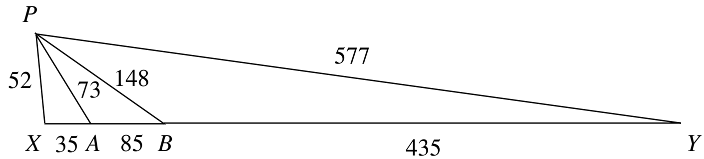
Problem Corner

Solutions are invited to the following problems. They should be addressed to Chris Starr, c/o Kintail, Longmorn, Elgin IV30 8RJ (e-mail: czqstarr@gmail.com) and should arrive not later than 10th August 2025. Proposals for problems are equally welcome. They should also be sent to Chris Starr at the above address and should be accompanied by solutions and any relevant background information.

Note: in order to fulfil any commitments to the publication of problems, I have used the problems suggested by the former editor, Nick Lord. In the transition between editors, if any solutions have not been accounted for then please inform me. CS.

108.I (Chris Starr)

Consider triangle PXY, with $PX = 52$ cm, $PY = 577$ cm and $XY = 555$ cm. The lines PA and PB are constructed such that $XA = 35$ cm and $AB = 85$ cm. It may then be verified that triangle PXY is split into three triangles, each with integer side lengths, and the areas of triangles ΔPXA, ΔPAB and ΔPBY are all integer values.

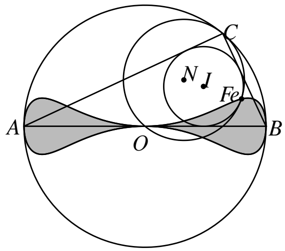


(a) Can you find a triangle PXY that can be split into four triangles with integer side lengths and integer areas?

(b) Is there a value N such that PXY cannot be split into N triangles with integer side lengths and integer areas?

108.J (Mark Hennings)

The points A and B lie on the diameter of a unit circle, and C is a third point on that circle, making a right-angled triangle ABC. The Feuerbach point Fe of a triangle is the point where the triangle's incircle (centre I) and the nine-point circle (centre N) are tangential to each other. The locus of the Feuerbach point as C varies forms an elegant bow-tie shape as below:



What is the area of the region enclosed by the locus?

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108.K (Toyesh Prakash Sharma)

Show that, in the acute-angled triangle ABC, the following inequality holds:

$$(\sin A)^{\cos A}(\sin B)^{\cos B}(\sin C)^{\cos C} \leqslant \left(\frac{27}{64}\right)^{1/4}.$$

108.L (Albert Natian)

You are invited to play the following two-stage game using a fair n-sided die labelled $1, 2, \ldots, n$.

Stage 1: You roll the die to get a number, say x, which is the number of gold coins that you win, and the possession of which is subject to the outcome(s) in Stage 2.

Stage 2: Now you roll the die x times. You win, in gold coins, all numbers that come up in the x rolls, except if any number is x, in which case you lose all your winnings, including that of Stage 1.

Find an expression for the expected winnings $\mathrm{E}[W]$ in this game, and determine

$$\lim_{n \to \infty} \frac{\mathrm{E}[W]}{n^2}.$$

Solutions and comments on **108.A**, **108.B**, **108.C**, **108.D** (March 2024).

108.A (W. C. Gosnell)

The right-angled triangle **T** with legs of length a, b and hypotenuse of length c has inradius r. These lengths satisfy the conditions:

- $r + ka = b$ for some positive rational number, k;
- $c^2 = a + b + c$.

(a) Show that **T** is similar to a triangle whose sides form a Pythagorean triple.

(b) Can all Pythagorean triples be created this way?

Solution:

Most solvers approached part (a) by finding a parametric expression for the three sides. The following is based on the solution by Nick Lord.

A standard result is that the inradius r of a right-angled triangle of hypotenuse c is given by $r = \frac{1}{2}(a + b - c)$. Combining this with the first condition, $r + ka = b$ gives $b + c = (1 + 2k)a$, (*) and using the second condition we obtain:

$$c^2 = a + b + c = a + (1 + 2k)a = (2 + 2k)a.$$

So $c = a\sqrt{(2 + 2k)}$, and using (*) we obtain $b = (1 + 2k)a + a\sqrt{(2 + 2k)}$.

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Substituting these into Pythagoras' Theorem we obtain, after some simplification, the following solutions:

$$a = \frac{2(1 + 2k)^2(1 + k)}{(2k^2 + 2k + 1)^2}, \qquad b = \frac{4k(1 + 2k)(1 + k)^2}{(2k^2 + 2k + 1)^2},$$

$$c = \frac{2(1 + 2k)(1 + k)\left(2k^2 + 2k + 1\right)}{(2k^2 + 2k + 1)^2}.$$

The triangle is therefore similar to that with sides $1 + 2k$, $2k^2 + 2k$, $2k^2 + 2k + 1$ and it may readily be verified that this is a Pythagorean triple.

For part (b), the answer is "yes". Stan Dolan observed that if (a, b, c) is a Pythagorean triple, then clearly $k = \frac{1}{a}(b - r)$ is rational, giving the first condition $r + ka = b$. If the sides are then scaled by $\frac{1}{c^2}(a + b + c)$ the same still holds true, and furthermore, if we define $A = \frac{a+b+c}{c^2}a$, $B = \frac{a+b+c}{c^2}b$, $C = \frac{a+b+c}{c^2}c$, then it may be verified that $A + B + C = C^2$.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliot, M. Hennings, P.F. Johnson, N. Lord, J.A. Mundie, Z. Retkes and the proposer W. C. Gosnell.

108.B (Luc Duc Binh and Dau Anh Hung)

Given a regular n-sided polygon $A_1A_2\ldots A_n$ with centre O and any straight line d. Let points $B_1, \ldots, B_n$ lie on d such that $\overrightarrow{A_1B_1}, \overrightarrow{A_2B_2}, \ldots, \overrightarrow{A_nB_n}$ are parallel vectors. Show that

$$\sum_{i=1}^{n} A_iB_i^2 = \sum_{i=1}^{n} OB_i^2.$$

Solution:

The most common techniques employed for this problem involved complex numbers or vectors, with careful use of trigonometric identities when computing the sums. The following solution, based on that by Zoltan Retkes was striking because of its unexpected physical interpretation at the end.

Let the centre O have coordinates $(0, a)$. For simplicity, relabel the vertices of the polygon $A_0, A_1, \ldots, A_{n-1}$, and let the starting angle of A_0 relative to the x-axis be a. The projections of $A_0, A_1, \ldots, A_{n-1}$ onto the x-axis are $T_0, T_1, \ldots, T_{n-1}$. Without loss of generality, the line d is represented by the x-axis, and the points $B_0, B_1, \ldots, B_{n-1}$ lie on this line. Define the angle between A_rB_r and the x-axis to be θ.

If the radius of the circumscribing circle is 1, then the coordinates of the r-th vertex, A_r are given by $\left(\cos\left(\alpha + \frac{2\pi r}{n}\right), a + \sin\left(\alpha + \frac{2\pi r}{n}\right)\right)$.

We now compute the following:

$$\begin{aligned}\sum_{r=0}^{n-1} A_rB_r^2 &= \sum_{r=0}^{n-1}\left(A_rT_r^2 + T_rB_r^2\right) \\ &= \sum_{r=0}^{n-1}\left[\left(a + \sin\left(\alpha + \frac{2\pi r}{n}\right)\right)^2 + \frac{1}{\tan^2\theta}\left(a + \sin\left(\alpha + \frac{2\pi r}{n}\right)\right)^2\right].\end{aligned}$$

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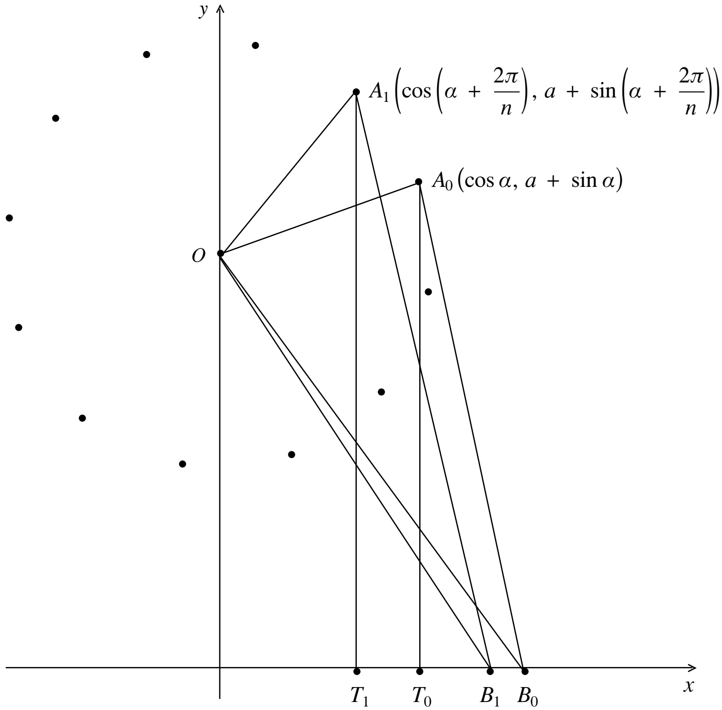

Expanding, and using the identities:

$$\sum_{r=0}^{n-1} \sin(u + rv) = \frac{\sin\frac{1}{2}nv \cos(u + (n-1)v)}{\sin\frac{1}{2}v}, \tag{1}$$

$$\sum_{r=0}^{n-1} \cos(u + rv) = \frac{\sin\frac{1}{2}nv \sin(u + (n-1)v)}{\sin\frac{1}{2}v}. \tag{2}$$

with $u = \alpha$, $v = \frac{2\pi r}{n}$, we obtain

$$\sum_{r=0}^{n-1} A_rB_r^2 = n^2a^2 + \sum_{r=0}^{n-1} \sin^2\left(\alpha + \frac{2\pi r}{n}\right) + \frac{1}{\tan^2\theta}\sum_{r=0}^{n-1}\left[a + \sin\left(\alpha + \frac{2\pi r}{n}\right)\right]^2.$$

In a similar manner,

$$\sum_{r=0}^{n-1} OB_r^2 = \sum_{r=0}^{n-1}\left[a^2 + \left(\cos\left(\alpha + \frac{2\pi r}{n}\right) + \frac{1}{\tan\theta}\left(a + \sin\left(\alpha + \frac{2\pi r}{n}\right)\right)\right)^2\right]$$

$$= n^2a^2 + \sum_{r=0}^{n-1} \cos^2\left(\alpha + \frac{2\pi r}{n}\right) + \frac{1}{\tan^2\theta}\sum_{r=0}^{n-1}\left(a + \sin\left(\alpha + \frac{2\pi r}{n}\right)\right)^2.$$

Subtracting these two gives

$$\sum_{r=0}^{n-1} OB_r^2 - \sum_{r=0}^{n-1} AB_r^2 = \sum_{r=0}^{n-1} \cos^2\left(\alpha + \frac{2\pi r}{n}\right) - \sum_{r=0}^{n-1} \sin^2\left(\alpha + \frac{2\pi r}{n}\right)$$
$$= \sum_{r=0}^{n-1} \cos 2\left(\alpha + \frac{2\pi r}{n}\right)$$

using identity (2) above, this sums to 0 and the equality is thus proved.

The identity, as mentioned previously, has the following physical interpretation. In Figure 1, we have a system of pendulums of mass m pivoted at the vertices of the polygon, and in Figure 2 we have the masses connected with rigid, weightless rods, and the system is allowed to oscillate about the centre of the polygon. The moment of inertia of the first system is $\sum mr_i^2 = m\sum A_iB_i^2$, and the moment of inertia of the second system is $\sum md_i^2 = m\sum OB_i^2$. The identity then shows that these two systems are physically equivalent.

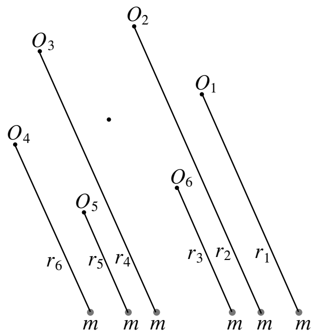

FIGURE 1

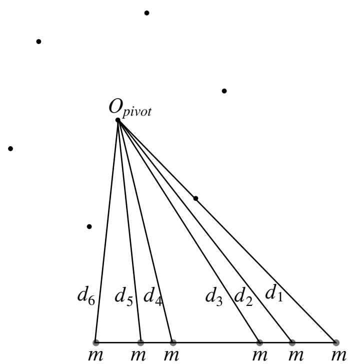

FIGURE 2

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliot, M. Hennings, P.F. Johnson, J. A. Mundie, Z. Retkes, S. Riccarelli and the proposers Luc Duc Binh and Dau Ann Hung.

108.C (George Stoica)

Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ with the property that, for any $a > 0$, the function $x \mapsto f(x).f(a - x)$ is constant on the interval $[0, a]$.

Answer: The zero function or the function $f(0)e^{kx}$.

Most solvers employed a similar strategy to deal with the case $f(x) = 0$, as in the following based on the solution offered by Stan Dolan.

Let $f(a) = 0$ for some $a \in [0, \infty)$. If we substitute a and $\frac{1}{2}a$ into the function, and remember from the definition that $f(x) \cdot f(a - x)$ is constant on the interval $[0, a]$, we obtain $f\left(\frac{1}{2}a\right) = f(a) \cdot f(0) = 0$, therefore

$f\left(\frac{1}{2}a\right) = 0$. Repeating this argument, we obtain the series of roots $\frac{1}{2}a$, $\frac{1}{4}a$, $\frac{1}{8}a$, … of roots of f which converge to 0. So, by continuity, $f(0) = 0$. But then, for any $y \in [0, \infty)$, $f(y)^2 = f(2y) \cdot f(0) = 0$, and $f(y) = 0$. Hence, f is identically zero.

Solvers used careful reasoning to deal with the case $f(x) \neq 0$. The proposer, George Stoica, offered this solution.

Using $a = x_1 + x_2, x = x_3$, then $a = x_1 + x_2, x = 0$ in the definition of the function, we obtain

$$f(x_1) \cdot f(x_2) = f(0) \cdot f(x_1 + x_2).$$

Consider the function $g(x) = \ln|f(x)| - \ln|f(0)|$. This function is well-defined and continuous on $[0, \infty)$ since $f(x) \neq 0$. We also have

$$g(x_1) + g(x_2) = \ln|f(x_1)| + \ln|f(x_2)| - 2\ln|f(0)|$$

$$\equiv \ln\left|\frac{f(x_1)f(x_2)}{f(0)}\right| - \ln|f(0)| = \ln|f(x_1 + x_2)| - \ln|f(0)| = g(x_1 + x_2).$$

Therefore, by a well-known result of Cauchy, we have $g(x) = kx$ for some real k. Therefore we must have $|f(x)| = |f(0)|e^{kx}$ or $f(x) = \pm|f(0)|e^{kx}$. If we let $x = 0$ in this equation we obtain $f(0) = \pm|f(0)|$, meaning the solution is $f(x) = f(0)e^{kx}$ as stated.

Correct solutions were received from: U. Abel, N. Curwen, S. Dolan, M. Ecker, M. G. Elliot, M. Hennings, P. F. Johnson, R. Mortini, J.A. Mundie, Z. Retkes, A. Sasane, and the proposer G. Stoica.

108.D (Toyesh Prakash Sharma)

(a) Show that $\int_0^\infty \left(1 - e^{-2x}\right)\frac{\sin^2 x}{x^3} = \frac{\pi}{2}$

(b) Show that $\int_{-\infty}^{\infty} \cos^2(\tan x)\frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}\left(1 + \frac{1}{e^2}\right) = \int_{-\infty}^{\infty} \sin^2(\tan x)\frac{\cos^2 x}{x^2}dx.$

Solution:

These integrals certainly provided a challenge, and solvers were ingenious in their varied approaches. By substituting x for $2x$ and employing half-angle formulae, the proposer established that the integral I can be rewritten in the form

$$I = 2\int_0^\infty \frac{(1 - e^{-x})}{x}\frac{(1 - \cos x)}{x^2}dx = 2\int_0^\infty \int_0^1 e^{-tx}\frac{(1 - \cos x)}{x^2}dt\,dx$$
$$= 2\int_0^1 \mathscr{L}\left[\frac{1 - \cos t}{t^2}\right]dx$$

where $\mathscr{L}[f(t)]$ represents the Laplace transform.

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Since $\mathcal{L}[1 - \cos t] = \dfrac{1}{s} - \dfrac{s}{1 + s^2}$, then an application of the division property gives

$$\mathcal{L}\left[\frac{1 - \cos t}{t}\right] = \int_t^\infty \left(\frac{1}{s} - \frac{s}{1 + s^2}\right) ds = \frac{1}{2}\log\left(1 + \frac{1}{t^2}\right).$$

A second application of the division property gives

$$\mathcal{L}\left[\frac{1 - \cos t}{t^2}\right] = \int_t^\infty \frac{1}{2}\log\left(1 + \frac{1}{s^2}\right) ds = \frac{1}{2}t\log\left(1 + \frac{1}{t^2}\right) + \cot^{-1} t$$

by using integration by parts. Finally,

$$I = 2\int_0^1 \left(\frac{1}{2}t\log\left(1 + \frac{1}{t^2}\right) + \cot^{-1} t\right) dt.$$

The second part of this integral is a standard result, and the first part may be tackled easily by integration by parts. The details are omitted, but we obtain

$$I = 2\left[-\frac{1}{4}t^2\log\left(1 + t^2\right) + \frac{1}{2}\log\left(1 + t^2\right) - \frac{1}{2}t^2\log t + t\cot^{-1} t - \frac{1}{4}\log\left(1 + t^2\right)\right]_0^1$$

which is indeed equal to $\frac{1}{2}\pi$.

For part (b), the equivalence of the two integrals was established in a very neat way by Nick Lord. First, define

$$I = \int_{-\infty}^\infty \sin^2(\tan x)\frac{\sin^2 x}{x^2}dx \text{ and } J = \int_{-\infty}^\infty \cos^2(\tan x)\frac{\cos^2 x}{x^2}dx.$$

Then by a simple application of Pythagorean identities, it can be proved that

$$J = \int_{-\infty}^\infty \frac{\sin^2 x - \sin^2(\tan x)}{x^2}dx + I.$$

Therefore, equivalence can be established if it can be proved that

$$\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}dx = \int_{-\infty}^\infty \frac{\sin^2(\tan x)}{x^2}dx.$$

We have

$$\int_{-\infty}^\infty \frac{\sin^2(\tan x)}{x^2}dx = \int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\cdot\frac{\sin^2(\tan x)}{\sin^2 x}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(\tan x)}{\sin^2 x}dx$$

by Lobachewsky's Theorem.

If we now substitute $t = \tan x$ and recall that $\sin^2 x = \dfrac{\tan^2 x}{1 + \tan^2 x}$, then the integral may be rewritten as

$$\int_{-\infty}^\infty \frac{\sin^2 t}{\frac{t^2}{1 + t^2}}\cdot\frac{1}{1 + t^2}\,dt = \int_{-\infty}^\infty \frac{\sin^2 t}{t^2}\,dt,$$

thus proving equivalence.

In order to evaluate the integral, apply Lobachewsky's theorem to J to obtain $J = \int_{-\pi/2}^{\pi/2} \cos^2(\tan x)\, dx$ and then substitute $t = \tan x$ and a double angle formula to get

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{1}{1 + t^2} + \frac{\cos 2t}{1 + t^2}\right) dt.$$

The first part of this integral is $\frac{1}{2}\pi$ using standard results, and the second part was attempted through some deft manipulation involving differentiating under the integral sign. However, contour integration can be used efficiently by considering the function $f(z) = \frac{e^{2iz}}{1 + z^2}$ and the closed semicircular contour Γ centre O extending from $-R$ to R on the real axis. We then have simple poles at $z = \pm i$, with only $z = i$ within the contour, and the residue is $e^{-2}/2i$ by the "g/h rule". The contribution to the integral of the circular arc can be estimated thus:

$$\left|\int_0^{\pi} \frac{e^{-2R\sin\theta + 2iR\cos\theta}}{R^2 e^{2i\theta} + 1} iRe^{i\theta}\, d\theta\right| \leqslant \int_0^{\pi} \frac{R}{R^2 - 1} d\theta \to 0 \text{ as } R \to \infty,$$

since $\sin\theta \geqslant 0$ on $[0, \pi]$.

So, using the residue theorem, we have $\int_\Gamma \frac{e^{2iz}}{1 + z^2} dz = 2\pi i \frac{e^{-2}}{2i} = \frac{\pi}{e^2}$, and evaluating real parts gives the result $\frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos 2x}{1 + x^2} dx = \frac{\pi}{2e^2}$. Combining this with the other result gives the answer $\frac{\pi}{2}\left(1 + \frac{1}{e^2}\right)$.

Correct solutions were received from: M. Hennings, N. Lord, R. Mortini, J. A. Mundie, Z. Retkes and R. Rupp and the proposer Toyesh Prakash Sharma.

C. STARR

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