Compositio Math. 142 (2006) 1063–1080
doi:10.1112/S0010437X06002053

Chow motives of twisted flag varieties

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Abstract

Let G be an adjoint simple algebraic group of inner type. We express the Chow motive (with integral coefficients) of an anisotropic projective G-homogeneous variety in terms of motives of simpler G-homogeneous varieties, namely, those that correspond to maximal parabolic subgroups of G. We decompose the motive of a generalized Severi–Brauer variety $\mathrm{SB}_2(A)$ of a division algebra A of degree 5 into a direct sum of twisted motives of the Severi–Brauer variety $\mathrm{SB}(B)$ of a division algebra B Brauer-equivalent to the tensor square $A^{\otimes 2}$. As an application we provide another counter-example to the uniqueness of a direct sum decomposition in the category of motives with integral coefficients.

1. Introduction

Let G be an adjoint simple algebraic group of inner type over a field F. Let X be a twisted flag variety, i.e. a projective G-homogeneous variety over F. The main purpose of the paper is to express the Chow motive of X in terms of motives of 'minimal' flags, i.e. those G-homogeneous varieties that correspond to maximal parabolic subgroups of G.

Observe that the motive of an *isotropic* G-homogeneous variety can be decomposed in terms of motives of simpler G-homogeneous varieties using the techniques developed by Chernousov *et al.* [CGM05] and Karpenko [Kar01]. For G-varieties, when G is *isotropic*, one obtains a similar decomposition following the arguments of Brosnan [Bro03]. In the case of G-varieties, where G is *anisotropic*, no general decomposition methods are known except several particular cases of quadrics (see, for example, Rost [Ros98]), Severi–Brauer varieties (see Karpenko [Kar95]) and exceptional varieties of type F_4 (see [NSZ05]).

In the present paper we provide methods that allow us to decompose the motives of some *anisotropic* twisted flag G-varieties, where the root system of G is of types A_n, B_n, C_n, G_2 and F_4, i.e. has a Dynkin diagram that does not branch.

As an application, we provide another counter-example to the uniqueness of a direct sum decomposition in the category of Chow motives with integral coefficients (see Corollary 2.7). Observe that such a counter-example was already given in [CM04, Example 9.4] using a G-homogeneous variety, where G is a product of two simple groups. Our example is given by a G-variety, where G is a *simple* group. Apart from this, our example involves a motivic decomposition

$$\mathcal{M}(\mathrm{SB}_2(A)) \simeq \mathcal{M}(\mathrm{SB}(B)) \oplus \mathcal{M}(\mathrm{SB}(B))(2),$$

of the motive of a generalized Severi–Brauer variety $\mathrm{SB}_2(A)$ of a division algebra A of degree 5 into a direct sum of twisted motives of the Severi–Brauer variety $\mathrm{SB}(B)$ of a division algebra B

Received 4 August 2005, accepted in final form 16 November 2005.
2000 Mathematics Subject Classification 57T15, 19E15.
Keywords: Chow motive, anisotropic projective homogeneous variety.
Partially supported by DAAD *Jahresstipendium* program, INTAS projects 00-566, 03-51-3251, RTN Network HPRN-CT-2002-00287 and AvH Foundation.

Brauer-equivalent to the tensor square $A^{\otimes 2}$. Observe that the motive $\mathcal{M}(\mathrm{SB}(B))$ is isomorphic to the motive $\mathcal{M}(\mathrm{SB}(A))$ over $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}[\frac{1}{3}]$, but not integrally.

The paper is organized as follows. In § 2 we state the main results. We then provide some technical facts that are extensively used in the proofs (§ 3). In the other sections we give proofs of the results for varieties of type A_n (§ 4), of types B_n and C_n (§ 6), and exceptional varieties of types G_2 and F_4 (§ 7). Section 5 is devoted to the motivic decomposition of generalized Severi–Brauer varieties.

Notation and conventions

By G we denote an adjoint simple algebraic group of inner type over a field F and by n its rank. By F_s we denote the separable closure of F and by X_s the respective base change $X_s = X \times_F F_s$ of a variety X. All varieties that appear in the paper are projective G-homogeneous varieties over F. They are twisted forms of the varieties G'/P, where G' is the split adjoint simple group of the same type as G and P its parabolic subgroup. The Chow motive of a variety X is denoted by $\mathcal{M}(X)$. By A we denote a central simple algebra over F of index $\mathrm{ind}(A)$ and by $\mathrm{SB}(A)$ the corresponding Severi–Brauer variety. I is always a right ideal of A and $\mathrm{rdim}\, I$ stands for its reduced dimension. V is a vector space over F.

By $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ we denote a partition $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l \geqslant 0$ with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. Integers $d_1, d_2, \ldots, d_k$ always satisfy the condition $1 \leqslant d_1 < d_2 < \cdots < d_k \leqslant n$ and are the dimensions of some flag. For each $i = 0, \ldots, k$ we define δ_i to be the difference $d_{i+1} - d_i$ (assuming here $d_0 = 0$ and $d_{k+1} = n + 1$).

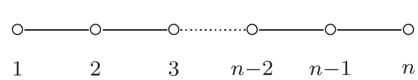
2. Statements of results

We follow [MPW96, Appendix] and [CG05] for the description of projective G-homogeneous varieties that appear below. According to the type of the group G, we obtain the following results.

$\mathbf{A}_{\boldsymbol{n}}$ In this case $G = \mathrm{PGL}_1(A)$, where A is a central simple algebra of degree $n + 1$, $n > 0$, and the set of F-points of a projective G-homogeneous variety X can be identified with the set of flags of (right) ideals

$$X(d_1, \ldots, d_k) = \{I_1 \subset I_2 \subset \cdots \subset I_k \subset A\}$$

of fixed reduced dimensions $1 \leqslant d_1 < d_2 < \cdots < d_k \leqslant n$. Observe that this variety is a twisted form of G'/P, where $G' = \mathrm{PGL}_{n+1}$ and P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i.



The following result reduces the computation of the motive of X to the motives of 'smaller' flags.

Theorem 2.1. *Suppose that* $\gcd(\mathrm{ind}(A), d_1, \ldots, \hat{d}_m, \ldots, d_k) = 1$, *then*

$$\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \ldots, \hat{d}_m, \ldots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_{m-1}})$ *such that* $\delta_m \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{\delta_{m-1}} \geqslant 0$.

Proof. See § 4.8. □

As a consequence, for the variety of complete flags we obtain the following corollary.

Corollary 2.2. *The motive of the variety $X = X(1,\dots,n)$ of complete flags is isomorphic to*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{n(n-1)/2} \mathcal{M}(\mathrm{SB}(A))(i)^{\oplus a_i},$$

where a_i are the coefficients of the polynomial $\varphi_n(z) = \sum_i a_i z^i = \prod_{k=2}^{n}(z^k-1)/(z-1)$.

Proof. Apply Theorem 2.1 recursively to the sequence of varieties $X(1,\dots,n), X(1,\dots,n-1),\dots,$ $X(1,2)$ and $X(1) = \mathrm{SB}(A)$. □

Another interesting example is the 'incidence' variety $X(1,n)$.

Corollary 2.3. *The motive of $X(1,n)$ is isomorphic to*

$$\mathcal{M}(X(1,n)) \simeq \bigoplus_{i=0}^{n-1} \mathcal{M}(\mathrm{SB}(A))(i).$$

In order to complete the picture we need to know how to decompose the motive of a 'minimal' flag, i.e. a generalized Severi–Brauer variety.

Note that for some rings of coefficients (fields, discrete valuation rings) one easily obtains the desired decomposition by using the Krull–Schmidt theorem (the uniqueness of a direct sum decomposition). More precisely, consider the subcategory $\mathcal{M}(G,R)$ of the category of motives with coefficients in a ring R that is a pseudo-abelian completion of the category of motives of projective G-homogeneous varieties (see [CM04, § 8]). Then we have the following.

Proposition 2.4. *Let $X(d) = \mathrm{SB}_d(A)$, $1 < d < n$, be a generalized Severi–Brauer variety for a central simple algebra A of degree $n+1$ such that $\gcd(\mathrm{ind}(A), d) = 1$. Let R be a ring such that the Krull–Schmidt theorem holds in the category $\mathcal{M}(G,R)$. Then the motive of $\mathrm{SB}_d(A)$ with coefficients in R is isomorphic to*

$$\mathcal{M}(\mathrm{SB}_d(A)) \simeq \bigoplus_{i\in\mathcal{I}} \mathcal{M}(\mathrm{SB}(A))(i)^{\oplus a_i},$$

where the integers a_i are the coefficients of the polynomial $\varphi_n(z)/\varphi_d(z)\varphi_{n+1-d}(z)$ at terms z^i and the set of indices $\mathcal{I} = \{i \mid a_i \neq 0\}$.

Proof. See § 4.10. □

It turns out that the motives of some generalized Severi–Brauer varieties with integral coefficients can still be decomposed.

Theorem 2.5. *Let $\mathrm{SB}_2(A)$ be a generalized Severi–Brauer variety for a division algebra A of degree 5. Then there is an isomorphism*

$$\mathcal{M}(\mathrm{SB}_2(A)) \simeq \mathcal{M}(\mathrm{SB}(B)) \oplus \mathcal{M}(\mathrm{SB}(B))(2),$$

where B is a division algebra Brauer-equivalent to the tensor square $A^{\otimes 2}$.

Proof. See § 5.11. □

Remark 2.6. It is expected that the mod-p version of this theorem can also be proven using techniques dealing with norm varieties. Namely, if the algebra A is cyclic, then it corresponds to a symbol in $\mathrm{K}_2^M(F)/5$ that is split by the variety $\mathrm{SB}_2(A)$ (see [Sus05]). By the results of Voevodsky [Voe03] the motive of $\mathrm{SB}_2(A)$ with $\mathbb{Z}/5\mathbb{Z}$-coefficients splits.

As an immediate consequence of Theorems 2.1 and 2.5 we obtain the following.

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Corollary 2.7. *The Krull–Schmidt theorem fails in the category of motives $\mathcal{M}(\mathrm{PGL}_1(A), \mathbb{Z})$ where A is a division algebra of degree 5.*

Proof. Apply Theorem 2.1 recursively to the sequences of varieties $X(1,2), X(1)$ and $X(1,2), X(2)$, where $X(1,2)$ is the twisted flag G-variety for $G = \mathrm{PGL}_1(A)$. We obtain two decompositions of the motive of $X(1,2)$

$$\bigoplus_{i=0}^{3} \mathcal{M}(\mathrm{SB}(A))(i) \simeq \mathcal{M}(X(1,2)) \simeq \mathcal{M}(\mathrm{SB}_2(A)) \oplus \mathcal{M}(\mathrm{SB}_2(A))(1).$$

Apply now Theorem 2.5 to the components of the second decomposition. We obtain two decompositions of the motive $\mathcal{M}(X(1,2))$

$$\bigoplus_{i=0}^{3} \mathcal{M}(\mathrm{SB}(A))(i) \simeq \mathcal{M}(X(1,2)) \simeq \bigoplus_{i=0}^{3} \mathcal{M}(\mathrm{SB}(B))(i). \qquad (*)$$

By [Kar95, Theorem. 2.2.1] and [Kar00, Criterion 7.1] the motives $\mathcal{M}(\mathrm{SB}(A))$ and $\mathcal{M}(\mathrm{SB}(B))$ are indecomposable and non-isomorphic. This finishes the proof of the corollary. □

Remark 2.8. Observe that the counter-example provided by Chernousov and Merkurjev (see [CM04, Example 9.4]) is the product of two Severi–Brauer varieties $X = \mathrm{SB}(A) \times \mathrm{SB}(B)$, which is a G-homogeneous variety for the *semi-simple* group $G = \mathrm{PGL}_1(A) \times \mathrm{PGL}_1(B)$, where A and B are two division algebras of degree 5 generating the same subgroup in the Brauer group. The example that we provide, i.e. the flag $X(1,2)$, is a G-homogeneous variety for the *simple* group $G = \mathrm{PGL}_1(A)$. Moreover, it implies that the *cancellation property* fails in the category of Chow motives $\mathcal{M}(G, \mathbb{Z})$. Indeed, on the one hand we have two different decompositions into indecomposable objects according to [CM04, Example 9.4]

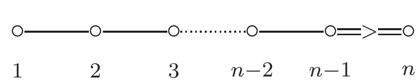
$$\bigoplus_{i=0}^{4} \mathcal{M}(\mathrm{SB}(A))(i) \simeq \mathcal{M}(\mathrm{SB}(A) \times \mathrm{SB}(B)) \simeq \bigoplus_{i=0}^{4} \mathcal{M}(\mathrm{SB}(B))(i),$$

where B is a division algebra of degree 5 Brauer-equivalent to $A^{\otimes 2}$. On the other hand we have two decompositions $(*)$ of Corollary 2.7.

$\mathbf{B}_{\boldsymbol{n}}$ We assume that the characteristic of the base field F is different from 2. It is known that $G = \mathrm{O}^+(V,q)$, where (V,q) is a regular quadratic space of dimension $2n+1$, $n > 0$, and projective G-homogeneous varieties can be described as flags of totally q-isotropic subspaces

$$X(d_1, \dots, d_k) = \{V_1 \subset \cdots \subset V_k \subset V\}.$$

of fixed dimensions $1 \leqslant d_1 < \cdots < d_k \leqslant n$. Observe that this variety is a twisted form of G'/P, where G' is a split group of the same type as G and P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i.



The following result shows that some motives of flag varieties can be decomposed into a direct sum of twisted motives of ‘smaller’ flags.

Theorem 2.9. *Suppose that $m < k$, then*

$$\mathcal{M}(X(d_1, \dots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \dots, \hat{d}_m, \dots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \dots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{\delta_{m-1}} \geqslant 0$.

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Proof. See § 6.5. ☐

In particular, for the variety of complete flags we obtain a formula similar to that of Corollary 2.2.

Corollary 2.10. *The motive of the variety of complete flags $X = X(1, 2, \ldots, n)$ is isomorphic to*

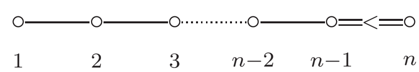
$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{n(n-1)/2} \mathcal{M}(X(n))(i)^{\oplus a_i},$$

where the a_i are the coefficients of the polynomial $\varphi_n(z) = \sum_i a_i z^i = \prod_{k=2}^{n}(z^k - 1)/(z - 1)$ and $X(n)$ is the twisted form of the maximal orthogonal Grassmannian.

$\mathbf{C}_n$ We assume that the characteristic of the base field F is different from 2. In this case $G = \mathrm{Aut}(A, \sigma)$, where A is a central simple algebra of degree $2n$, $n \geqslant 2$, with an involution σ of symplectic type on A, and a projective G-homogeneous variety can be described as the set of flags of (right) ideals

$$X(d_1, \ldots, d_k) = \{I_1 \subset \cdots \subset I_k \subset A \mid I_i \subseteq I_i^{\perp}\}$$

of fixed reduced dimensions $1 \leqslant d_1 < \cdots < d_k \leqslant n$, where $I^{\perp} = \{x \in A \mid \sigma(x)I = 0\}$ is the right ideal of reduced dimension $2n - \mathrm{rdim}\, I$. Observe that this variety is a twisted form of G'/P, where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i.



Again, the motives of some flag varieties can be decomposed into a direct sum of twisted motives of 'smaller' flags.

Theorem 2.11. *Suppose that d_i is odd for some $i < k$ and $d_k - d_{k-1} = 1$. Then*

$$\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{i=0}^{2n-2d_{k-1}-1} \mathcal{M}(X(d_1, \ldots, d_{k-1}))(i).$$

In particular, for the variety of complete flags we obtain the following.

Corollary 2.12. *The motive of the variety of complete flags $X = X(1, 2, \ldots, n)$ is isomorphic to*

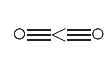
$$\mathcal{M}(X(1, \ldots, n)) \simeq \bigoplus_{i=0}^{n(n-1)} \mathcal{M}(\mathrm{SB}(A))(i)^{\oplus a_i},$$

where a_i are the coefficients of the polynomial $\psi_n(z) = \prod_{k=1}^{n-1}(z^{2k} - 1)/(z - 1)$.

$\mathbf{G_2}$ We suppose that the characteristic of F is not 2. It is known that $G = \mathrm{Aut}(C)$, where C is a Cayley algebra over F. By an *i-space*, where $i = 1, 2$, we mean an i-dimensional subspace V_i of C such that $uv = 0$ for every $u, v \in V_i$. The only flag variety corresponding to a non-maximal parabolic subgroup is the variety of complete flags $X(1, 2)$, which is described as follows

$$X(1, 2) = \{V_1 \subset V_2 \mid V_i \text{ is a } i\text{-subspace of } C\}.$$

We enumerate the simple roots on the Dynkin diagram as follows.



In this case we obtain the following.

Theorem 2.13. *The motive of the variety of complete flags $X = X(1,2)$ is isomorphic to*

$$\mathcal{M}(X) \simeq \mathcal{M}(X(2)) \oplus \mathcal{M}(X(2))(1).$$

Proof. See § 7.5. □

Observe that by the result of Bonnet [Bon03] the motives of $X(1)$ and $X(2)$ are isomorphic (here $X(1)$ is a five-dimensional quadric).

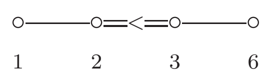
$\mathbf{F_4}$ We suppose that the characteristic of F is neither 2 nor 3. It is known that $G = \mathrm{Aut}(J)$, where J is an exceptional Jordan algebra of dimension 27 over F. Set $\mathcal{I} = \{1, 2, 3, 6\}$. By an *$i$-space*, $i \in \mathcal{I}$, we mean an i-dimensional subspace V of J such that every $u, v \in V$ satisfy the following condition:

$$\mathrm{Tr}(u) = 0, \quad u \times v = 0, \quad \text{and if } i < 6, \text{ then } u(va) = v(ua) \text{ for all } a \in J.$$

A projective G-homogeneous variety can be described as the set of flags of subspaces

$$X(d_1, \ldots, d_k) = \{V_1 \subset \cdots \subset V_k \mid V_i \text{ is a } d_i\text{-subspace of } J\}.$$

where the integers $d_1 < \cdots < d_k$ are taken from the set $\mathcal{I}$. Observe that this variety is a twisted form of G'/P, where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i.



In this case we obtain the following.

Theorem 2.14. *Suppose that $m < k$ and either $d_{m+1} < 6$ or $d_m = 1$, then*

$$\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \ldots, \hat{d}_m, \ldots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{\delta_{m-1}} \geqslant 0$.

Proof. See § 7.10. □

3. Preliminaries

In the present section, we introduce the category of Chow motives following [Man68]. We formulate the Grassmann bundle theorem (see [Koc91, Theorem 3.2]) and recall the notion of a functor of points following [Kar01, § 8].

3.1 Chow motives

Let F be a field and $\mathcal{V}ar_F$ be the category of smooth projective varieties over F. We define the category $\mathcal{C}or_F$ of *correspondences* over F. Its objects are non-singular projective varieties over F. For morphisms, called correspondences, we set $\mathrm{Mor}(X, Y) := \mathrm{CH}^{\dim X}(X \times Y)$. For any two correspondences $\alpha \in \mathrm{CH}(X \times Y)$ and $\beta \in \mathrm{CH}(Y \times Z)$ we define the composition $\beta \circ \alpha \in \mathrm{CH}(X \times Z)$

$$\beta \circ \alpha = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\alpha) \cdot \mathrm{pr}_{23}^*(\beta)),$$

where pr_{ij} denotes the projection on product of the ith and jth factors of $X \times Y \times Z$ and pr_{ij*}, pr_{ij}^* denote the induced push-forwards and pull-backs for Chow groups, respectively. Observe that the composition $\circ$ induces a ring structure on the abelian group $\mathrm{CH}^{\dim X}(X \times X)$. The unit element of this ring is the class of the diagonal Δ_X.

The pseudo-abelian completion of $\mathcal{C}or_F$ is called the category of *Chow motives* and is denoted by $\mathcal{M}_F$. The objects of $\mathcal{M}_F$ are pairs (X, p), where X is a non-singular projective variety and p is a projector, that is, $p \circ p = p$. The motive (X, Δ_X) will be denoted by $\mathcal{M}(X)$.

3.2 By the construction $\mathcal{M}_F$ is a self-dual tensor additive category, where the duality is given by the transposition of cycles $\alpha \mapsto \alpha^t$ and the tensor product is given by the usual fiber product $(X, p) \otimes (Y, q) = (X \times Y, p \times q)$. Moreover, the Chow functor $\mathrm{CH} : \mathcal{V}ar_F \to \mathbb{Z}\text{-}\mathcal{A}b$ (to the category of $\mathbb{Z}$-graded abelian groups) factors through $\mathcal{M}_F$, i.e. one has the commutative diagram of functors

$$\begin{array}{ccc} \mathcal{V}ar_F & \xrightarrow{\ \mathrm{CH}\ } & \mathbb{Z}\text{-}\mathcal{A}b \\ & {}_{\Gamma}\searrow \quad \nearrow_{R} & \\ & \mathcal{M}_F & \end{array}$$

where $\Gamma : f \mapsto \Gamma_f$ is the graph and the functor R is given by $R : (X, p) \mapsto \mathrm{Im}(p^*)$, where p^* is the composition

$$p^* : \mathrm{CH}(X) \xrightarrow{\mathrm{pr}_1^*} \mathrm{CH}(X \times X) \xrightarrow{\cdot p} \mathrm{CH}(X \times X) \xrightarrow{\mathrm{pr}_{2*}} \mathrm{CH}(X).$$

3.3 Consider the morphism $(\mathrm{id}, e) : \mathbb{P}^1 \times \{pt\} \to \mathbb{P}^1 \times \mathbb{P}^1$. The image of the induced push-forward $(\mathrm{id}, e)_*$ does not depend on the choice of a point $e : \{pt\} \to \mathbb{P}^1$ and defines the projector in $\mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ denoted by p_1. The motive $L = (\mathbb{P}^1, p_1)$ is called the Lefschetz motive. For a motive M and an nonnegative integer i we denote by $M(i) = M \otimes L^{\otimes i}$ its twist. Observe that

$$\mathrm{Mor}((X, p)(i), (Y, q)(j)) = q \circ \mathrm{CH}^{\dim X + i - j}(X, Y) \circ p.$$

3.4 Grassmann bundle theorem

Let X be a variety over F and $\mathcal{E}$ be a vector bundle over X of rank n. Then the motive of the Grassmann bundle $\mathrm{Gr}(d, \mathcal{E})$ over X is isomorphic to

$$\mathcal{M}(\mathrm{Gr}(d, \mathcal{E})) \simeq \bigoplus_{\lambda} \mathcal{M}(X)(d(n-d) - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \dots, \lambda_d)$ such that $n - d \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_d \geqslant 0$.

3.5 Functors of points

In §§ 4, 6 and 7 we use the functorial language, that is, consider F-schemes as functors from the category of F-algebras to the category of sets. Fix a scheme X. By an X-algebra we mean a pair (R, x), where R is a F-algebra and x is an element of $X(R)$. X-algebras form a category with obvious morphisms. Morphisms $\varphi \colon Y \to X$ can be considered as functors from the category of X-algebras to the category of sets, by sending a pair (R, x) to its preimage in $Y(R)$.

3.6 Let X be a variety over F. To any vector bundle $\mathcal{F}$ over X we can associate the Grassmann bundle $Y = \mathrm{Gr}(d, \mathcal{F})$. Fix an X-algebra (R, x). The value of the functor corresponding to $\mathrm{Gr}(d, \mathcal{F})$ at (R, x) is the set of direct summands of rank d of the projective R-module $\mathcal{F}_x \otimes_F R$, where $\mathcal{F}_x = \mathcal{F}(R, x)$.

4. Groups of type $\mathbf{A}_n$

The goal of the present section is to prove Theorem 2.1 and Proposition 2.4. We use the notation of § 2.

4.1 Let G be an adjoint group of inner type A_n defined over a field F. It is well known that $G = \mathrm{PGL}_1(A)$, where A is a central simple algebra of degree $n+1$ and points of projective G-homogeneous varieties are flags of (right) ideals of A

$$X(d_1,\dots,d_k) = \{I_1 \subset \cdots \subset I_k \subset A \mid \operatorname{rdim} I_i = d_i\}.$$

For convenience we set $d_0 = 0$, $d_{k+1} = n+1$, $I_0 = 0$, $I_{k+1} = A$.

4.2 The value of the functor of points corresponding to the variety $X(d_1,\dots,d_k)$ at a F-algebra R (see § 3.5) equals the set of all flags $I_1 \subset \cdots \subset I_k$ of right ideals of $A_R = A \otimes_F R$ having the following properties (see [IK00, § 4]):

- the injection of A_R-modules $I_i \hookrightarrow A_R$ splits;
- $\operatorname{rdim} I_i = d_i$ (rdim means reduced rank over R).

4.3 On the scheme $X = X(d_1,\dots,d_k)$ there are 'tautological' vector bundles $\mathcal{J}_i$, $i = 0,\dots,k+1$, of ranks $(n+1)d_i$. The value of $\mathcal{J}_i$ on an X-algebra (R,x), where $x = (I_1,\dots,I_k)$, is the ideal I_i considered as a projective R-module. The bundle $\mathcal{J}_i$ also has a structure of right A_X-module, where A_X is the constant sheaf of algebras on X determined by A.

For every $m \in \{1,\dots,k\}$ there exists an obvious morphism

$$X(d_1,\dots,d_k) \to X(d_1,\dots,\hat{d}_m,\dots,d_k)$$
$$(I_1,\dots,I_k) \mapsto (I_1,\dots,\hat{I}_m,\dots,I_k)$$

that turns $X(d_1,\dots,d_k)$ into a $X(d_1,\dots,\hat{d}_m,\dots,d_k)$-scheme.

Lemma 4.4. *Denote $X(d_1,\dots,d_k)$ by Y and $X(d_1,\dots,\hat{d}_m,\dots,d_k)$ by X. Assume there exists a vector bundle $\mathcal{E}$ over X such that $A_X \simeq \operatorname{End}_{\mathcal{O}_X}(\mathcal{E})$. Consider the vector bundle*

$$\mathcal{F} = \mathcal{J}_{m+1}\mathcal{E}/\mathcal{J}_{m-1}\mathcal{E} = \mathcal{J}_{m+1}/\mathcal{J}_{m-1} \otimes_{A_X} \mathcal{E}$$

of rank $d_{m+1} - d_{m-1}$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z = \mathrm{Gr}(d_m - d_{m-1}, \mathcal{F})$ over X.

Proof. We use essentially the same method as in [IK00, Proposition 4.3].

Fix an X-algebra (R,x) where $x = (I_1,\dots,\hat{I}_m,\dots,I_k)$. The fiber of Y over x, i.e. the value at (R,x), can be identified with the set of all ideals I_m satisfying the conditions in § 4.2 such that $I_{m-1} \subset I_m \subset I_{m+1}$. The fiber of Z over x is the set of all R-submodules N of $\mathcal{F}_y = \mathcal{F}(R,y)$ such that the injection $N \hookrightarrow \mathcal{F}_y$ splits and $\operatorname{rk}_R N = d_m - d_{m-1}$.

We define a natural bijection between the fibers of Y and Z over x as follows.

Consider the following mutually inverse bijections between the set of all right ideals of reduced dimension r in A_R (satisfying § 4.2) and the set of all direct summands of rank r of the R-module $\mathcal{E}_x$

$$\Phi : I \mapsto I\mathcal{E}_x$$
$$\Psi : N \mapsto \operatorname{Hom}_R(\mathcal{E}_x, N) \subset \operatorname{End}_R(\mathcal{E}_x) \simeq A_R.$$

Observe that these bijections preserve the respective inclusions of ideals and modules. So ideals of reduced dimension d_m between I_{m-1} and I_m correspond to submodules of rank d_m between $I_{m-1}\mathcal{E}_x$ and $I_{m+1}\mathcal{E}_x$, and, therefore, to submodules of rank $d_{m+1}-d_{m-1}$ in $I_{m+1}\mathcal{E}_x/I_{m-1}\mathcal{E}_x=\mathcal{F}_x$. This gives the desired natural bijection on the fibers. □

Lemma 4.5. *Suppose that* $\gcd(\operatorname{ind}(A),d_1,\dots,d_k)=1$. *Then there exists a vector bundle* $\mathcal{E}$ *over* $X=X(d_1,\dots,d_k)$ *of rank* $n+1$ *such that* $A_X\simeq\operatorname{End}_{\mathcal{O}_X}(\mathcal{E})$.

Proof. We have to prove that the class $[A_X]$ in $\operatorname{Br}(X)$ is trivial. Since X is a regular Noetherian scheme the canonical map

$$\operatorname{Br}(X)\to\operatorname{Br}(F(X))$$

where $F(X)$ is the function field of X, is injective by [Gro65, 1.10] and [AG60, Theorem 7.2]. So it is enough to prove that $A\otimes_F F(X)$ splits. However, the generic point of X defines a flag of ideals of $A\otimes_F F(X)$ of reduced dimensions $d_1,\dots,d_k$. Since the index $\operatorname{ind}(A\otimes_F F(X))$ divides $d_1,\dots,d_k$ and $\operatorname{ind}A$, by the assumption of the lemma it must be equal to 1. So $A\otimes_F F(X)$ is split and this finishes the proof of the lemma. □

Remark 4.6. In the case $d_1=1$ one can take $\mathcal{E}=\mathcal{J}_1^\vee$.

Remark 4.7. It can be shown using the index reduction formula (see [MPW96]) that the condition on the greatest common divisor is necessary and sufficient for the central simple algebra $A_{F(X)}$ to be split.

We are now ready to finish the proof of Theorem 2.1.

4.8 Proof of Theorem 2.1

By Lemma 4.5 there exists a vector bundle $\mathcal{E}$ over variety $X=X(d_1,\dots,\hat{d}_m,\dots,d_k)$ of rank $n+1$ such that $A_X\simeq\operatorname{End}_{\mathcal{O}_X}(\mathcal{E})$. By Lemma 4.4 we conclude that $Y=X(d_1,\dots,d_k)$ is a Grassmann bundle over X. Now by § 3.4 we obtain the isomorphism of Theorem 2.1.

Remark 4.9. Note that the assumption of Theorem 2.1 on the reduced dimensions $d_1,\dots,d_k$ is essential. Indeed, suppose that Theorem 2.1 holds for any twisted flag variety. Consider the flag $X=X(1,d)$ with $\gcd(\operatorname{ind}(A),d)>1$. Then we have an isomorphism of motives

$$\mathcal{M}(X)\simeq\bigoplus_{i=0}^{d-1}\mathcal{M}(\operatorname{SB}_d(A))(i)$$

which appears after applying Theorem 2.1 to the flags $X(1,d)$ and $X(d)$. Consider the group $\operatorname{CH}_0(X)=\operatorname{Mor}_{\mathcal{M}}(\mathcal{M}(pt),\mathcal{M}(X))$. The isomorphism above induces the isomorphism of groups

$$\begin{aligned}\operatorname{Coker}(\operatorname{CH}_0(X)\xrightarrow{\text{res}}\operatorname{CH}_0(X_s))&\cong\operatorname{Coker}(\operatorname{CH}_0(\operatorname{SB}_d(A))\xrightarrow{\text{res}}\operatorname{CH}_0(\operatorname{Gr}(d,n+1)))\\&\cong\mathbb{Z}\Big/\left(\frac{\operatorname{ind}(A)}{\gcd(\operatorname{ind}(A),d)}\right)\mathbb{Z},\end{aligned}$$

where res is the pull-back induced by the scalar extension F_s/F and the last isomorphism follows by [Bla91, Theorem 3]. On the other hand, applying Theorem 2.1 to the flags $X(1,d)$ and $X(1)$ we obtain the isomorphism

$$\mathcal{M}(X)\simeq\bigoplus_{\lambda}\mathcal{M}(\operatorname{SB}(A))((n+1-d)(d-1)-|\lambda|),$$

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which induces the isomorphism of groups

$$\operatorname{Coker}(\mathrm{CH}_0(X) \xrightarrow{\text{res}} \mathrm{CH}_0(X_s)) \cong \operatorname{Coker}(\mathrm{CH}_0(\mathrm{SB}(A)) \xrightarrow{\text{res}} \mathrm{CH}_0(\mathbb{P}^n)) \cong \mathbb{Z}/\operatorname{ind}(A)\mathbb{Z},$$

that leads to a contradiction.

We now prove Proposition 2.4.

4.10 Proof of Proposition 2.4

Let $G = \mathrm{PGL}_1(A)$ and let $\mathcal{M}(G, R)$ be the symmetric tensor category of motives of G-homogeneous varieties with coefficients in a ring R for which the Krull–Schmidt theorem holds. It is the case, for example, when R is a field or, more generally, a discrete valuation ring (see [CM04, Theorem 9.6]).

Consider the G-homogeneous variety $X(1, d)$, $1 < d < n$. Apply Theorem 2.1 to the sequences of flags $X(1, d)$, $X(d)$ and $X(1, d)$, $X(1)$. We obtain two isomorphisms in $\mathcal{M}(G, R)$

$$\bigoplus_{i=0}^{d-1} \mathcal{M}(\mathrm{SB}_d(A))(i) \simeq \mathcal{M}(X) \simeq \bigoplus_{\lambda} \mathcal{M}(\mathrm{SB}(A))((n+1-d)(d-1) - |\lambda|), \qquad (**)$$

where the sum on the right-hand side is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{d-1})$ such that $n + 1 - d \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{d-1} \geqslant 0$. Since the Krull–Schmidt theorem holds in $\mathcal{M}(G, R)$, the motive $\mathrm{SB}(A)$ has a unique decomposition into the direct sum of indecomposable objects H_i, $i \in \mathcal{I}$, and their twists

$$\mathcal{M}(\mathrm{SB}(A)) \simeq \bigoplus_{i \in \mathcal{I}} \biggl(\bigoplus_{j \in \mathcal{J}_i} H_i(j) \biggr).$$

Consider the subcategory $\mathcal{M}(G, R)_{\mathcal{I}}$ additively generated by the motives H_i, $i \in \mathcal{I}$, and their twists. The abelian group of isomorphism classes of objects of this category can be equipped with a structure of a free module over the polynomial ring $R[z]$. Namely, multiplication by z is given by the twist. Clearly, the classes $[H_i]$, $i \in \mathcal{I}$, form the basis of this $R[z]$-module.

By (**) we have $\mathcal{M}(\mathrm{SB}_d(A)) \in \mathcal{M}(G, R)_{\mathcal{I}}$ and the isomorphisms (**) can be rewritten as

$$\begin{aligned} \frac{z^d - 1}{z - 1} [\mathrm{SB}_d(A)] &= \frac{\varphi_n(z)}{\varphi_{d-1}(z)\varphi_{n+1-d}(z)} [\mathrm{SB}(A)] \\ &= \frac{z^d - 1}{z - 1} \frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)} [\mathrm{SB}(A)] \end{aligned}$$

where $\varphi_n(z) = \prod_{k=2}^{n} (z^k - 1)/(z - 1)$. This immediately implies the equality

$$[\mathrm{SB}_d(A)] = \frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)} [\mathrm{SB}(A)],$$

i.e. the isomorphism in $\mathcal{M}(G, R)_{\mathcal{I}}$ between $\mathcal{M}(\mathrm{SB}_d(A))$ and the respective sum of twists of $\mathcal{M}(\mathrm{SB}(A))$. This finishes the proof of the proposition.

5. Motivic decomposition of $\mathbf{SB_2(A)}$

This section is devoted to the proof of Theorem 2.5.

First, we need to recall some properties of rational cycles on projective homogeneous varieties.

5.1 Let G be a split linear algebraic group over F. Let X be a projective G-homogeneous variety, i.e. $X = G/P$, where P is a parabolic subgroup of G. The abelian group structure of $\mathrm{CH}(X)$,

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as well as its ring structure, is well known. Namely, X has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the free additive generators of $\mathrm{CH}(X)$ (see [Kar01]). Note that the product of two projective homogeneous varieties $X \times Y$ has a cellular filtration as well, and $\mathrm{CH}^*(X \times Y) \cong \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y)$ as graded rings. The correspondence product of two cycles $\alpha = f_\alpha \times g_\alpha \in \mathrm{CH}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \mathrm{CH}(Y \times X)$ is given by (cf. [Bon03, Lemma 5])

$$(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \deg(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta),$$

where $\deg : \mathrm{CH}(Y) \to \mathrm{CH}(\{pt\}) = \mathbb{Z}$ is the degree map.

5.2 Let X be a projective variety of dimension n over a field F. Let F_s be the separable closure of the field F. Consider the scalar extension $X_s = X \times_F F_s$. We say a cycle $J \in \mathrm{CH}(X_s)$ is *rational* if it lies in the image of the pull-back homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(X_s)$. For instance, there is an obvious rational cycle Δ_{X_s} on $\mathrm{CH}^n(X_s \times X_s)$ that is given by the diagonal class. Clearly, linear combinations, intersections and correspondence products of rational cycles are rational.

5.3 We use the following fact (see [CGM05, Corollary 8.3]) that follows from the Rost nilpotence theorem. Let p_s be a non-trivial rational projector in $\mathrm{CH}^n(X_s \times X_s)$, i.e. $p_s \circ p_s = p_s$. Then there exists a non-trivial projector p on $\mathrm{CH}^n(X \times X)$ such that $p \times_F F_s = p_s$. Hence, the existence of a non-trivial rational projector p_s on $\mathrm{CH}^n(X_s \times X_s)$ gives rise to the decomposition of the Chow motive of X

$$\mathcal{M}(X) \cong (X, p) \oplus (X, \Delta_X - p).$$

Our goal is to find such a projector in the case $X = \mathrm{SB}_d(A)$.

5.4 An isomorphism between the twisted motives $(X, p)(m)$ and $(Y, q)(l)$ is given by correspondences $j_1 \in \mathrm{CH}^{\dim X - l + m}(X \times Y)$ and $j_2 \in \mathrm{CH}^{\dim Y - m + l}(Y \times X)$ such that $q \circ j_1 = j_1 \circ p$, $p \circ j_2 = j_2 \circ q$ and $j_1 \circ j_2 = q$, $j_2 \circ j_1 = p$. If X and Y lie in the category $\mathcal{M}(G, \mathbb{Z})$, then by the Rost nilpotence theorem (see [CM04, Theorem 8.2] and [CGM05, Corollary 8.4]) it suffices to give a rational j_1 and some j_2 satisfying these conditions over separable closure (note that j_2 will automatically be rational).

We now recall some properties of Grassmann varieties and describe their Chow rings.

5.5 Consider the Grassmann variety $\mathrm{Gr}(d, n+1)$, $1 \leqslant d \leqslant n$, of d-planes in the $(n+1)$-dimensional affine space. It has the dimension $d(n+1-d)$. A twisted form of it is a generalized Severi–Brauer variety $\mathrm{SB}_d(A)$, where A is a central simple algebra of degree $n+1$. For any two integers d and d', $1 \leqslant d, d' \leqslant n$, there is the fiber product diagram

$$\begin{array}{ccc} \mathrm{Gr}(d, n+1) \times \mathrm{Gr}(d', n+1) & \xrightarrow{\mathrm{Seg}} & \mathrm{Gr}(dd', (n+1)^2) \\ \downarrow & & \downarrow \\ \mathrm{SB}_d(A) \times \mathrm{SB}_{d'}(A^{\mathrm{op}}) & \xrightarrow{\mathrm{Seg}} & \mathrm{SB}_{dd'}(A \otimes A^{\mathrm{op}}) \end{array} \qquad (1)$$

where the horizontal arrows are Segre embeddings given by the tensor product of ideals (respectively linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension F_s/F.

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5.6 The diagram (1) induces the commutative diagram of rings

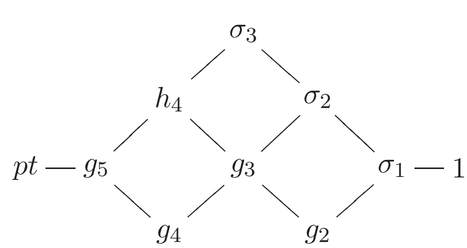
$$\begin{array}{ccc} \mathrm{CH}(\mathrm{Gr}(d,n+1)\times \mathrm{Gr}(d',n+1)) & \xleftarrow{\mathrm{Seg}^*} & \mathrm{CH}(\mathrm{Gr}(dd',(n+1)^2)) \\ \uparrow \mathrm{res} & & \simeq \uparrow \mathrm{res} \\ \mathrm{CH}(\mathrm{SB}_d(A)\times \mathrm{SB}_{d'}(A^{\mathrm{op}})) & \xleftarrow{\mathrm{Seg}^*} & \mathrm{CH}(\mathrm{SB}_{dd'}(A\otimes A^{\mathrm{op}})) \end{array} \tag{2}$$

where all maps are the induced pull-backs. Observe that the right vertical arrow is an isomorphism since $A\otimes A^{\mathrm{op}}$ splits. Consider a vector bundle E over $\mathrm{Gr}(dd',(n+1)^2)$. It is easy to see that the pull-back of the total Chern class $\mathrm{Seg}^*(c(E))$ is a rational cycle on $\mathrm{CH}(\mathrm{Gr}(d,n+1)\times\mathrm{Gr}(d',n+1)) = \mathrm{CH}(\mathrm{Gr}(d,n+1))\otimes\mathrm{CH}(\mathrm{Gr}(d',n+1))$. In particular, if $E=\tau_{dd'}$ is the tautological bundle of $\mathrm{Gr}(dd',(n+1)^2)$ we obtain the following.

Lemma 5.7. *The total Chern class $c(\mathrm{pr}_1^*\tau_d\otimes\mathrm{pr}_2^*\tau_{d'})$ of the tensor product of the pull-backs (induced by the projection maps) of the tautological bundles τ_d and $\tau_{d'}$ of $\mathrm{Gr}(d,n+1)$ and $\mathrm{Gr}(d',n+1)$ respectively, is rational.*

From now on we restrict ourselves to the case $n=4$, $d=2$ and $d'=1$, i.e. to the Grassmannian $\mathrm{Gr}(2,5)$ and the projective space $\mathbb{P}^4=\mathrm{Gr}(1,5)$.

5.8 We describe the generators and relations of the Chow ring $\mathrm{CH}(\mathrm{Gr}(2,5))$ following [Ful98, § 14.7]. Set $\sigma_m=c_m(Q)$, $m=1,2,3$, where $Q=\mathcal{O}^5/\tau_2$ is the universal quotient bundle of rank 3 over $\mathrm{Gr}(2,5)$. It is known that the elements σ_m generate the Chow ring $\mathrm{CH}(\mathrm{Gr}(2,5))$. More precisely, as an abelian group this ring is generated by the Schubert cycles $\varDelta_\lambda(\sigma)$ that are parameterized by all partitions $\lambda=(\lambda_1,\lambda_2)$ such that $3\geqslant\lambda_1\geqslant\lambda_2\geqslant 0$. In particular, $\sigma_m=\varDelta_{(m,0)}$, $m=1,2,3$. For other generators we set the following notation $g_2=\varDelta_{(1,1)}$, $g_3=\varDelta_{(2,1)}$, $h_4=\varDelta_{(3,1)}$, $g_4=\varDelta_{(2,2)}$, $g_5=\varDelta_{(3,2)}$, $pt=\varDelta_{(3,3)}$. These generators corresponds to the vertices of the Hasse diagram of $\mathrm{Gr}(2,5)$ (see [Hil82])



The multiplication rules can be determined using Pieri formulae

$$\varDelta_\lambda\cdot\sigma_m=\sum_\mu \varDelta_\mu,$$

where the sum is taken over all partitions $\mu=(\mu_1,\mu_2)$ such that $3\geqslant\mu_1\geqslant\lambda_1\geqslant\mu_2\geqslant\lambda_2\geqslant 0$.

5.9 Consider the tautological bundle τ_2 of the Grassmannian $\mathrm{Gr}(2,5)$. Its total Chern class is

$$c(\tau_2)=c(Q)^{-1}=\frac{1}{1+\sigma_1+\sigma_2+\sigma_3}=1-\sigma_1-\sigma_2+\sigma_1^2+\cdots$$

where the rest consists of the summands of degree greater than 2. Hence, we obtain $c_1(\tau_2)=-\sigma_1$ and $c_2(\tau_2)=-\sigma_2+\sigma_1^2=g_2$.

5.10 The Chow ring of the projective space $\mathbb{P}^4$ can be identified with the factor ring $\mathbb{Z}[H]/(H^5)$, where $H=c_1(\mathcal{O}(1))$ is the class of a hyperplane section. Thus, the first Chern class of the tautological bundle of $\mathbb{P}^4$ is equal to $c_1(\tau_1)=c_1(\mathcal{O}(-1))=-H$.

We are now ready to prove Theorem 2.5.

5.11 Proof of Theorem 2.5

By Lemma 5.7 we obtain the following rational cycles in the group $\mathrm{CH}^*(\mathrm{Gr}(2,5)\times\mathbb{P}^4)$

$$
\begin{aligned}
r &= c_1(\mathrm{pr}_1^*(\tau_2)\otimes\mathrm{pr}_2^*(\tau_1)) = c_1(\mathrm{pr}_1^*(\tau_2)) + 2c_1(\mathrm{pr}_2^*(\tau_1)) = -\sigma_1\times 1 - 2(1\times H),\\
\rho &= c_2(\mathrm{pr}_1^*(\tau_2)\otimes\mathrm{pr}_2^*(\tau_1)) = c_2(\mathrm{pr}_1^*(\tau_2)) + c_1(\mathrm{pr}_1^*(\tau_2))c_1(\mathrm{pr}_2^*(\tau_1)) + c_1(\mathrm{pr}_2^*(\tau_1))^2\\
&= g_2\times 1 + \sigma_1\times H + 1\times H^2.
\end{aligned}
$$

For two cycles x and y we write $x\equiv y$ if there exists a cycle z such that $x-y=5z$. Note that $\equiv$ is an equivalence relation that preserves rationality of cycles. Then the following cycles are rational

$$
\begin{aligned}
\rho^2 &\equiv 1\times H^4 + 2\sigma_1\times H^3 + (\sigma_2+3g_2)\times H^2 + 2g_3\times H + g_4\times 1,\\
\rho^3 &\equiv (3\sigma_2+g_2)\times H^4 + (\sigma_3+3g_3)\times H^3 + (g_4+3h_4)\times H^2 + 3g_5\times H + pt\times 1.
\end{aligned}
$$

Consider the composition

$$(\rho^2)^t\circ\rho^3 \equiv (3\sigma_2+g_2)\times g_4 + (2\sigma_3+g_3)\times g_3 + (g_4+3h_4)\times(\sigma_2-2g_2) + g_5\times\sigma_1 + pt\times 1.$$

Note that the right-hand side is a rational projector (over $\mathbb{Z}$) and, therefore, by the Rost nilpotence theorem (see [CGM05, Corollary 8.3]) has a form $p\times_F F_s$ where p is a projector in $\mathrm{End}(\mathcal{M}(\mathrm{SB}_2(A)))$. The latter determines an object $(\mathrm{SB}_2(A),p)$ in the category of motives (actually in $\mathcal{M}(G,\mathbb{Z})$), which we denote by $\mathcal{H}$.

Set $q=\Delta_{\mathrm{SB}_2(A)}-p$. We then show that

$$(\mathcal{M}(\mathrm{SB}_2(A)),q)\simeq(\mathcal{M}(\mathrm{SB}_2(A)),p^t)\simeq\mathcal{H}(2),$$

which gives the decomposition $\mathcal{M}(\mathrm{SB}_2(A))\simeq\mathcal{H}\oplus\mathcal{H}(2)$.

Observe that an isomorphism $(\mathrm{SB}_2(A),q)\simeq(\mathrm{SB}_2(A),p^t)$ is given by the two mutually inverse motivic isomorphisms $p_s^t\circ q_s$ and $q_s\circ p_s^t$ over F_s that are rational. An isomorphism $\mathcal{H}(2)\simeq(\mathcal{M}(\mathrm{SB}_2(A)),p^t)$ is given by the following two cycles

$$
\begin{aligned}
j_1 &= (3\sigma_2+g_2)\times pt - (2\sigma_3+g_3)\times g_5 + (g_4+3h_4)\times(g_4+3h_4) - g_5\times(2\sigma_3+g_3) + pt\times(3\sigma_2+g_2),\\
j_2 &= 1\times g_4 - \sigma_1\times g_3 + (\sigma_2-2g_2)\times(\sigma_2-2g_2) - g_3\times\sigma_1 + g_4\times 1.
\end{aligned}
$$

Note that j_1 is rational, since $j_1\equiv(1\times(3\sigma_2+g_2))p$, and $1\times(3\sigma_2+g_2)\equiv 3(\rho+r^2)^t\circ\rho^2$ is rational.

Since A is a division algebra of degree 5, there is a division algebra B of degree 5 Brauer-equivalent to the tensor square $A^{\otimes 2}$. We claim that $\mathcal{H}\simeq\mathcal{M}(\mathrm{SB}(B))$. By the exact sequence (see [Kar00, Remark 7.17])

$$\mathrm{CH}^1(\mathrm{SB}(A^{\mathrm{op}})\times\mathrm{SB}(B)) \xrightarrow{\mathrm{res}_{F_s/F}} \mathrm{CH}^1(\mathbb{P}^4\times\mathbb{P}^4) \xrightarrow[1\times H\mapsto[B]]{H\times 1\mapsto[A^{\mathrm{op}}]} \mathrm{Br}(F) \tag{3}$$

the following cycle in $\mathrm{CH}^1(\mathbb{P}^4\times\mathbb{P}^4)$ is rational

$$u = 2H\times 1 + 1\times H.$$

Therefore, the cycles

$$
\begin{aligned}
\alpha &= pt\times 1 + g_5\times H - (g_4+3h_4)\times H^2 - (g_3+2\sigma_3)\times H^3 + (3\sigma_2+g_2)\times H^4 \equiv u^4\circ\rho^3,\\
\beta &= 1\times g_4 - H\times g_3 - H^2\times(\sigma_2-2g_2) + H^3\times\sigma_1 + H^4\times 1 \equiv(\rho^2)^t\circ(u^4)^t
\end{aligned}
$$

are rational. A direct computation shows that $\alpha\circ\beta=\Delta_{\mathbb{P}^4}$ and $\beta\circ\alpha=p_s$. Therefore, by the Rost nilpotence theorem $\mathcal{H}\simeq\mathcal{M}(\mathrm{SB}(B))$. This finishes the proof of the theorem.

6. Groups of types B_n and C_n

The goal of the present section is to prove Theorems 2.9 and 2.11.

6.1 Let G be an adjoint group of type B_n. From now on we suppose that the characteristic of F is not 2. It is known that $G = O^+(V, q)$, where (V, q) is a regular quadratic space of dimension $2n + 1$ and projective G-homogeneous varieties can be described as flags of q-totally isotropic subspaces

$$X(d_1, \dots, d_k) = \{V_1 \subset \dots \subset V_k \subset V \mid \dim V_i = d_i\}.$$

6.2 The value of the functor corresponding to the variety $X(d_1, \dots, d_k)$ at a F-algebra R equals the set of all flags $V_1 \subset \dots \subset V_k$, where V_i is a q_R-totally isotropic direct summand of V_R of rank d_i.

For convenience we set $d_0 = 0$, $V_0 = 0$.

6.3 On the scheme $X = X(d_1, \dots, d_k)$ there are 'tautological' vector bundles $\mathcal{V}_i$ of ranks d_i. The value of $\mathcal{V}_i$ on an X-algebra (R, x) is V_i, where $x = (V_1, \dots, V_k)$. For every m there exists an obvious morphism

$$X(d_1, \dots, d_k) \to X(d_1, \dots, \hat{d}_m, \dots, d_k)$$
$$(V_1, \dots, V_k) \mapsto (V_1, \dots, \hat{V}_m, \dots, V_k)$$

which makes $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, \hat{d}_m, \dots, d_k)$-scheme.

LEMMA 6.4. *Denote the variety $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ by X. Suppose that $m < k$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z = \mathrm{Gr}(d_m - d_{m-1}, \mathcal{V}_{m+1}/\mathcal{V}_{m-1})$ over X.*

Proof. Fix an X-algebra (R, x), where $x = (V_1, \dots, \hat{V}_m, \dots, V_k)$. We define a natural bijection between the fibers over the point x of Y and Z as follows. The fiber of Y over x can be identified with the set of all direct summands V_m of V_R of rank d_m such that $V_{m-1} \subset V_m \subset V_{m+1}$ (note that V_m is automatically q_R-isotropic since V_{m+1} is so). This fiber is clearly isomorphic to the fiber of Z over x, which is the set of all direct summands of $(\mathcal{V}_{m+1}/\mathcal{V}_{m-1})_x = V_{m+1}/V_{m-1}$ of rank d_m. □

6.5 Proof of Theorem 2.9

Apply Lemma 6.4 to the varieties $Y = X(d_1, \dots, d_k)$ and $X = X(d_1, \dots, \hat{d}_m, \dots, d_k)$. We obtain that Y is a Grassmann bundle over X. To finish the proof apply § 3.4.

6.6 Let G be an adjoint group of type C_n over F. It is known that $G = \mathrm{Aut}(A, \sigma)$, where A is a central simple algebra of degree $2n$ with an involution σ of symplectic type on A, and projective G-homogeneous varieties can be described as flags of (right) ideals of A

$$X(d_1, \dots, d_k) = \{I_1 \subset \dots \subset I_k \subset A \mid I_i \subseteq I_i^{\perp},\ \mathrm{rdim}\, I_i = d_i\}.$$

Here $I^{\perp} = \{x \in A \mid \sigma(x)I = 0\}$ is a right ideal of reduced dimension $2n - \mathrm{rdim}\, I$.

6.7 The value of the functor corresponding to the variety $X(d_1, \dots, d_k)$ at a F-algebra R is equal to the set of all flags $I_1 \subset \dots \subset I_k$ of right ideals of $A_R = A \otimes_F R$ having the following properties:

- the injection of A_R-modules $I_i \hookrightarrow A_R$ splits;
- $I_i \subseteq I_i^{\perp}$;
- $\mathrm{rdim}\, I_i = d_i$.

For convenience we set $I_0 = 0$.

6.8 On the scheme $X = X(d_1, \dots, d_k)$ there are 'tautological' vector bundles $\mathcal{J}_i$ of ranks $2nd_i$ and their 'orthogonal complements' $\mathcal{J}_i^{\perp}$ of rank $2n(2n - d_i)$. The value of $\mathcal{J}_i$ (respectively $\mathcal{J}_i^{\perp}$) on an X-algebra (R, x), where $x = (I_1, \dots, I_k)$, is I_i (respectively $I_i^{\perp}$) considered as a projective R-module. The bundles $\mathcal{J}_i$ and $\mathcal{J}_i^{\perp}$ also have structures of right A_X-modules, where A_X is a constant sheaf of algebras on X determined by A. There exists an obvious morphism

$$\begin{aligned} X(d_1, \dots, d_k) &\to X(d_1, \dots, d_{k-1}) \\ (I_1, \dots, I_k) &\mapsto (I_1, \dots, I_{k-1}), \end{aligned}$$

which makes $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, d_{k-1})$-scheme.

Lemma 6.9. *Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, d_{k-1})$ by X. Suppose that $d_k = d_{k-1} + 1$ and there exists a vector bundle $\mathcal{E}$ over X such that $A_X \simeq \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$. Consider the vector bundle*

$$\mathcal{F} = \mathcal{J}_{k-1}^{\perp}\mathcal{E} / \mathcal{J}_{k-1}\mathcal{E} = \mathcal{J}_{k-1}^{\perp} / \mathcal{J}_{k-1} \otimes_{A_X} \mathcal{E}$$

of rank $2(n - d_{k-1})$. Then Y as a scheme over X can be identified with the projective bundle $Z = \mathbb{P}(\mathcal{F}) = \mathrm{Gr}(1, \mathcal{F})$ over X.

Proof. Fix an X-algebra (R, x), where $x = (I_1, \dots, I_{k-1})$. We define a natural bijection between the fibers over the point x of Y and Z. The fiber of Y can be identified with the set of all ideals I_k containing I_{k-1} and satisfying the conditions in § 6.7. The fiber of Z is the set of all direct summands of $\mathcal{F}_x = \mathcal{F}(R, x)$ of rank 1.

The involution σ induces an isomorphism $h\colon \mathcal{E}_x \otimes \mathcal{L} \to \mathcal{E}_x^*$ for some invertible R-module $\mathcal{L}$ (see [Knu91, Lemma III.8.2.2]) such that

$$\begin{aligned} \sigma(f) \otimes 1 &= h^{-1} f^* h \quad \text{for all } f \in A \\ h^* \mathrm{can} \otimes 1 &= -h \end{aligned}$$

where $\mathrm{can} : \mathcal{E}_x \to \mathcal{E}_x^{**}$ is the canonical isomorphism.

Let U_1 and U_2 be direct summands of $\mathcal{E}_x$. We write $U_2 \subseteq U_1^{\perp}$ if $h(u \otimes l)(v) = 0$ for all $u \in U_1$, $v \in U_2$, $l \in \mathcal{L}$. We call a direct summand U of $\mathcal{E}_x$ *totally isotropic* if $U \subseteq U^{\perp}$. Note that any direct summand of rank 1 is totally isotropic (it can be proved easily using localization).

Define Φ and Ψ as in the proof of Theorem 4.4. Direct computations show that $I_1 \subseteq I_2^{\perp}$ if and only if $\Phi(I_1) \subseteq \Phi(I_2)^{\perp}$.

So the fiber of Y over x is naturally isomorphic to the set of all totally isotropic direct summands U_k of $\mathcal{E}_x$ of rank d_k containing $U_{k-1} = \Phi(I_k)$. One can represent U_k as the direct sum $U_{k-1} \oplus U$ where U is a direct summand of rank 1 (since $d_k = d_{k-1} + 1$). This U is totally isotropic and, therefore, U_k is totally isotropic if and only if $U_k \subseteq U_{k-1}^{\perp}$. Hence, the set of all U_{k-1} is naturally isomorphic to the set of all direct summands of $\Phi(I_{k-1}^{\perp})$ of rank d_k containing $\Phi(I_{k-1})$. The latter can be identified with $\mathbb{P}(\mathcal{F}_x)$. This finishes the proof of the lemma. □

We are now ready to prove Theorem 2.11.

6.10 Proof of Theorem 2.11

Consider the flag varieties $Y = X(d_1, \dots, d_k)$ and $X(d_1, \dots, d_{k-1})$. Since $\mathrm{ind}\, A = 2^r$ for some r and there is an odd d_i, we have $\gcd(\mathrm{ind}(A), d_1, \dots, d_{k-1}) = 1$. By Lemma 4.5 there exists a bundle $\mathcal{E}$ over X such that $A_X = \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$. Applying Lemma 6.9 to the varieties X, Y and the bundle $\mathcal{E}$, we obtain that Y is a projective bundle over X. Now we use § 3.4 to finish the proof.

7. Groups of types G_2 and F_4

This section is devoted to the proofs of Theorems 2.13 and 2.14.

7.1 Let G be a group of type G_2. We suppose that the characteristic of F is not 2. It is known that $G = \mathrm{Aut}(C)$ where C is a Cayley algebra over F. By *i-space* where $i = 1, 2$ we mean an i-dimensional subspace V_i of C such that $uv = 0$ for every $u, v \in V_i$.

The only flag variety corresponding to a non-maximal parabolic is the full flag variety $X(1, 2)$, which is described as follows (see [Bon03]):

$$X(1, 2) = \{V_1 \subset V_2 \mid V_i \text{ is an } i\text{-subspace of } C\}.$$

Similarly one can describe the homogeneous flag variety corresponding to the maximal parabolic

$$X(2) = \{V \mid V \text{ is a 2-subspace of } C\}.$$

7.2 Let R be a F-algebra. By an i-submodule in $C_R = C \otimes_F R$ we mean a direct summand V_i of C_R of rank i such that $uv = 0$ for every two elements $u, v \in V_i$. The value of the functor corresponding to the variety $X(1, 2)$ (respectively $X(2)$) at a F-algebra R equals the set of all flags $V_1 \subset V_2$ (respectively submodules V_2) where V_i is an i-submodule of C_R.

7.3 On the scheme $X = X(2)$ there is a 'tautological' vector bundle $\mathcal{V}$ of rank 2. The value of $\mathcal{V}$ on an X-algebra (R, x) is V, where $x = V$.

There exists an obvious morphism

$$\begin{aligned} X(1, 2) &\to X(2), \\ (V_1, V_2) &\mapsto V_2, \end{aligned}$$

which makes $X(1, 2)$ into a $X(2)$-scheme.

Lemma 7.4. *The scheme $X(1, 2)$ as a scheme over $X(2)$ can be identified with the projective bundle $\mathbb{P}(\mathcal{V}) = \mathrm{Gr}(1, \mathcal{V})$ over $X(2)$.*

Proof. The proof goes as in the B_n case (note that if V_2 is a 2-submodule then each of its direct summands of rank 1 is a 1-submodule). □

7.5 Proof of Theorem 2.13

Apply Lemmas 7.4 and 3.4.

7.6 Let G be a group of type F_4. We suppose that the characteristic of F is not $2, 3$. It is known that $G = \mathrm{Aut}(J)$ where J is an exceptional Jordan algebra of dimension 27 over F. Set $I = \{1, 2, 3, 6\}$. By an *i-space* where $i \in I$ we mean an i-dimensional subspace V_i of J such that every $u, v \in V_i$ satisfy the following condition:

$$\mathrm{Tr}(u) = 0, \quad u \times v = 0, \quad \text{and if } i < 6, \text{ then } u(va) = v(ua) \text{ for all } a \in J.$$

It is known that projective G-homogeneous varieties are parameterized by sequences of numbers $d_1 < \cdots < d_k$ from I and can be described as follows:

$$X(d_1, \ldots, d_k) = \{V_1 \subset \cdots \subset V_k \mid V_i \text{ is a } d_i\text{-subspace of } A\}.$$

7.7 Let R be a F-algebra. By an i-submodule in $J_R = J \otimes_F R$ we mean a direct summand V_i of J_R of rank i such that every two elements $u, v \in V_i$ satisfy the conditions above. The value of

https://doi.org/10.1112/S0010437X06002053 Published online by Cambridge University Press

the functor corresponding to the variety $X(d_1,\dots,d_k)$ at a F-algebra R equals the set of all flags $V_1\subset\cdots\subset V_k$ where V_i is a d_i-submodule of J_R.

For convenience we set $d_0=0$, $V_0=0$.

7.8 On the scheme $X=X(d_1,\dots,d_k)$ there are 'tautological' vector bundles $\mathcal{V}_i$ of rank d_i. The value of $\mathcal{V}_i$ on an X-algebra (R,x) is V_i, where $x=(V_1,\dots,V_k)$.

There exists an obvious morphism

$$X(d_1,\dots,d_k)\to X(d_1,\dots,\hat{d}_m,\dots,d_k),$$
$$(V_1,\dots,V_k)\mapsto(V_1,\dots,\hat{V}_m,\dots,V_k),$$

which makes $X(d_1,\dots,d_k)$ into a $X(d_1,\dots,\hat{d}_m,\dots,d_k)$-scheme.

Lemma 7.9. *Denote $X(d_1,\dots,d_k)$ by Y and $X(d_1,\dots,\hat{d}_m,\dots,d_k)$ by X. Suppose that $m<k$ and either $d_{m+1}<6$ or $d_m=1$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z=\mathrm{Gr}(d_m-d_{m-1},\mathcal{V}_{m+1}/\mathcal{V}_{m-1})$ over X.*

Proof. The proof goes as in the B_n case (note that under our restrictions if V_{m+1} is a d_{m+1}-submodule then each of its direct summands of rank d_m is a d_m-submodule). □

7.10 Proof of Theorem 2.14

Apply Lemma 7.9 to the varieties $Y=X(d_1,\dots,d_k)$ and $X=X(d_1,\dots,\hat{d}_m,\dots,d_k)$. We obtain that Y is a Grassmann bundle over X. To finish the proof apply § 3.4.

Acknowledgements

We are sincerely grateful to N. Karpenko and M. Rost for the comments concerning rational cycles and Severi–Brauer varieties. The first and the last authors would like to thank U. Rehmann for his attention and hospitality. The second and the third authors highly appreciate A. Bak's hospitality and involvements with their activity.

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