)$: the discriminant of binary cubic forms $\leftrightarrow v_3 + v_4 + v_5 \in S_{8,5}^*$

iii) $(S_{x_7''}^*)_2 \leftrightarrow d\rho_2(\tilde{A}) = \varepsilon - \eta \leftrightarrow f_2^*(y) = y_5 \leftrightarrow v_1 + v_4 \in S_{12,6}^*$

iv) $-\delta\chi = d\rho_1 + 3d\rho_2$, $\mathrm{tr}_{V_{x_7''}} = \frac{3}{2}d\rho_1 + 3d\rho_2$.

The conormal bundle $\Lambda_{7,7}^{(2)}$ is a good holonomic variety with $\mathrm{ord}_{\Lambda_{7,7}^{(2)}} f^s = -7s - \frac{11}{2}$. We have $\dim \Lambda_5 \cap \Lambda_{7,7}^{(2)} = \dim \Lambda_6 \cap \Lambda_{7,7}^{(2)} = \dim V - 1$, $b_{\Lambda_{7,7}^{(2)}}(s)/b_{\Lambda_5}(s) = (s + \frac{3}{2})$ and $b_{\Lambda_{7,7}^{(2)}}(s)/b_{\Lambda_6}(s) = (s+1)(s + \frac{4}{3})(s + \frac{5}{3})$. The intersections are regular and G_0-prehomogeneous.

(12) We shall calculate the isotropy subalgebra at $\tilde{x}_8 = (u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, 0)$ instead of x_8.

$$(11.12) \quad \mathfrak{g}_{\tilde{x}_8} = \left\{ \tilde{A} = \left[\begin{array}{cc|ccc} 3\varepsilon+\alpha & \alpha_{12} & \gamma_1 & \gamma_2 & \gamma_3 \\ \alpha_{21} & 3\varepsilon-\alpha & \gamma_3 & \gamma_1 & \gamma_4 \\ \hline & & -2\varepsilon & -2\alpha_{12} & -2\alpha_{21} \\ & \mathbf{0} & -\alpha_{21} & -2\varepsilon-2\alpha & \\ & & -\alpha_{12} & & -2\varepsilon+2\alpha \end{array} \right] \right.$$

$$\left. \oplus \left(\begin{array}{cc|c} -\varepsilon+\alpha & \alpha_{12} & \gamma_5 \\ \alpha_{21} & -\varepsilon-\alpha & \alpha_6 \\ \hline & 0 & \eta \end{array} \right) \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \hat{\mathfrak{sl}}(2)) \oplus V(6) .$$

Then $V_{\tilde{x}_8}^*$ is spanned by $v_1 = (0, 0, u_1 \wedge u_2)$, $v_2 = (0, 0, u_1 \wedge u_5)$, $v_3 = (0, 0, u_1 \wedge u_3 - u_2 \wedge u_5)$, $v_4 = (0, 0, u_1 \wedge u_4 - u_2 \wedge u_3)$, $v_5 = (0, 0, u_2 \wedge u_4)$, $v_6 = (0, 0, u_4 \wedge u_5)$, $v_7 = (0, 0, u_3 \wedge u_5)$, $v_8 = (0, 0, u_3 \wedge u_4)$ and the action $d\rho_{\tilde{x}_8}$ of $\mathfrak{g}_{\tilde{x}_8}$ on $V_{\tilde{x}_8}^*$ is given by

$$d\rho_{\tilde{x}_8}(\tilde{A})(v_1, \cdots, v_8) = (v_1, \cdots, v_8)\left(\begin{array}{c|c|c} -6\varepsilon - \eta & 0 & 0 \\ \hline C_1 & -(\varepsilon + \eta)I_4 + A_1 & 0 \\ \hline 0 & C_2 & (4\varepsilon - \eta)I_3 + A_2 \end{array}\right)\begin{array}{l} \}1 \\ \\ \}4 \\ \\ \}3 \end{array}$$

where

$$(C_2, A_2) = \begin{pmatrix} -\gamma_2 & \gamma_1 & \gamma_3 & \gamma_4 & 0 & 2\alpha_{12} & -2\alpha_{21} \\ -\gamma_1 & 2\gamma_3 & -\gamma_4 & 0 & \alpha_{21} & -2\alpha & 0 \\ 0 & \gamma_2 & -2\gamma_1 & -\gamma_3 & -\alpha_{12} & 0 & 2\alpha \end{pmatrix}$$

and

$$(C_1, A_1) = \begin{bmatrix} -\gamma_4 & -3\alpha & 3\alpha_{21} & 0 & 0 \\ -\gamma_3 & \alpha_{12} & -\alpha & 2\alpha_{21} & 0 \\ -\gamma_1 & 0 & 2\alpha_{12} & \alpha & -\alpha_{21} \\ \gamma_2 & 0 & 0 & -3\alpha_{12} & 3\alpha \end{bmatrix}.$$

i) $V_{\tilde{x}_8}^* - S_{\tilde{x}_8}^* \leftrightarrow v_1 + v_6 = (0, 0, u_1 \wedge u_2 + u_4 \wedge u_5) \in S_{18,8}^*$

ii) $(S_{\tilde{x}_8}^*)_1 \leftrightarrow d\rho_1(\tilde{A}) = -4\varepsilon - 4\eta \leftrightarrow f_1^*(y_2, \cdots, y_5)$: the discriminant of binary cubic forms $\leftrightarrow v_1 + v_7 = (0, 0, u_1 \wedge u_2 + u_3 \wedge u_5) \in S_{18,8}^*$

iii) $(S_{\tilde{x}_8}^*)_2 \leftrightarrow d\rho_2(\tilde{A}) = -6\varepsilon - \eta \leftrightarrow f_2^*(y) = y_1 \leftrightarrow v_2 + v_5 = (0, 0, u_1 \wedge u_5 + u_2 \wedge u_4) \in S_{18,8}^*$

iv) $-\delta\chi = 10\varepsilon - 5\eta = -3d\rho_1 + 2d\rho_2$, $\mathrm{tr}_{V_{\tilde{x}_8}^*} = 2\varepsilon - 8\eta = 2d\rho_1 + \frac{5}{2}d\rho_2$.

The conormal bundle $\Lambda_{8,18}$ is a good holonomic variety with $\mathrm{ord}_{\Lambda_{8,18}}f^s = -5s - \frac{8}{2}$. The conormal vector space $(G_{x_8}, \rho_{x_8}, V_{x_8}^*)$ is a regular P.V. In fact, for $z = \sum_{i=1}^8 z_i v_i \in V_{\tilde{x}_8}$, the localization $f_{\tilde{x}_8}(z)$ of $f(x)$ is given by $f_{\tilde{x}_8}(z) = z_1 z_6^4 + z_1 z_7^2 z_8^2 + 2z_1 z_6^2 z_7 z_8 + z_3^2 z_8^3 + z_3^2 z_6^2 z_8 + 2z_2 z_3 z_6 z_8^2 + z_2 z_5 z_6^3 + z_2 z_4 z_6^2 z_8 + z_3 z_4 z_6^3 + 3z_2 z_5 z_6 z_7 z_8 + z_3 z_5 z_6^2 z_7 - z_2 z_4 z_7 z_8^2 - z_3 z_4 z_6 z_7 z_8 - z_3 z_5 z_7^2 z_8 - z_5^2 z_7^3 - z_4^2 z_6^2 z_7 - 2z_4 z_5 z_6 z_7^2$, and hence its Hessian is not identically zero. Now we shall show that $\dim \Lambda_{7,7}^{(1)} \cap \Lambda_{8,18} = \dim V - 1$. From iii) above, $\Lambda = \overline{G(\tilde{x}_8, v_2 + v_5)}$ is one-codimensional and $\Lambda \subset \Lambda_{8,18}$. It is enough to show $(\tilde{x}_8, v_2 + v_5) = \{(u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, 0), (0, 0, u_1 \wedge u_5 + u_2 \wedge u_4)\} \in \Lambda_{7,7}^{(1)}$. Put $z = \{(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4, u_2 \wedge u_3 + u_1 \wedge u_5), (u_1 \wedge u_4 - u_2 \wedge u_5, u_3 \wedge u_5, u_3 \wedge u_4)\}$. Then $z \in \Lambda_{7,7}^{(1)}$ (See (10)). Then for $\varepsilon > 0$, put

$$g_\varepsilon = \begin{bmatrix} -\varepsilon^3 & & & & \\ & \varepsilon^3 & & & \\ & & \varepsilon^{-2} & & \\ \hline & & & -\varepsilon^{-2} & \\ & & & & -\varepsilon^{-2} \end{bmatrix} \times \begin{pmatrix} & & \varepsilon^{-1} \\ & -\varepsilon^{-1} & \\ \varepsilon^{-1} & & \end{pmatrix} \in G = SL(5) \times GL(3).$$

Then $g_\varepsilon \cdot z = \{(u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, -\varepsilon^5 u_1 \wedge u_2), (-\varepsilon^5 u_3 \wedge u_5, \varepsilon^5 u_3 \wedge u_4, u_1 \wedge u_5 + u_2 \wedge u_4)\}$. Since $g_\varepsilon \cdot z \in \Lambda_{7,7}^{(1)}$ and $\Lambda_{7,7}^{(1)}$ is closed, we have $(\tilde{x}_8, v_2 + v_5) = \lim_{\varepsilon \to 0} g_\varepsilon \cdot z \in \Lambda_{7,7}^{(1)}$ and hence $\dim \Lambda_{7,7}^{(1)} \cap \Lambda_{8,18} = \dim V - 1$. One can see easily that their intersection is regular and G_0-prehomogeneous. We have $b_{\Lambda_{7,7}^{(1)}}(s)/b_{\Lambda_{8,18}}(s) = (s+1)(s+\frac{4}{3})(s+\frac{5}{3})$. Next we shall show that $\dim \Lambda_{8,18} \cap \Lambda_{3,13} = \dim V - 1$. From ii) above, $\Lambda = \overline{G(\tilde{x}_8, v_1 + v_7)}$ is one-codimensional and $\Lambda \subset \Lambda_{8,18}$, where $(\tilde{x}_8, v_1 + v_7) = \{(u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, 0), (0, 0, u_1 \wedge u_2 + u_3 \wedge u_5)\}$. Put $w = \{(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5), (u_2 \wedge u_5 + u_3 \wedge u_4, -u_3 \wedge u_5, 0)\}$. Then $w \in \Lambda_{3,13}$ (See (5)). For $\varepsilon > 0$, put

$$
g_\varepsilon = \begin{bmatrix} \rule[0.5ex]{1em}{0.4pt} & & & -\sqrt{-1}\varepsilon^{-4} & \\ & & & & \varepsilon \\ & -\sqrt{-1}\varepsilon & & & \\ \varepsilon^6 & & & & \\ & & & & \varepsilon^{-4} \end{bmatrix} \times \begin{pmatrix} 0 & -\sqrt{-1}\varepsilon^{-2} & 0 \\ 0 & 0 & \varepsilon^3 \\ \sqrt{-1}\varepsilon^3 & 0 & 0 \end{pmatrix}
$$

$$\in G = SL(5) \times GL(3).$$

Then $g_\varepsilon \cdot w = \{(u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, -\varepsilon^{10} u_3 \wedge u_4), (\varepsilon^{10} u_3 \wedge u_5, 0, u_1 \wedge u_2 + u_3 \wedge u_5)\}$. Since $g_\varepsilon \cdot w \in \Lambda_{3,13}$ and $\Lambda_{3,13}$ is closed, we have $(\tilde{x}_8, v_1 + v_7) = \lim_{\varepsilon \to 0} g_\varepsilon \cdot w \in \Lambda_{3,13}$, i.e., $\Lambda \subset \Lambda_{3,13} \cap \Lambda_{8,18}$. Hence we have $\dim \Lambda_{3,13} \cap \Lambda_{8,18} = \dim V - 1$. The intersection is G_0-prehomogeneous and regular, and hence we have $b_{\Lambda_{8,18}}(s)/b_{\Lambda_{3,13}}(s) = (s+\frac{5}{4})(s+\frac{7}{4})$.

(13) The isotropy subalgebra $\mathfrak{g}_{x_8'}$ at x_8' is given as follows.

$$
(11.13) \quad \mathfrak{g}_{x_8'} = \left\{ A = \begin{bmatrix} \varepsilon+\alpha+\beta & \alpha_{12} & & \beta_{12} & \gamma_1 \\ \alpha_{21} & \varepsilon-\alpha+\beta & -\beta_{12} & & \gamma_2 \\ & \beta_{21} & \varepsilon-\alpha-\beta & & \gamma_3 \\ -\beta_{21} & & & \varepsilon+\alpha-\beta & \gamma_4 \\ & & 0 & & -4\varepsilon \end{bmatrix} \right.
$$

$$
\left. \oplus \begin{pmatrix} -2\varepsilon-2\beta & & -\beta_{21} \\ & -2\varepsilon+2\beta & \beta_{12} \\ 2\beta_{12} & -2\beta_{21} & -2\varepsilon \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(4).
$$

Then $V_{x_8'}^*$ is spanned by $v_1 = (u_3 \wedge u_5, 0, 0)$, $v_2 = (u_2 \wedge u_5, 0, -u_3 \wedge u_5)$, $v_3 = (0, u_3 \wedge u_5, -u_2 \wedge u_5)$, $v_4 = (0, u_2 \wedge u_5, 0)$, $v_5 = (u_4 \wedge u_5, 0, 0)$, $v_6 = (u_1 \wedge u_5, 0, u_4 \wedge u_5)$, $v_7 = (0, u_4 \wedge u_5, u_1 \wedge u_5)$, $v_8 = (0, u_1 \wedge u_5, 0)$. We have $(G_{x_8'}, \rho_{x_8'}, V_{x_8'}^*) \cong (GL(1) \times SL(2) \times SL(2), 5\Lambda_1 \otimes 3\Lambda_1 \otimes \Lambda_1, V(1) \otimes V(4) \otimes V(2))$. Since $\dim \rho_{x_8'}(G_{x_8'}) = 7 < \dim V_{x_8'}^* = 8$, this is not a P.V. The dual of the orbit

$S_{8,14}$ is $S_{14,8}^*$ in V^*, i.e., $\Lambda_{8,14} = \Lambda_{14,8}^*$.

(14) The isotropy subalgebra $\mathfrak{g}_{x_8''}$ at x_8'' is given as follows.

(11.14)
$$\mathfrak{g}_{x_8''} = \left\{ A = \begin{bmatrix} \varepsilon & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & \eta & \gamma_5 & \gamma_6 & \gamma_7 \\ 0 & 0 & -2(\eta+\xi) & 0 & \gamma_8 \\ 0 & 0 & 0 & \xi & -\gamma_1 \\ 0 & 0 & 0 & 0 & -\varepsilon+\eta+\xi \end{bmatrix} \right.$$

$$\oplus \begin{pmatrix} -(\varepsilon+\eta) & 0 & 0 \\ -\gamma_5 & -\varepsilon+2\eta+2\xi & 0 \\ \gamma_3-\gamma_7 & -\gamma_8 & -\eta-\xi \end{pmatrix} \left. \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(8) .$$

Then $V_{x_8''}^*$ is spanned by $v_1 = (u_1 \wedge u_5 - u_2 \wedge u_4, 2u_3 \wedge u_4, u_4 \wedge u_5)$, $v_2 = (0, u_2 \wedge u_5, 0)$, $v_3 = (u_2 \wedge u_3, 0, u_3 \wedge u_5)$, $v_4 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_5 = (0, u_4 \wedge u_5, 0)$, $v_6 = (u_3 \wedge u_4, 0, 0)$, $v_7 = (u_4 \wedge u_5, 0, 0)$, $v_8 = (u_3 \wedge u_5, 0, 0)$.

The action $d\rho_{x_8''}$ of $\mathfrak{g}_{x_8''}$ on $V_{x_8''}^*$ is given by

$$d\rho_{x_8''}(A)(v_1, \cdots, v_8) = (v_1, \cdots, v_8) \left[\begin{array}{cccc|cccc} A_1 & & & & & & & \\ & A_2 & & & & & & \\ & & A_3 & & & & 0 & \\ -2\gamma_1 & \gamma_5 & -\gamma_8 & A_4 & & & & \\ \hline 3\gamma_8 & -\gamma_6 & & & A_5 & & & \\ 3\gamma_5 & & \gamma_6 & & & A_6 & & \\ -2\gamma_3 & & & -\gamma_6 & \gamma_5 & \gamma_8 & A_7 & \\ -\gamma_2 & & 2\gamma_7-\gamma_3 & -2\gamma_5 & & \gamma_1 & & A_8 \end{array} \right]$$

where $A_1 = \varepsilon - \xi$, $A_2 = 2\varepsilon - 4\eta - 3\xi$, $A_3 = \varepsilon + 2\eta + 2\xi$, $A_4 = 2\varepsilon - \eta - \xi$, $A_5 = 2\varepsilon - 3\eta - 4\xi$, $A_6 = \varepsilon + 3\eta + \xi$, $A_7 = 2\varepsilon - 2\xi$ and $A_8 = 2\varepsilon + 2\eta + \xi$.

i) $V_{x_8''}^* - S_{x_8''}^* \leftrightarrow v_1 + v_2 + v_3 \in S_{5,8}^*$

ii) $(S_{x_8''}^*)_1 \leftrightarrow d\rho_1(A) = \varepsilon - \xi \leftrightarrow f_1^*(y) = y_1(y = \sum y_i v_i) \leftrightarrow v_2 + v_3 + v_5 \in S_{9,7}^*$

iii) $(S_{x_8''}^*)_2 \leftrightarrow d\rho_2(A) = 2\varepsilon - 4\eta - 3\xi \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 + v_3 \in S_{7,7}^{(2)*}$

iv) $(S_{x_8''}^*)_3 \leftrightarrow d\rho_3(A) = \varepsilon + 2\eta + 2\xi \leftrightarrow f_3^*(y) = y_3 \leftrightarrow v_1 + v_2 \in S_{7,7}^{(1)*}$

v) $-\delta\chi = 6d\rho_1 + d\rho_2 + 2d\rho_3$, $\operatorname{tr}_{V_{x_8''}^*} = \frac{15}{2}d\rho_1 + \frac{3}{2}d\rho_2 + \frac{5}{2}d\rho_3$.

Since the intersection of $\Lambda_{8,5}^{(3)}$ and $\Lambda_{7,7}^{(1)}$ is G_0-prehomogeneous, the conormal bundle $\Lambda_{8,5}^{(3)}$ is a good holonomic variety by Proposition 1–5. The order is given by $\operatorname{ord}_{\Lambda_{8,5}} f^s = -9s - \frac{15}{2}$. We have $\dim \Lambda_{8,5}^{(3)} \cap \Lambda_{7,9} = \dim \Lambda_{8,5}^{(3)} \cap \Lambda_{7,7}^{(i)} = \dim V - 1$ $(i = 1, 2)$, $b_{\Lambda_{8,5}^{(3)}}(s)/b_{\Lambda_{7,7}^{(1)}}(s) = s + \frac{3}{2}$ and $b_{\Lambda_{8,5}^{(3)}}(s)/b_{\Lambda_{7,7}^{(2)}}(s) =$

$(s + \frac{5}{4})(s + \frac{7}{4})$. In (15), we shall prove that $\dim \varLambda^{(3)}_{8,5} \cap \varLambda^{(2)}_{7,9} \cap \varLambda^{(2)}_{5,8} = \dim \varLambda^{(3)}_{8,5}$ $\cap \varLambda^{(4)}_{9,7} \cap \varLambda^{(2)}_{5,8} = \dim V - 1$.

(15) The isotropy subalgebra $\mathfrak{g}_{x_9}$ at x_9 is given as follows.

$$\mathfrak{g}_{x_9} = \left\{ A = \begin{bmatrix} \varepsilon_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & \varepsilon_2 & \gamma_4 & \gamma_5 & \gamma_6 \\ 0 & 0 & \varepsilon_3 & 0 & \gamma_7 \\ 0 & 0 & 0 & \varepsilon_4 & \gamma_8 \\ 0 & 0 & 0 & 0 & -(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4) \end{bmatrix} \right.$$

(11.15)

$$\left. \oplus \begin{pmatrix} -\varepsilon_1-\varepsilon_2 & 0 & 0 \\ -\gamma_4 & -\varepsilon_1-\varepsilon_3 & 0 \\ \gamma_2 & 0 & -\varepsilon_2-\varepsilon_4 \end{pmatrix} \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(8) \ .$$

Then $V^*_{x_9}$ is spanned by $v_1 = (0, u_2 \wedge u_5, 0)$, $v_2 = (0, 0, u_1 \wedge u_5)$, $v_3 = (u_2 \wedge u_5, - u_3 \wedge u_5, 0)$, $v_4 = (u_1 \wedge u_5, 0, u_4 \wedge u_5)$, $v_5 = (0, u_4 \wedge u_5, 0)$, $v_6 = (0, 0, u_3 \wedge u_5)$, $v_7 = (u_3 \wedge u_4, 0, 0)$, $v_8 = (u_3 \wedge u_5, 0, 0)$, $v_9 = (u_4 \wedge u_5, 0, 0)$.

The action $d\rho_{x_9}$ of $\mathfrak{g}_{x_9}$ on $V^*_{x_9}$ is given by

$$d\rho_{x_9}(A)(v_1, \cdots, v_9) = (v_1, \cdots, v_9) \begin{bmatrix} A_1 & & & & & & & & \\ & A_2 & & & & & & & \\ \gamma_4 & & A_3 & & & & & & \\ & -\gamma_2 & & A_4 & & & & & \\ -\gamma_5 & & & & A_5 & & & & \\ & -\gamma_1 & & & & A_6 & & & \\ & & & & & & A_7 & & \\ & & -2\gamma_4 & -\gamma_1 & & -\gamma_2 & -\gamma_8 & A_8 & \\ & & -\gamma_5 & -2\gamma_2 & \gamma_4 & & \gamma_7 & & A_9 \end{bmatrix}$$

where $A_1 = 2\varepsilon_1 + 2\varepsilon_3 + \varepsilon_4$, $A_2 = 2\varepsilon_2 + \varepsilon_3 + 2\varepsilon_4$, $A_3 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, $A_4 = \varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4$, $A_5 = 2\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3$, $A_6 = \varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_4$, $A_7 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$, $A_8 = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_4$ and $A_9 = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$.

 i) $V^*_{x_9} - S^*_{x_9} \leftrightarrow v_1 + v_2 + v_7 = (u_3 \wedge u_4, u_2 \wedge u_5, u_1 \wedge u_5) \in S^*_{7,9}$

 ii) $(S^*_{x_9})_1 \leftrightarrow d\rho_1(A) = 2\varepsilon_1 + 2\varepsilon_3 + \varepsilon_4 \leftrightarrow f^*_1(y) = y_1 \leftrightarrow v_2 + v_3 + v_5 + v_7 \in S^*_{8,5}$

 iii) $(S^*_{x_9})_2 \leftrightarrow d\rho_2(A) = 2\varepsilon_2 + \varepsilon_3 + 2\varepsilon_4 \leftrightarrow f^*_2(y) = y_2 \leftrightarrow v_1 + v_4 + v_6 + v_7 \in S^*_{8,5}$

 iv) $(S^*_{x_9})_3 \leftrightarrow d\rho_3(A) = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \leftrightarrow f^*_3(y) = y_7 \leftrightarrow v_1 + v_2 + v_8 + v_9 \in S^*_{14,8}$

 v) $-\delta\chi = 3d\rho_1 + 3d\rho_2 + 4d\rho_3$, $\mathrm{tr}_{V^*_{x_9}} = 4d\rho_1 + 4d\rho_2 + 5d\rho_3$.

The conormal bundle $\varLambda_{9,7}$ is a good holonomic variety with $\mathrm{ord}_{\varLambda_{9,7}} f^s = -10s - \frac{17}{2}$.

Put $p = (u_1 \wedge u_2, u_1 \wedge u_3, u_2 \wedge u_4; u_1 \wedge u_5 + u_3 \wedge u_4, u_2 \wedge u_5, u_3 \wedge u_5 + u_4 \wedge u_5)$. Then by iii), we have $p \in \varLambda_{9,7}^{(4)} \cap \varLambda_{5,8}^{(2)}$ and $\dim G \cdot p = \dim V - 1$. We shall prove that $p \in \varLambda_{8,5}^{(3)}$, i.e., $\dim \varLambda_{5,8}^{(2)} \cap \varLambda_{8,5}^{(3)} \cap \varLambda_{9,7}^{(4)} = \dim V - 1$. By (14), for any $\varepsilon > 0$, we have $(u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_5 + u_2 \wedge u_4; u_1 \wedge u_5 - u_2 \wedge u_4 + (1/\varepsilon)u_2 \wedge u_3 + (1/\varepsilon^2)u_3 \wedge u_4, 2u_3 \wedge u_4 + u_2 \wedge u_5, u_4 \wedge u_5 + (1/\varepsilon)u_3 \wedge u_5) \in$

$(x_8'', V_{x_8''}^*) \subset \varLambda_{8,5}^{(3)}$. Therefore, by the action of $g_\varepsilon = \begin{bmatrix} \varepsilon & & & \\ & 1 & & \\ & & \varepsilon^{-1} & \\ & & & 1 \\ & & & & 1 \end{bmatrix} \times \begin{pmatrix} \varepsilon^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$\in G = SL(5) \times GL(3)$, we have $p_\varepsilon = (u_1 \wedge u_2, u_1 \wedge u_3, \varepsilon u_1 \wedge u_5 + u_2 \wedge u_4; u_1 \wedge u_5 - \varepsilon u_2 \wedge u_4 + \varepsilon u_2 \wedge u_3 + u_3 \wedge u_4, 2\varepsilon u_3 \wedge u_4 + u_2 \wedge u_5, u_3 \wedge u_5 + u_4 \wedge u_5)$ $\in \varLambda_{8,5}^{(3)}$. Hence, we have $p = \lim_{\varepsilon \to 0} p_\varepsilon \in \varLambda_{8,5}^{(3)}$. Since $\varLambda_{5,8}^{(2)} = \varLambda_{8,5}^{(3)*}$, $\varLambda_{8,5}^{(3)} = \varLambda_{5,8}^{(2)*}$, $\varLambda_{9,7}^{(4)} = \varLambda_{7,9}^{(2)*}$ and $(G, \rho, V) \cong (G, \rho^*, V^*)$, we have also $\dim \varLambda_{5,8}^{(2)} \cap \varLambda_{8,5}^{(3)} \cap \varLambda_{7,9}^{(2)} = \dim V - 1$.

(16) The isotropy subalgebra $\mathfrak{g}_{x_{10}}$ at x_{10} is given as follows.

$$(11.16) \quad \mathfrak{g}_{x_{10}} = \left\{ A = \left[\begin{array}{cc|cc|c} \eta & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ & -4\varepsilon & 0 & 0 & \gamma_5 \\ \hline & & \varepsilon+\alpha & \alpha_{12} & \gamma_3 \\ \multicolumn{1}{c}{0} & & \alpha_{21} & \varepsilon-\alpha & -\gamma_2 \\ \hline 0 & & 0 & & 2\varepsilon-\eta \end{array} \right] \oplus \left(\begin{array}{cc|c} 4\varepsilon-\eta & 0 & \gamma_6 \\ -\gamma_5 & -2\varepsilon & \gamma_7 \\ \hline 0 & & \xi \end{array} \right) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(7) .$$

Then $V_{x_{10}}^*$ is spanned by $v_1 = (0, 0, u_1 \wedge u_3)$, $v_2 = (0, 0, u_1 \wedge u_4)$, $v_3 = (0, 0, u_1 \wedge u_5 - u_3 \wedge u_4)$, $v_4 = (0, 0, u_2 \wedge u_3)$, $v_5 = (0, 0, u_2 \wedge u_4)$, $v_6 = (u_3 \wedge u_5, 0, 0)$, $v_7 = (u_4 \wedge u_5, 0, 0)$, $v_8 = (0, 0, u_3 \wedge u_5)$, $v_9 = (0, 0, u_4 \wedge u_5)$, $v_{10} = (0, 0, u_2 \wedge u_5)$.

$$V_{x_{10}}^* - S_{x_{10}}^* \leftrightarrow v_1 + v_5 + v_7 = (u_4 \wedge u_5, 0, u_1 \wedge u_3 + u_2 \wedge u_4) = y_{10}.$$

Let A_0 be an element of $\mathfrak{g}_{x_{10}}$ with $\alpha = -\frac{1}{3} - 5\varepsilon$, $\eta = \frac{2}{3} + 6\varepsilon$, $\xi = -\frac{4}{3} - 2\varepsilon$, all remaining parts zero in (5.16). Then $d\rho(A_0)x_{10} = 0$ and $d\rho^*(A_0)y_{10} = y_{10}$. Since $-\delta\chi(A_0) = 10(1 + 3\varepsilon)$ is not definite, the conormal bundle $\varLambda_{10,10}$ is *not* a good holonomic variety.

(17) The isotropy subalgebra $\mathfrak{g}_{x_{11}}$ at x_{11} is given as follows.

$$(11.17) \quad \mathfrak{g}_{x_{11}} = \left\{ A = \left[\begin{array}{c|cc|c|c} \varepsilon+\eta & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \hline & \varepsilon+\alpha & \alpha_{12} & \gamma_5 & \gamma_6 \\ \multicolumn{1}{c|}{0} & \alpha_{21} & \varepsilon-\alpha & \gamma_7 & \gamma_8 \\ \hline & & & \varepsilon-\eta & \gamma_9 \\ 0 & \multicolumn{2}{c|}{0} & n_0 & -4\varepsilon \end{array} \right] \oplus \left(\begin{array}{c|c|c} -2\varepsilon-\eta-\alpha & -\alpha_{21} & 0 \\ -\alpha_{12} & -2\varepsilon-\eta+\alpha & \\ \hline \gamma_2-\gamma_5 & -\gamma_1-\gamma_7 & -2\varepsilon \end{array} \right) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(9) .$$

The conormal vector space $V_{x_{11}}^*$ is spanned by $v_1 = (0, u_1 \wedge u_5, -u_2 \wedge u_5)$, $v_2 = (u_1 \wedge u_5, 0, u_3 \wedge u_5)$, $v_3 = (u_2 \wedge u_5, 0, -u_4 \wedge u_5)$, $v_4 = (u_2 \wedge u_4, -u_3 \wedge u_4, 0)$, $v_5 = (0, u_2 \wedge u_4, 0)$, $v_6 = (u_3 \wedge u_4, 0, 0)$, $v_7 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_8 = (0, u_2 \wedge u_5, 0)$, $v_9 = (u_3 \wedge u_5, 0, 0)$, $v_{10} = (0, u_4 \wedge u_5, 0)$, $v_{11} = (u_4 \wedge u_5, 0, 0)$.

The action $d\rho_{x_{11}}$ of $\mathfrak{g}_{x_{11}}$ on $V_{x_{11}}^*$ is given by

$$
d\rho_{x_{11}}(A)(v_1, \cdots, v_{11})
\begin{bmatrix}
A_1 & & & & \\
B_1 & A_2 & & & \\
& & A_3 & & \\
B_2 & A_6 & B_3 & A_4 & \\
B_4 & B_5 & B_6 & B_7 & A_5
\end{bmatrix}
\begin{matrix} \}\,2 \\ \}\,1 \\ \}\,3 \\ \}\,3 \\ \}\,2 \end{matrix}
$$
$$
\underbrace{}_{2}\ \underbrace{}_{1}\ \underbrace{}_{3}\ \underbrace{}_{3}\ \underbrace{}_{2}
$$

where $(B_1, A_2) = (-\gamma_5, \gamma_7, 5\varepsilon + \eta)$, $B_3 = -\gamma_9 \cdot I_3$, $B_4 = -\gamma_3 \cdot I_2$, $A_1 = 5\varepsilon I_2 + A'$, $A_5 = (5\varepsilon + 2\eta)I_2 + A'$ with $A' = \begin{pmatrix} -\alpha & \alpha_{21} \\ \alpha_{12} & \alpha \end{pmatrix}$,

$$
(B_5, B_6, B_7) = \begin{pmatrix} -\gamma_1 - \gamma_7 & -\gamma_8 & \gamma_6 & 0 & -\gamma_7 & -\gamma_5 & 0 \\ \gamma_2 - 2\gamma_5 & \gamma_6 & 0 & \gamma_8 & -\gamma_5 & 0 & -\gamma_7 \end{pmatrix},
$$

$A_3 = 2\eta \cdot I_3 + A''$, $A_4 = (5\varepsilon + \eta)I_3 + A''$ with $A'' = \begin{pmatrix} 0 & \alpha_{12} & -\alpha_{21} \\ 2\alpha_{21} & -2\alpha & 0 \\ -2\alpha_{12} & 0 & 2\alpha \end{pmatrix}$,

$$
(B_2, A_6) = \begin{pmatrix} -\gamma_2 & -\gamma_1 - \gamma_7 & \\ -2\gamma_1 - \gamma_7 & & \alpha_{21} \\ 0 & \gamma_5 - 2\gamma_2 & -\alpha_{12} \end{pmatrix}.
$$

Note that A_6 will disappear if we take $v_3 - \frac{1}{2}v_7$ instead of v_3.

i) $V_{x_{11}}^* - S_{x_{11}}^* \leftrightarrow v_1 + v_2 + v_4 = (u_1 \wedge u_5 + u_2 \wedge u_4, u_1 \wedge u_5 - u_3 \wedge u_4, u_3 \wedge u_5 - u_2 \wedge u_5) \in S_{4,11}^{(3)*}$

ii) $(S_{x_{11}}^*)_1 \leftrightarrow v_1 + v_4 + v_9 = (u_2 \wedge u_4 + u_3 \wedge u_5, u_1 \wedge u_5 - u_3 \wedge u_4, -u_2 \wedge u_5) \in S_{5,8}^{(2)*} \leftrightarrow d\rho_1 = 10\varepsilon + 2\eta$

iii) $(S_{x_{11}}^*)_2 \leftrightarrow v_2 + v_5 = (u_1 \wedge u_5, u_2 \wedge u_4, u_3 \wedge u_5) \in S_{7,9}^{(2)*} \leftrightarrow d\rho_2 = 4\eta \leftrightarrow f_2^*(y) = y_4^2 + y_5 y_6$

iv) $-\delta\chi = 30\varepsilon + 10\eta = 3d\rho_1 + d\rho_2$, $\mathrm{tr}_{V_{x_{11}}^*} = 40\varepsilon + 14\eta = 4d\rho_1 + \frac{3}{2}d\rho_2$.

Remark for calculation of $d\rho_1$. Let f_1^* be the relative invariant on $V_{x_{11}}^*$ corresponding to $d\rho_1$. Since $f_1^*(v_2 + v_5) \neq 0$, the restriction of $d\rho_1$ to the isotropy subalgebra of $\mathfrak{g}_{x_{11}}$ at $v_2 + v_5$ should be zero. Hence $d\rho_1$ must be of the form $d\rho_1 = 5\lambda\varepsilon + \lambda\eta$ for some λ. Take an element A_0 in $\mathfrak{g}_{x_{11}}$ satisfying $d\rho^*(A_0)x_{11}^* = x_{11}^*$ where $x_{11}^* = v_1 + v_2 + v_4 \in V_{x_{11}}^* - S_{x_{11}}^*$. Then, by the Euler's identity, we have $(\deg f_1^*) \cdot f_1^*(x_{11}^*) = \langle d\rho^*(A_0)y, D_y \rangle f_1^*(y)|_{y = x_{11}^*} =$

$d\rho_1(A_0)f_1^*(x_{11})$ and hence $\deg f_1^* = d\rho_1(A_0) = \frac{3}{2}\lambda \in \mathbf{N}$. Therefore $d\rho_1 = (10\varepsilon + 2\eta)\mu$ where μ is a natural number. Since $-\delta\chi$ is a linear combination of $d\rho_1$ and $d\rho_2$ with coefficients in $\mathbf{Z}$, we have $\mu = 1$ or 3. On the other hand, $2\,\mathrm{tr}_{V^*_{x_{11}}}$ is also a linear combination of $d\rho_1$ and $d\rho_2$ with coefficients in $\mathbf{Z}$, μ is a divisor of 8, and hence $\mu = 1$, i.e., $d\rho_1 = 10\varepsilon + 2\eta$.

(18) The isotropy subalgebra $\mathfrak{g}_{x_{12}}$ at x_{12} is given as follows.

$$(11.18) \qquad \mathfrak{g}_{x_{12}} = \left\{ A = \left[\begin{array}{cc|cc|c} \varepsilon+\alpha & \alpha_{12} & \multicolumn{2}{c|}{0} & \gamma_1 \\ \alpha_{21} & \varepsilon-\alpha & & & \gamma_2 \\ \hline \multicolumn{2}{c|}{0} & \eta+\beta & \beta_{12} & \gamma_3 \\ & & \beta_{21} & \eta-\beta & \gamma_4 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & -2(\varepsilon+\eta) \end{array} \right] \oplus \left(\begin{array}{cc|c} -2\varepsilon & 0 & \gamma_5 \\ 0 & -2\eta & \gamma_6 \\ \multicolumn{2}{c|}{0} & \xi \end{array} \right) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(6) .$$

The conormal vector space $V^*_{x_{12}}$ is spanned by $v_1 = (0, 0, u_1 \wedge u_5)$, $v_2 = (0, 0, u_2 \wedge u_5)$, $v_3 = (0, 0, u_3 \wedge u_5)$, $v_4 = (0, 0, u_4 \wedge u_5)$, $v_5 = (0, u_1 \wedge u_5, 0)$, $v_6 = (0, u_2 \wedge u_5, 0)$, $v_7 = (u_3 \wedge u_5, 0, 0)$, $v_8 = (u_4 \wedge u_5, 0, 0)$, $v_9 = (0, 0, u_1 \wedge u_3)$, $v_{10} = (0, 0, u_1 \wedge u_4)$, $v_{11} = (0, 0, u_2 \wedge u_3)$, $v_{12} = (0, 0, u_2 \wedge u_4)$.

The action $d\rho_{x_{12}}$ of $\mathfrak{g}_{x_{12}}$ on $V^*_{x_{12}}$ is given by

$$d\rho_{x_{12}}(A)(v_1, \cdots, v_{12}) = (v_1, \cdots, v_{12}) = \left[\begin{array}{ccccc} A_1 & & B_1 & & B_2 \\ & A_2 & & B_3 & B_4 \\ & & A_3 & & \\ & & & A_4 & \\ & & & & A_5 \end{array} \right] \begin{array}{l} \}2 \\ \}2 \\ \}2 \\ \}2 \\ \}4 \end{array}$$

$$\underbrace{}_{2} \underbrace{}_{2} \underbrace{}_{2} \underbrace{}_{2} \underbrace{}_{4}$$

where $B_1 = -\gamma_6 I_2$, $B_3 = -\gamma_5 I_2$, $A_1 = (\varepsilon + 2\eta - \xi)I_2 + A'$, $A_2 = (2\varepsilon + \eta - \xi)I_2 + B'$, $A_3 = (\varepsilon + 4\eta)I_2 + A'$, $A_4 = (4\varepsilon + \eta)I_2 + B'$ with $A' = \begin{pmatrix} -\alpha & -\alpha_{21} \\ -\alpha_{12} & \alpha \end{pmatrix}$, $B' = \begin{pmatrix} -\beta & -\beta_{21} \\ -\beta_{12} & \beta \end{pmatrix}$, $A_5 = -(\varepsilon + \eta + \xi)I_4 + \begin{pmatrix} -\alpha I_2 + B' & -\alpha_{21}I_2 \\ -\alpha_{12}I_2 & \alpha I_2 + B' \end{pmatrix}$, $B_2 = \begin{pmatrix} -\gamma_3 & -\gamma_4 & 0 & 0 \\ 0 & 0 & -\gamma_3 & -\gamma_4 \end{pmatrix}$ and $B_4 = \begin{pmatrix} \gamma_1 & 0 & \gamma_2 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 \end{pmatrix}$.

i) $V^*_{x_{12}} - S^*_{x_{12}} \leftrightarrow v_6 + v_7 + v_{10} + v_{11} = (u_3 \wedge u_5, u_2 \wedge u_5, u_1 \wedge u_4 + u_2 \wedge u_3)$
$\in S^{(1)*}_{6,12}$

ii) $(S^*_{x_{12}})_1 \leftrightarrow v_6 + v_8 + v_9 = (u_4 \wedge u_5, u_2 \wedge u_5, u_1 \wedge u_3) \in S^{(2)*}_{7,9} \leftrightarrow d\rho_1$
$= -2(\varepsilon + \eta + \xi) \leftrightarrow f_1^*(y) = y_{10}y_{11} - y_9 y_{12}$

iii) $(S^*_{x_{12}})_2 \leftrightarrow v_5 + v_7 + v_{10} + v_{11} = (u_3 \wedge u_5, u_1 \wedge u_5, u_2 \wedge u_3 + u_2 \wedge u_4)$
$\in S^{(2)*}_{7,7} \leftrightarrow d\rho_2 = 4(\varepsilon + \eta) - \xi \leftrightarrow \deg f_2^*(y) = 3$

iv) $-\delta\chi = 10(\varepsilon + \eta) - 5\xi = d\rho_1 + 3d\rho_2$, $\mathrm{tr}_{V^*_{x_{12}}} = 12(\varepsilon + \eta) - 8\xi = 2d\rho_1 + 4d\rho_2$.

(19) The isotropy subalgebra $\mathfrak{g}_{x_{13}}$ at x_{13} is given as follows.

$$(11.19) \quad \mathfrak{g}_{x_{13}} = \left\{ A = \begin{bmatrix} \varepsilon+\alpha & \alpha_{12} & \gamma_1 & \gamma_2 & \gamma_5 \\ \alpha_{21} & \varepsilon-\alpha & \gamma_3 & \gamma_4 & \gamma_6 \\ \hline & 0 & -(\varepsilon+\eta)-\alpha & -\alpha_{21} & \gamma_7 \\ & & -\alpha_{12} & -(\varepsilon+\eta)+\alpha & \gamma_8 \\ \hline & 0 & & 0 & 2\eta \end{bmatrix} \right.$$

$$\left. \oplus \begin{pmatrix} -2\varepsilon & 0 & \delta_1 \\ \gamma_2-\gamma_3 & \eta & \delta_2 \\ 0 & 0 & \xi \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(10) \,.$$

The conormal vector space $V_{x_{13}}^*$ is spanned by $v_1 = (0, 0, -u_4 \wedge u_5)$, $v_2 = (0, 0, u_3 \wedge u_5)$, $v_3 = (0, 0, u_1 \wedge u_5)$, $v_4 = (0, 0, u_2 \wedge u_5)$, $v_5 = (0, 0, u_3 \wedge u_4)$, $v_6 = (0, 0, -u_1 \wedge u_4)$, $v_7 = (0, 0, u_1 \wedge u_3 - u_2 \wedge u_4)$, $v_8 = (0, 0, u_2 \wedge u_3)$, $v_9 = (-u_4 \wedge u_5, 0, 0)$, $v_{10} = (u_3 \wedge u_5, 0, 0)$, $v_{11} = (u_3 \wedge u_4, 0, 0)$, $v_{12} = (u_1 \wedge u_5, u_4 \wedge u_5, 0)$, $v_{13} = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$.

The action $d\rho_{x_{13}}$ of $\mathfrak{g}_{x_{13}}$ on $V_{x_{13}}^*$ is given by

$$d\rho_{x_{13}}(A)(v_1, \cdots, v_{13}) = (v_1, \cdots, v_{13}) \begin{bmatrix} A_1 & B_1 & B_2 & B_3 & B_4 & & B_5 \\ & A_2 & & B_6 & & & B_7 \\ & & A_3 & B_8 & & B_9 & \\ & & & A_4 & & & \\ & & & & A_5 & B_{10} & B_{11} \\ & & & & & A_6 & \\ & & & & & & A_7 \end{bmatrix} \begin{matrix} \}2 \\ \}2 \\ \}1 \\ \}3 \\ \}2 \\ \}1 \\ \}2 \end{matrix}$$

$$\underbrace{}_{2} \underbrace{}_{2} \underbrace{}_{1} \underbrace{}_{3} \underbrace{}_{2} \underbrace{}_{1} \underbrace{}_{2}$$

where $A_3 = 2\varepsilon + 2\eta - \xi$, $B_8 = (\gamma_1, \gamma_2 + \gamma_3, \gamma_4)$, $B_9 = -\delta_1$, $A_6 = 4\varepsilon + 2\eta$, $B_4 = B_7 = -\delta_1 I_2$, $B_5 = \delta_2 I_2$, $A_1 = (\varepsilon - \eta - \xi)I_2 + A'$ with $A' = \begin{pmatrix} -\alpha & -\alpha_{21} \\ -\alpha_{12} & \alpha \end{pmatrix}$, $(B_1, B_2, B_3) = \begin{pmatrix} \gamma_2 & \gamma_4 & -\gamma_7 & \gamma_5 & \gamma_6 & 0 \\ -\gamma_1 & -\gamma_3 & -\gamma_8 & 0 & \gamma_5 & \gamma_6 \end{pmatrix}$, $B_6 = \begin{pmatrix} \gamma_8 & -\gamma_7 \\ \gamma_8 & -\gamma_7 \end{pmatrix}$, $(B_{10}, B_{11}) = \begin{pmatrix} -\gamma_7 & 2\gamma_2-\gamma_3 & \gamma_4 \\ -\gamma_8 & -\gamma_1 & \gamma_2-2\gamma_3 \end{pmatrix}$, $A_2 = -(\varepsilon + 2\eta + \xi)I_2 + A'$, $A_5 = (3\varepsilon - \eta)I_2 + A'$, $A_7 = (\varepsilon - 2\eta)I_2 + A'$ and $A_4 = (\eta - \xi)I_3 + \begin{pmatrix} -2\alpha & -2\alpha_{21} & 0 \\ -\alpha_{12} & 0 & -\alpha_{21} \\ 0 & -2\alpha_{12} & 2\alpha \end{pmatrix}$.

  i) $V_{x_{13}}^* - S_{x_{13}}^* \leftrightarrow v_7 + v_{11} + v_{12} + v_{13} \in S_{3,13}^{(2)*}$

  ii) $(S_{x_{13}}^*)_1 \leftrightarrow v_7 + v_{12} + v_{13} \in S_{6,12}^{(1)*} \leftrightarrow d\rho_1 = 4\varepsilon + 2\eta \leftrightarrow f_1^*(y) = y_{11}$

 iii) $(S_{x_{13}}^*)_2 \leftrightarrow v_8 + v_{11} + v_{12} \in S_{4,11}^{(3)*} \leftrightarrow d\rho_2 = 2(\eta - \xi) \leftrightarrow f_2^*(y) = y_7^2 - y_6 y_8$

 iv) $(S_{x_{13}}^*)_3 \leftrightarrow v_7 + v_{11} + v_{12} \in S_{5,8}^{(2)*} \leftrightarrow d\rho_3 = 2\varepsilon - 3\eta - \xi \leftrightarrow \deg f_2^*(y) = 3$

  v) $-\delta\chi = d\rho_1 + d\rho_2 + 3d\rho_3$, $\mathrm{tr}_{V_{x_{13}}^*} = \frac{3}{2}d\rho_1 + 2d\rho_2 + 4d\rho_3$.

    (20)   The isotropy subalgebra $\mathfrak{g}_{x_{14}}$ at x_{14} is given as follows.

$$(11.20) \quad \mathfrak{g}_{x_{14}} = \left\{ A = \left(\begin{array}{c|c|c} \varepsilon & * & * \\ \hline 0 & \eta I_3 + X & * \\ \hline 0 & 0 & -\varepsilon - 3\eta \end{array} \right) \oplus (-(\varepsilon + \eta) I_3 - {}^t X); \ X \in \mathfrak{sl}(3) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(3)) \oplus \mathfrak{u}(7) \ .$$

The conormal vector space $V_{x_{14}}^*$ is spanned by $v_1 = (u_3 \wedge u_4, 0, 0)$, $v_2 = (0, u_2 \wedge u_4, 0)$, $v_3 = (0, 0, u_2 \wedge u_3)$, $v_4 = (0, u_2 \wedge u_3, -u_2 \wedge u_4)$, $v_5 = (u_2 \wedge u_3, 0, u_3 \wedge u_4)$, $v_6 = (u_2 \wedge u_4, -u_3 \wedge u_4, 0)$, $v_7 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_8 = (0, u_3 \wedge u_5, -u_4 \wedge u_5)$, $v_9 = (0, u_2 \wedge u_5, 0)$, $v_{10} = (0, 0, u_3 \wedge u_5)$, $v_{11} = (u_4 \wedge u_5, 0, 0)$, $v_{12} = (u_3 \wedge u_5, 0, 0)$, $v_{13} = (0, u_4 \wedge u_5, 0)$, $v_{14} = (0, 0, u_2 \wedge u_5)$.

Since $\dim \rho_{x_{14}} (G_{x_{14}}) = 13$ and $\dim V_{x_{14}}^* = 14$, the conormal vector space $(G_{x_{14}}, \rho_{x_{14}}, V_{x_{14}}^*)$ is not a P.V. Note that it is also obtained from the fact that $\varLambda_{8,14} = \varLambda_{14,8}^*$ is not G-prehomogeneous (See (13)).

(21) The isotropy subalgebra $\mathfrak{g}_{x_{15}}$ at x_{15} is given as follows.

$$(11.21) \quad \mathfrak{g}_{x_{15}} = \left\{ A = \left(\begin{array}{c|c} 2\varepsilon I_3 + X & Z \\ \hline 0 & -3\varepsilon I_2 + Y \end{array} \right) \oplus (-4\varepsilon I_3 + S^{-1} X S); \ X \in \mathfrak{sl}(3), \right.$$

$$\left. Y \in \mathfrak{sl}(2), \ Z \in V(6), \ S = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(3) \oplus \mathfrak{sl}(2)) \oplus V(6) \ .$$

The conormal vector space $V_{x_{15}}^*$ is spanned by $v_1 = (0, 0, u_1 \wedge u_4)$, $v_2 = (0, u_2 \wedge u_4, 0)$, $v_3 = (u_3 \wedge u_4, 0, 0)$, $v_4 = (0, u_1 \wedge u_4, -u_2 \wedge u_4)$, $v_5 = (u_2 \wedge u_4, -u_3 \wedge u_4, 0)$, $v_6 = (u_1 \wedge u_4, 0, u_3 \wedge u_4)$, $v_7 = (0, 0, u_1 \wedge u_5)$, $v_8 = (0, u_2 \wedge u_5, 0)$, $v_9 = (u_3 \wedge u_5, 0, 0)$, $v_{10} = (0, u_1 \wedge u_5, -u_2 \wedge u_5)$, $v_{11} = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_{12} = (u_1 \wedge u_5, 0, u_3 \wedge u_5)$, $v_{13} = (0, 0, u_4 \wedge u_5)$, $v_{14} = (0, u_4 \wedge u_5, 0)$, $v_{15} = (u_4 \wedge u_5, 0, 0)$. Then the action $d\rho_{x_{15}}$ of $\mathfrak{g}_{x_{15}}$ on $V_{x_{15}}^*$ is given by

$$(v_1, \cdots, v_{15}) \mapsto (v_1, \cdots, v_{15}) \left[\begin{array}{c|c} \begin{array}{c} GL(1) \times SL(3) \times SL(2) \\ \hline 5\varLambda_1 \otimes 2\varLambda_1^* \otimes \varLambda_1^* \\ \hline * \end{array} & \begin{array}{c} 0 \\ \hline GL(1) \times SL(3) \\ 10\varLambda_1 \otimes \varLambda_1^* \end{array} \end{array} \right]$$

   i) $V_{x_{15}}^* - S_{x_{15}}^* \leftrightarrow v_1 + v_2 + v_7 + v_9 = (u_3 \wedge u_5, u_2 \wedge u_4, u_1 \wedge u_4 + u_1 \wedge u_5)$
           $\in S_{3,15}^{(3)*}$

  ii) $(S_{x_{15}}^*)_1 \leftrightarrow v_4 + v_{11} + v_{12} \in S_{4,11}^{(3)*} \leftrightarrow d\rho_1 = 60\varepsilon$

 iii) $-\delta\chi = 60\varepsilon = d\rho_1$, $\mathrm{tr}_{V_{x_{15}}^*} = 90\varepsilon = \frac{3}{2}d\rho_1$.

       (22) The isotropy subalgebra $\mathfrak{g}_{x_{16}}$ at x_{16} is given as follows.

$$(11.22) \qquad \mathfrak{g}_{x_{16}} = \left\{ A = \begin{bmatrix} \begin{array}{c|cc|cc} -2(\varepsilon+\eta) & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \cline{1-5} & \varepsilon+\alpha_1 & \alpha_{12} & \gamma_5 & \gamma_6 \\ 0 & \alpha_{21} & \varepsilon-\alpha_1 & \gamma_7 & \gamma_8 \\ \cline{1-5} & & & \eta+\beta_1 & \beta_{12} \\ 0 & & 0 & \beta_{21} & \eta-\beta_1 \end{array} \end{bmatrix} \right.$$

$$\oplus \left(\begin{array}{cc|c} \varepsilon+2\eta-\alpha_1 & -\alpha_{21} & \gamma_9 \\ -\alpha_{12} & \varepsilon+2\eta+\alpha_1 & \gamma_{10} \\ \cline{1-3} \multicolumn{2}{c|}{0} & \xi \end{array} \right) \left. \vphantom{\begin{array}{c}1\\2\\3\end{array}} \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(10) .$$

The conormal vector space $V_{x_{16}}^*$ is spanned by $v_1 = (0, 0, u_4 \wedge u_5)$, $v_2 = (0, -u_4 \wedge u_5, 0)$, $v_3 = (u_4 \wedge u_5, 0, 0)$, $v_4 = (0, 0, u_2 \wedge u_4)$, $v_5 = (0, 0, u_2 \wedge u_5)$, $v_6 = (0, 0, u_3 \wedge u_4)$, $v_7 = (0, 0, u_3 \wedge u_5)$, $v_8 = (0, 0, u_2 \wedge u_3)$, $v_9 = (0, 0, u_1 \wedge u_4)$, $v_{10} = (0, 0, u_1 \wedge u_5)$, $v_{11} = (0, -u_2 \wedge u_4, 0)$, $v_{12} = (u_2 \wedge u_4, -u_3 \wedge u_4, 0)$, $v_{13} = (u_3 \wedge u_4, 0, 0)$, $v_{14} = (0, -u_2 \wedge u_5, 0)$, $v_{15} = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_{16} = (u_3 \wedge u_5, 0, 0)$.

The action $d\rho_{x_{16}}$ of $\mathfrak{g}_{x_{16}}$ on $V_{x_{16}}^*$ is given by

$$d\rho(A)(v_1, \cdots, v_{16}) = (v_1, \cdots, v_{16}) \begin{bmatrix} A_1 & B_1 & B_2 & & B_3 & \\ & A_2 & & & & B_4 \\ & & A_3 & B_5 & B_6 & B_7 \\ & & & A_4 & & \\ & & & & A_5 & \\ & & & & & A_6 \end{bmatrix} \begin{matrix} \}1 \\ \}2 \\ \}4 \\ \}1 \\ \}2 \\ \}6 \end{matrix}$$
$$\underbrace{}_{1} \underbrace{}_{2} \underbrace{}_{4} \underbrace{}_{1} \underbrace{}_{2} \underbrace{}_{6}$$

where $(A_1, B_1, B_2) = (-2\eta - \xi, \gamma_{10}, -\gamma_9, \gamma_6, -\gamma_5, \gamma_8, -\gamma_7)$, $B_3 = (\gamma_4, -\gamma_3)$, $A_4 = -2\varepsilon - \xi$, $A_2 = -(\varepsilon + 4\eta)I_2 + \begin{pmatrix} -\alpha_1 & -\alpha_{21} \\ -\alpha_{12} & \alpha_1 \end{pmatrix}$, $B_4 = \begin{pmatrix} \gamma_6 & \gamma_8 & 0 & -\gamma_5 & -\gamma_7 & 0 \\ 0 & \gamma_6 & \gamma_8 & 0 & -\gamma_5 & -\gamma_7 \end{pmatrix}$, $A_5 = (2\varepsilon + \eta - \xi)I_2 + B$ with $B = \begin{pmatrix} -\beta_1 & -\beta_{21} \\ -\beta_{12} & \beta_1 \end{pmatrix}$, $A_3 = -(\varepsilon + \eta + \xi)I_4 + \begin{pmatrix} -\alpha_1 I_2 + B & -\alpha_{21} I_2 \\ -\alpha_{12} I_2 & \alpha_1 I_2 + B \end{pmatrix}$, $A_6 = -(2\varepsilon + 3\eta)I_6 + \begin{pmatrix} -\beta_1 I_3 + A' & -\beta_{21} I_3 \\ -\beta_{12} I_3 & \beta_1 I_3 + A' \end{pmatrix}$ with $A' = \begin{pmatrix} -2\alpha_1 & -2\alpha_{21} & 0 \\ -\alpha_{12} & 0 & -\alpha_{21} \\ 0 & -2\alpha_{12} & 2\alpha_1 \end{pmatrix}$ and

$$(B_5, B_6, B_7) = \begin{bmatrix} \begin{array}{cc|cc|cccc} -\gamma_7 & & -\gamma_1 & & \gamma_{10} & -\gamma_9 & & \\ -\gamma_8 & & & -\gamma_1 & & & \gamma_{10} & -\gamma_9 \\ \gamma_5 & & -\gamma_2 & & & \gamma_{10} & -\gamma_9 & \\ \gamma_6 & & & -\gamma_2 & & & & \gamma_{10} & -\gamma_9 \end{array} \end{bmatrix}$$

i) $V_{x_{16}}^* - S_{x_{16}}^* \leftrightarrow v_8 + v_9 + v_{11} + v_{15} + v_{16} = (u_2 \wedge u_5 + u_3 \wedge u_5,$
$\qquad - u_2 \wedge u_4 - u_3 \wedge u_5, u_2 \wedge u_3 + u_1 \wedge u_4) \in S_{2,16}^{(2)*}$

ii) $(S_{x_{16}}^*)_1 \leftrightarrow v_9 + v_{10} + v_{11} + v_{16} = (u_3 \wedge u_5, - u_2 \wedge u_4, u_1 \wedge u_4 + u_1 \wedge u_5)$
$\qquad \in S_{3,15}^{(3)*} \leftrightarrow d\rho_1 = - 2\varepsilon - \xi \leftrightarrow f_1^*(y) = y_8$

iii) $(S_{x_{16}}^*)_2 \leftrightarrow v_8 + v_9 + v_{10} + v_{11} + v_{15} = (u_2 \wedge u_5, - u_2 \wedge u_4 - u_3 \wedge u_5,$
$\qquad u_2 \wedge u_3 + u_1 \wedge u_4 + u_1 \wedge u_5) \in S_{3,13}^{(2)*} \leftrightarrow d\rho_2 = - 8\varepsilon - 12\eta$
$\qquad \leftrightarrow f_2^*(y) = \det\begin{pmatrix} y_{11} & y_{14} \\ y_{13} & y_{16} \end{pmatrix}^2 - 4\det\begin{pmatrix} y_{12} & y_{15} \\ y_{13} & y_{16} \end{pmatrix} \cdot \det\begin{pmatrix} y_{11} & y_{14} \\ y_{12} & y_{15} \end{pmatrix}$

iv) $(S_{x_{16}}^*)_3 \leftrightarrow v_8 + v_{10} + v_{11} + v_{16} = (u_3 \wedge u_5, - u_2 \wedge u_4, u_1 \wedge u_5 + u_2 \wedge u_3)$
$\qquad \in S_{4,11}^{(3)*} \leftrightarrow d\rho_3 = - 4\eta - 2\xi$
$\qquad \leftrightarrow f_3^*(y) = \det\begin{pmatrix} y_9 & y_{12} \\ y_{10} & y_{15} \end{pmatrix}^2 - \det\begin{pmatrix} y_9 & y_{11} \\ y_{10} & y_{14} \end{pmatrix} \cdot \det\begin{pmatrix} y_9 & y_{13} \\ y_{10} & y_{16} \end{pmatrix}$

v) $- \delta\chi = d\rho_1 + d\rho_2 + 2d\rho_3,\ \mathrm{tr}_{V_{x_{16}}^*} = 2d\rho_1 + \frac{3}{2}d\rho_2 + 3d\rho_3.$

(23) The isotropy subalgebra $\mathfrak{g}_{x_{18}}$ at x_{18} is given as follows.

(11.23) $\mathfrak{g}_{x_{18}} = \left\{ A = \left(\begin{array}{c|c} \varepsilon I_4 + X & * \\ \hline 0 & -4\varepsilon \end{array} \right) \oplus \left(\begin{array}{c|c} -2\varepsilon & * \\ \hline 0 & Y \end{array} \right); X \in \mathfrak{sp}(2),\ Y \in \mathfrak{gl}(2) \right\}$
$\qquad\qquad \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sp}(2) \oplus \mathfrak{sl}(2)) \oplus V(6)\ .$

The conormal vector space $V_{x_{18}}^*$ is spanned by $v_1 = (0, u_1 \wedge u_3 - u_2 \wedge u_4, 0)$, $v_2 = (0, u_1 \wedge u_2, 0)$, $v_3 = (0, u_1 \wedge u_4, 0)$, $v_4 = (0, u_3 \wedge u_4, 0)$, $v_5 = (0, u_2 \wedge u_3, 0)$, $v_6 = (0, 0, u_1 \wedge u_3 - u_2 \wedge u_4)$, $v_7 = (0, 0, u_1 \wedge u_2)$, $v_8 = (0, 0, u_1 \wedge u_4)$, $v_9 = (0, 0, u_3 \wedge u_4)$, $v_{10} = (0, 0, u_2 \wedge u_3)$, $v_{11} = (0, u_1 \wedge u_5, 0)$, $v_{12} = (0, u_2 \wedge u_5, 0)$, $v_{13} = (0, u_3 \wedge u_5, 0)$, $v_{14} = (0, u_4 \wedge u_5, 0)$, $v_{15} = (0, 0, u_1 \wedge u_5)$, $v_{16} = (0, 0, u_2 \wedge u_5)$, $v_{17} = (0, 0, u_3 \wedge u_5)$, $v_{18} = (0, 0, u_4 \wedge u_5)$. The action $d\rho_{x_{18}}$ of $\mathfrak{g}_{x_{18}}$ on $V_{x_{18}}^*$ is given by

$(v_1, \cdots, v_{18})$

$\mapsto (v_1, \cdots, v_{18}) \left(\begin{array}{c|c} \begin{array}{c} GL(1) \times GL(1) \times Sp(2) \times SL(2) \\ 2\Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_2 \otimes \Lambda_1^* \end{array} & 0 \\ \hline * & \begin{array}{c} GL(1) \times GL(1) \times Sp(2) \times SL(2) \\ 3\Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1 \otimes \Lambda_1^* \end{array} \end{array} \right)$

i) $V_{x_{18}}^* - S_{x_{18}}^* \leftrightarrow v_2 + v_4 + v_8 + v_{10} + v_{15} \in S_{8,18}^*$

ii) $(S_{x_{18}}^*)_1 \leftrightarrow v_1 + v_3 + v_{10} + v_{11} \in S_{8,18}^* \leftrightarrow d\rho_1 = - 2\varepsilon - 6\eta \leftrightarrow \deg f_1^* = 6$

iii) $(S_{x_{18}}^*)_2 \leftrightarrow v_1 + v_7 + v_{17} + v_{18} \in S_{8,18}^* \leftrightarrow d\rho_2 = - 8\varepsilon - 4\eta \leftrightarrow \deg f_2^* = 4$

iv) $- \delta\chi = 3d\rho_1 - 2d\rho_2,\ \mathrm{tr}_{V_{x_{18}}^*} = 4d\rho_1 - \frac{3}{2}d\rho_2.$

(24) The isotropy subalgebra $\mathfrak{g}_{x_{21}}$ at x_{21} is given as follows.

$$(11.24) \quad \mathfrak{g}_{x_{21}} = \left\{ A = \left(\begin{array}{c|c} 3\varepsilon I_3 + X & * \\ \hline 0 & -2\varepsilon I_3 + Y \end{array} \right) \oplus \left(\begin{array}{c|c} -6\varepsilon & * \\ \hline 0 & \eta I_2 + Z \end{array} \right); \right.$$

$$\left. X, Z \in \mathfrak{sl}(2), \ Y \in \mathfrak{sl}(3) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(3)) \oplus V(8) \ .$$

Then $V_{x_{21}}^*$ is spanned by $v_1 = (0, u_4 \wedge u_5, 0)$, $v_2 = (0, u_3 \wedge u_5, 0)$, $v_3 = (0, u_3 \wedge u_4, 0)$, $v_4 = (0, 0, u_4 \wedge u_5)$, $v_5 = (0, 0, u_3 \wedge u_5)$, $v_6 = (0, 0, u_3 \wedge u_4)$, $v_7 = (u_4 \wedge u_5, 0, 0)$, $v_8 = (u_3 \wedge u_5, 0, 0)$, $v_9 = (u_3 \wedge u_4, 0, 0)$, $v_{10} = (0, u_1 \wedge u_3, 0)$, $v_{11} = (0, u_1 \wedge u_4, 0)$, $v_{12} = (0, u_1 \wedge u_5, 0)$, $v_{13} = (0, u_2 \wedge u_3, 0)$, $v_{14} = (0, u_2 \wedge u_4, 0)$, $v_{15} = (0, u_2 \wedge u_5, 0)$, $v_{16} = (0, 0, u_1 \wedge u_3)$, $v_{17} = (0, 0, u_1 \wedge u_4)$, $v_{18} = (0, 0, u_1 \wedge u_5)$, $v_{19} = (0, 0, u_2 \wedge u_3)$, $v_{20} = (0, 0, u_2 \wedge u_4)$, $v_{21} = (0, 0, u_2 \wedge u_5)$. The action $d\rho_{x_{21}}$ of $\mathfrak{g}_{x_{21}}$ on $V_{x_{21}}^*$ is given by

$(v_1, \cdots, v_{21})$

$$\mapsto (v_1, \cdots, v_{21}) \left[\begin{array}{c|c|c} \begin{array}{c} GL(1) \times GL(1) \times SL(2) \\ \times SL(3) \\ 4\Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1 \end{array} & * & * \\ \hline 0 & \begin{array}{c} GL(1) \times SL(3) \\ (10\Lambda_1) \otimes \Lambda_1 \end{array} & 0 \\ \hline 0 & 0 & \begin{array}{c} GL(1) \times GL(1) \times SL(2) \\ \times SL(2) \times SL(3) \\ \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1^* \end{array} \end{array} \right]$$

i) $V_{x_{21}}^* - S_{x_{21}}^* \leftrightarrow v_7 + v_{10} + v_{14} + v_{18} + v_{19} = (u_4 \wedge u_5, u_1 \wedge u_3 + u_2 \wedge u_4,$
$\qquad u_1 \wedge u_5 + u_2 \wedge u_3) \in S_{1,21}^{(2)*}$

ii) $(S_{x_{21}}^*)_1 \leftrightarrow v_8 + v_{10} + v_{14} + v_{18} + v_{19} = (u_3 \wedge u_5, u_1 \wedge u_3 + u_2 \wedge u_4,$
$\qquad u_1 \wedge u_5 + u_2 \wedge u_3) \in S_{3,13}^{(2)} \leftrightarrow d\rho_1 = 18\varepsilon - 2\eta \leftrightarrow \deg f_1^* = 4$

iii) $(S_{x_{21}}^*)_2 \leftrightarrow v_8 + v_9 + v_{10} + v_{14} + v_{18} = (u_3 \wedge u_4 + u_3 \wedge u_5, u_1 \wedge u_3$
$\qquad + u_2 \wedge u_4, u_1 \wedge u_5) \in S_{2,16}^{(2)*} \leftrightarrow d\rho_2 = -6\varepsilon - 6\eta \leftrightarrow \deg f_2^* = 6$

iv) $-\delta\chi = 2d\rho_1 + d\rho_2$, $\operatorname{tr}_{V_{x_{21}}^*} = 3d\rho_1 + 2d\rho_2$.

(25) The isotropy subalgebra $\mathfrak{g}_{x_{30}}$ at $x_{30} = 0$ is $\mathfrak{g}$ itself.

This is a good holonomic variety and $\operatorname{ord}_\Lambda f^s = -15s - \frac{3 \cdot 0}{2}$. Thus we obtain the holonomy diagram (Figure 11–1). From this diagram, we obtain the b-function $b(s) = ((s + 1)(s + \frac{3}{2})(s + 2))^3 \cdot ((s + \frac{4}{3})(s + \frac{5}{3}))^2 \cdot (s + \frac{5}{4})(s + \frac{7}{4})$.

Remark. Let $\Lambda_0 = \overline{G(x_0, y_0)}$ and $\Lambda_1 = \overline{G(x_1, y_1)}$ be good holonomic varieties satisfying $(x_0, y_1) \in \Lambda_0 \cap \Lambda_1$ and $\dim G(x_0, y_1) = \dim V - 1$. Then we can calculate β by Proposition 1–4. It is known that if β depends on the choice of Λ_1, then (x_0, y_1) is not contained in other Λ_i $(i \neq 0, 1)$, i.e., there are no three Λ_i's which intersect at (x_0, y_1) with codimension one. (If more
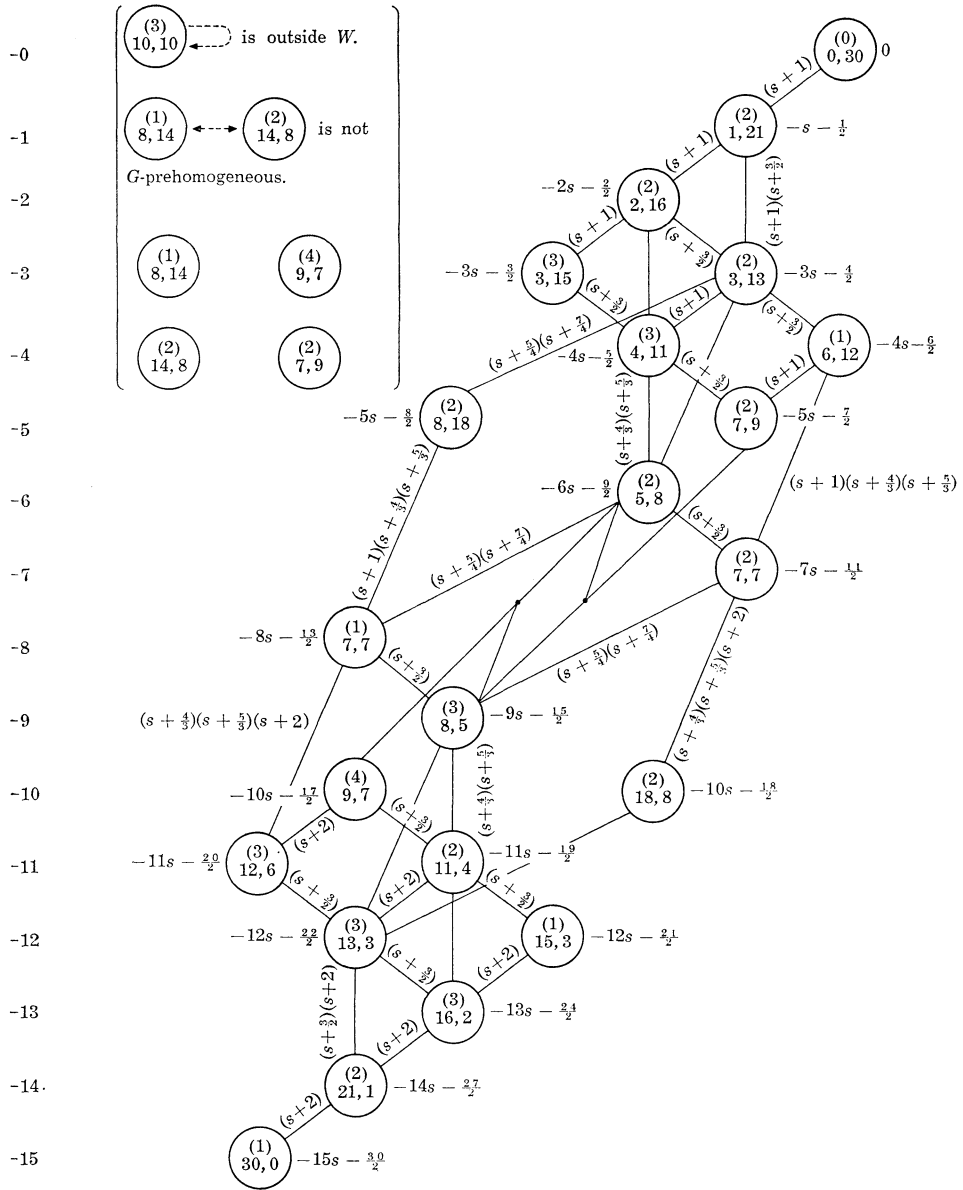
Fig. 11–1. Holonomy diagram of $(SL(5) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3))$ where $\binom{(k)}{i\,j}$ denotes the conormal bundles of the orbit $S_{ij}^{(k)}$ in Proposition 6-1.

than two Λ_i's intersect with codimension one, then $\beta = 1$ and it does not depend on Λ_1.) All one-codimensional intersections obtained from (1)$\sim$(14) satisfy this condition except $\Lambda_{5,8}^{(2)} \cap \Lambda_{4,11}^{(3)}$ and $\Lambda_{8,5}^{(3)} \cap \Lambda_{7,9}^{(2)}$. In general, if (x_0, y_1) $\in \overline{G(x_2, y_2)}$, then we have $\mathrm{codim}_V \rho(G) x_0 \geqq \mathrm{codim}_V \rho(G) x_1$ and $\mathrm{codim}_{V*} \rho^*(G) y_1$ $\geqq \mathrm{codim}_{V*} \rho^*(G) y_2$. From this, there are no other Λ's satisfying $\dim \Lambda \cap \Lambda_{5,8}^{(2)} \cap$ $\Lambda_{4,11}^{(3)} = \dim V - 1$. For $\Lambda_{8,5}^{(3)} \cap \Lambda_{7,9}^{(2)}$, it is enough to check $\Lambda_{7,7}^{(1)}$, $\Lambda_{7,7}^{(2)}$ and $\Lambda_{5,8}^{(2)}$. By using the duality, i.e., $(G, \rho, V) \cong (G, \rho^*, V^*)$, we get all one-codimensional intersections of three good holonomic varieties.

§ 12. Table of the *b*-functions of irreducible reduced regular P.V.'s

(1) $(G \times GL(m), \rho \otimes \Lambda_1, V(m) \otimes V(m))$ where $\rho : G \to GL(V(m))$ is an m-dimensional irreducible representation of a connected semi-simple algebraic group G (or $G = \{1\}$ and $m = 1$).

$b(s) = (s + 1)(s + 2) \cdots (s + m)$ (See Figure 2–1 and 2–4).

(2) $(GL(n), 2\Lambda_1, V(\frac{1}{2} n(n + 1)))$ $(n \geqq 2)$

$$b(s) = \prod_{\nu=1}^{n} \left(s + \frac{\nu + 1}{2}\right) = (s + 1)\left(s + \frac{3}{2}\right) \cdots \left(s + \frac{n + 1}{2}\right)$$

(See Figure 2–2 and 2–4).

(3) $(GL(2m), \Lambda_2, V(m(2m - 1)))$ $(m \geqq 3)$

$$b(s) = \prod_{k=1}^{m} (s + 2k - 1) = (s + 1)(s + 3) \cdots (s + 2m - 1)$$

(See Figure 2–3 and 2–4).

(4) $(GL(2), 3\Lambda_1, V(4))$

$b(s) = (s + 1)^2 (s + \frac{5}{6})(s + \frac{7}{6})$ (See [2]).

(5) $(GL(6), \Lambda_3, V(20))$

$b(s) = (s + 1)(s + \frac{5}{2})(s + \frac{7}{2})(s + 5)$ (See Figure 8–1).

(6) $(GL(7), \Lambda_3, V(35))$

$b(s) = (s + 1)(s + 2)(s + \frac{5}{2})(s + \frac{7}{2})(s + 3)(s + 4)(s + 5)$
(See Figure 10–1).

(7) $(GL(8), \Lambda_3, V(56))$

$b(s) = (s+1)(s+\frac{3}{2})^2(s+\frac{11}{6})(s+2)^3(s+\frac{13}{6})(s+\frac{7}{3})(s+\frac{5}{2})^3(s+\frac{8}{5})(s+3)^2(s+\frac{7}{2})$
(See [10]).

(8) $(SL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2))$

$b(s) = \{(s + 1)^2(s + \frac{5}{6})(s + \frac{7}{6})(s + \frac{3}{4})(s + \frac{5}{4})\}^2$ (See [12]).

(9) $(SL(6) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(15) \otimes V(2))$

$b(s) = (s + 1)^2(s + \frac{5}{6})(s + \frac{7}{6})(s + \frac{3}{2})^2(s + 2)^2(s + \frac{5}{2})^2(s + \frac{7}{3})(s + \frac{8}{3})$
(See [12]).

(10) $(SL(5) \times GL(3), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(3))$

$b(s) = ((s + 1)(s + \frac{3}{2})(s + 2))^3 \cdot ((s + \frac{4}{3})(s + \frac{5}{3}))^2 \cdot (s + \frac{5}{4})(s + \frac{7}{4})$

(See Figure 11–1).

(11) $(SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$ (See [11]).

(12) $(SL(3) \times SL(3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2))$

$b(s) = (s + 1)^4(s + \frac{3}{2})^4(s + \frac{4}{3})(s + \frac{5}{3})(s + \frac{5}{6})(s + \frac{7}{6})$ (See [12]).

(13) $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ $(n \geqq 2m \geqq 2)$

$b(s) = \prod_{k=1}^{m} (s + 2k - 1) \prod_{\ell=0}^{m-1} (s + 2n - 2\ell)$

$= (s + 1)(s + 3) \cdots (s + 2m - 1)(s + 2n)(s + 2n - 2) \cdots$

$(s + 2n - 2m + 2)$ (See Figure 3–1 and 3–2).

(14) $(GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$

$b(s) = (s + 1)(s + 2)(s + \frac{5}{2})(s + \frac{7}{2})$ (See Figure 9–1).

(15) $(SO(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m))$ $\left(n \geq 3, \frac{n}{2} \geqq m \geqq 1\right)$

$b(s) = \prod_{k=1}^{m} \left(s + \frac{k + 1}{2}\right) \prod_{\ell=1}^{m} \left(s + \frac{n - \ell + 1}{2}\right)$

$= (s + 1)\left(s + \frac{3}{2}\right) \cdots \left(s + \frac{m + 1}{2}\right)\left(s + \frac{n}{2}\right)\left(s + \frac{n - 1}{2}\right) \cdots$

$\left(s + \frac{n - m + 1}{2}\right)$ (See [2]).

(16) $(GL(1) \times \text{Spin}(7), \square \otimes \text{spin rep.}, V(1) \otimes V(8))$

$b(s) = (s + 1)(s + 4)$ (See Remark in § 5).

(17) $\text{Spin}(7) \times GL(2), \text{spin rep.} \otimes \Lambda_1, V(8) \otimes V(2))$

$b(s) = (s + 1)(s + \frac{3}{2})(s + 4)(s + \frac{7}{2})$ (See Remark in § 5).

(18) $(\text{Spin}(7) \times GL(3), \text{spin rep.} \otimes \Lambda_1, V(8) \otimes V(3))$

$b(s) = (s + 1)(s + \frac{3}{2})(s + 2)(s + 4)(s + \frac{7}{2})(s + 3)$ (See Remark in § 5).

(19) $(GL(1) \times \text{Spin}(9), \square \otimes \text{spin rep.}, V(1) \otimes V(16))$

$b(s) = (s + 1)(s + 8)$ (See Remark in § 5).

(20) $(\text{Spin}(10) \times GL(2), \text{half-spin rep.} \otimes \Lambda_1, V(16) \otimes V(2))$

$b(s) = (s + 1)(s + 4)(s + 5)(s + 8)$ (See Figure 4–1).

(21) $(\text{Spin}(10) \times GL(3), \text{half-spin rep.} \otimes \Lambda_1, V(16) \otimes V(3))$

$b(s) = (s + 1)(s + \frac{3}{2})(s + 2)(s + 3)(s + \frac{7}{2})(s + 4)(s + \frac{5}{3})(s + \frac{6}{3})(s + \frac{7}{3}) \times$

$\times (s + \frac{8}{3})(s + \frac{9}{3})(s + \frac{10}{3})$ (See [15]).

(22) $(GL(1) \times \text{Spin}(11), \square \otimes \text{spin rep.}, V(1) \otimes V(32))$

$b(s) = (s + 1)(s + \frac{7}{2})(s + \frac{11}{2})(s + 8)$ (See Remark in § 5).

(23) $(GL(1) \times \text{Spin}(12), \square \otimes \text{half-spin rep.}, V(1) \otimes V(32))$

$b(s) = (s + 1)(s + \frac{7}{2})(s + \frac{11}{2})(s + 8)$ (See Figure 5–1).

(24) $(GL(1) \times \text{Spin}\,(14),\ \square \otimes \text{half-spin rep.},\ V(1) \otimes V(64))$
$b(s) = (s + 1)(s + \frac{5}{2})(s + \frac{7}{2})(s + 4)(s + 5)(s + \frac{11}{2})(s + \frac{13}{2})(s + 8)$
(See Appendix).

(25) $(GL(1) \times (G_2),\ \square \otimes \Lambda_2,\ V(1) \otimes V(7))$
$b(s) = (s + 1)(s + \frac{7}{2})$ (See Remark in § 5).

(26) $((G_2) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(2))$
$b(s) = (s + 1)(s + \frac{3}{2})(s + \frac{7}{2})(s + 3)$ (See Remark in § 5).

(27) $(GL(1) \times E_6,\ \square \otimes \Lambda_1,\ V(1) \otimes V(27))$
$b(s) = (s + 1)(s + 5)(s + 9)$ (See Figure 6–1).

(28) $(E_6 \times GL(2),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(2))$
$b(s) = (s + 1)^2(s + \frac{5}{6})(s + \frac{7}{6})(s + \frac{5}{2})^2(s + 3)^2(s + \frac{9}{2})^2(s + \frac{13}{3})(s + \frac{14}{3})$
(See [12]).

(29) $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$
$b(s) = (s + 1)(s + \frac{11}{2})(s + \frac{19}{2})(s + 14)$ (See Figure 7–1).

We can obtain the b-functions of all irreducible regular P.V.'s, except for those in the castling class of (11), from the Table above and the following theorem due to T. Shintani.

THEOREM (T. Shintani).   *Let (G', ρ', V') be a castling transform of an irreducible regular P.V. (G, ρ, V), i.e., there exists a triplet $(\tilde{G}, \tilde{\rho}, V(m))$ and a positive number n with $m > n \geqq 1$ such that*

$$(G, \rho, V) \cong (\tilde{G} \times GL(n),\ \tilde{\rho} \otimes \Lambda_1,\ V(m) \otimes V(n))$$
$$(G', \rho', V') \cong (\tilde{G} \times GL(m - n),\ \tilde{\rho}^* \otimes \Lambda_1,\ V(m)^* \otimes V(m - n))\ .$$

Then the b-functions $b(s)$ and $b'(s)$ of them satisfy

$$b(s) \prod_{i=1}^{d} (ds - i)(ds - i + 1) \cdots (ds - i + m - n - 1)$$

$$= b'(s) \prod_{i=1}^{d} (ds - i)(ds - i + 1) \cdots (ds - i + n - 1)$$

where $\deg f = dm$ and $\deg f' = d(m - n)$. Here f and f' are the basic relative invariants of (G, ρ, V) and (G', ρ', V') respectively.

Appendix with I. Ozeki

Here we consider the regular irreducible P.V. $(GL(1) \times \text{Spin}\,(14),\ \square \otimes \text{half-spin rep.},\ V(1) \otimes V(64))$. The orbital decomposition of this space has been done by the author and I. Ozeki ([7]), by Popov ([9]), by V. Gatti and E. Viniberghi ([10]). There exist ten orbits, and the conormal bundle

of each orbit is a good Lagrangian variety. The relative invariant of this space is of degree eight ([1]), and its b-function is given by $b(s) = (s + 1)$ $(s + \frac{5}{2})(s + \frac{7}{2})(s + 4)(s + 5)(s + \frac{11}{2})(s + \frac{13}{2})(s + 8)$. Its holonomy diagram is given by Figure A, where we denote by $\boxed{m}$ the conormal bundle Λ of the m-codimensional orbit.
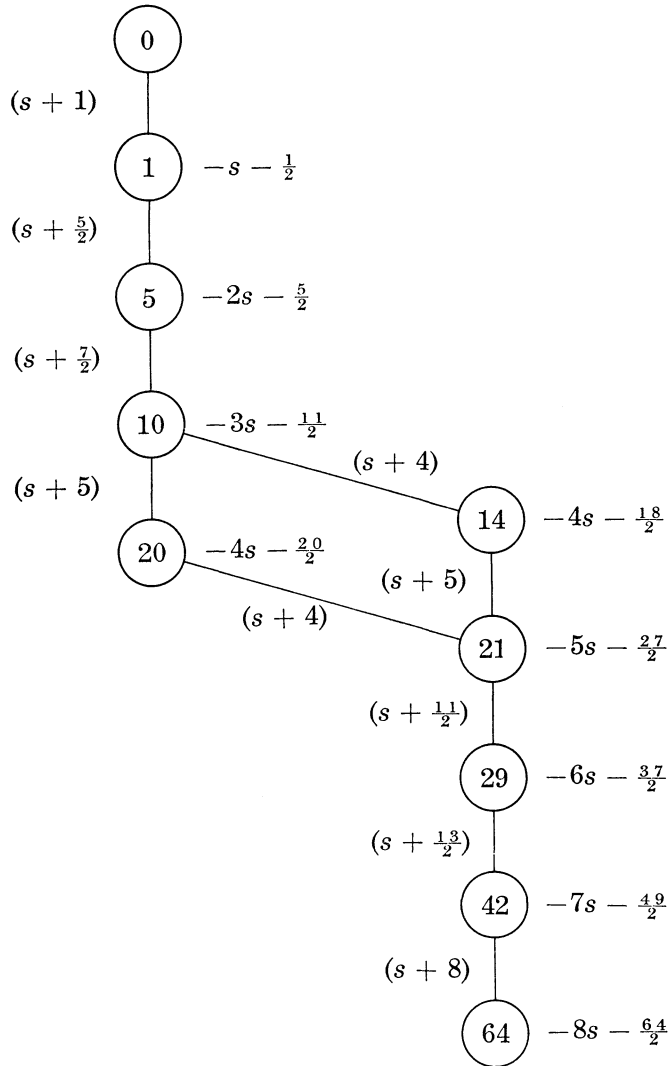

Figure A. Holonomy diagram of $(GL(1) \times \mathrm{Spin}(14),$
$\square \otimes$ half-spin rep., $V(1) \otimes V(64))$

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