CONCERNING SUMMABLE SZLENK INDEX

RYAN MICHAEL CAUSEY

Department of Mathematics, Miami University, Oxford, OH 45056, USA
e-mail: causeyrm@miamioh.edu

Abstract. We generalize the notion of summable Szlenk index from a Banach space to an arbitrary weak*-compact set. We prove that a weak*-compact set has summable Szlenk index if and only if its weak*-closed, absolutely convex hull does. As a consequence, we offer a new, short proof of a result from Draga and Kochanek [J. Funct. Anal. **271** (2016), 642–671] regarding the behavior of summability of the Szlenk index under c_0 direct sums. We also use this result to prove that the injective tensor product of two Banach spaces has summable Szlenk index if both spaces do, which answers a question from Draga and Kochanek [Proc. Amer. Math. Soc. **145** (2017), 1685–1698]. As a final consequence of this result, we prove that a separable Banach space has summable Szlenk index if and only if it embeds into a Banach space with an asymptotic c_0 finite dimensional decomposition, which generalizes a result from Odell et al. [Q. J. Math. **59**, (2008), 85–122]. We also introduce an ideal norm $\mathfrak{s}$ on the class $\mathfrak{S}$ of operators with summable Szlenk index and prove that $(\mathfrak{S}, \mathfrak{s})$ is a Banach ideal. For $1 \leqslant p \leqslant \infty$, we prove precise results regarding the summability of the Szlenk index of an ℓ_p direct sum of a collection of operators.

2010 *Mathematics Subject Classification*. Primary: 46B03, 47L20; Secondary: 46B28

1. Introduction. Since its inception in [**19**], the Szlenk index has been a fundamental object in the geometry of Banach spaces, including the non-linear theory (see [**9**] and [**10**]). The Szlenk index and Szlenk power type are fundamentally connected to asymptotically uniformly smooth renorming properties of spaces and operators, as was shown in [**4**, **6**, **13**, **17**]. Such properties have seen significant recent use in the non-linear asymptotic theory (see [**2**, **12**, **14**]). Of particular importance is the notion of a Banach space having summable Szlenk index. In [**10**], a characterization is given of those separable Banach spaces which have summable Szlenk index in terms of the behavior of the modulus of asymptotic uniform smoothness under equivalent norms. Furthermore, it is shown there that if X has summable Szlenk index, and if Y is uniformly homeomorphic to X, then Y has summable Szlenk index.

In this work, we define what it means for a weak*-compact set to have summable Szlenk index, which generalizes the notion of a Banach space having summable Szlenk index. Our first result is the following.

THEOREM 1.1. *Let X be a Banach space and let $K \subset X^*$ be weak*-compact. Then, K has summable Szlenk index if and only if $\overline{\text{abs co}}^{\text{weak}^*}(K)$ does.*

Our first application of this result is the following embedding result, which generalizes a result from [**16**].

THEOREM 1.2. *If X is a separable Banach space, then X has summable Szlenk index if and only if there exists a Banach space Z with finite dimensional decomposition (FDD) E such that E is asymptotic c_0 in Z and Z admits a subspace isometric to X.*

Our second application answers a question posed in [8].

THEOREM 1.3. *Let $A_0 : X_0 \to Y_0$, $A_1 : X_1 \to Y_1$ be bounded, linear operators. If A_0, A_1 have summable Szlenk index, then so does the induced operator $A_0 \otimes A_1 : X_0 \hat{\otimes}_\varepsilon X_1 \to Y_0 \hat{\otimes}_\varepsilon Y_1$ between the injective tensor products. If neither A_0 nor A_1 is the zero operator, then the converse holds as well.*

One last application of Theorem 1.1 is a short proof of an operator version of a result from [7].

THEOREM 1.4. *Let Λ be a non-empty set and let $A_\lambda : X_\lambda \to Y_\lambda$ be a uniformly bounded collection of linear operators. Then, the induced operator $A : (\oplus_{\lambda \in \Lambda} X_\lambda)_{c_0(\Lambda)} \to (\oplus_{\lambda \in \Lambda} Y_\lambda)_{c_0(\Lambda)}$ has summable Szlenk index if and only if the operators A_λ have uniformly summable Szlenk index.*

We also study the ideal properties of the class $\mathfrak{S}$ of operators with summable Szlenk index, as well as introduce a way to assign to each operator A a value $\Sigma(A) \in [0, \infty]$ such that A has summable Szlenk index if and only if $\Sigma(A) < \infty$. Moreover, the quantity $\mathfrak{s}(A) := \|A\| + \Sigma(A)$ defines an ideal norm on $\mathfrak{S}$. In this direction, we prove the following.

THEOREM 1.5. *The class $(\mathfrak{S}, \mathfrak{s})$ is a Banach ideal.*

We also study the behavior of summable Szlenk index of ℓ_p direct sums of operators for $1 \leqslant p \leqslant \infty$. Such a study is trivial in the setting of spaces, since the norm of the identity operator of a Banach space is either 0 or 1, but non-trivial for operators. We prove the following.

THEOREM 1.6. *Let Λ be a non-empty set and let $A_\lambda : X_\lambda \to Y_\lambda$ be a uniformly bounded collection of linear operators. Then, for any $1 \leqslant p \leqslant \infty$, the induced operator $A : (\oplus_{\lambda \in \Lambda} X_\lambda)_{\ell_p(\Lambda)} \to (\oplus_{\lambda \in \Lambda} Y_\lambda)_{\ell_p(\Lambda)}$ has summable Szlenk index if and only if $(\|A_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$ and $(\Sigma(A_\lambda))_{\lambda \in \Lambda} \in \ell_p(\Lambda)$.*

2. Definitions.   Throughout, $\mathbb{K}$ will denote the scalar field (either $\mathbb{R}$ or $\mathbb{C}$), and "operator" will mean "bounded, linear operator."

Given a directed set D and $n \in \mathbb{N}$, we let $D^{\leqslant n} = \cup_{i=1}^n D^i$. Given $t = (u_i)_{i=1}^k \in D^{\leqslant n}$, we let $|t| = k$, $t^- = (u_i)_{i=1}^{k-1}$ (where $(u_i)_{i=1}^0 = \varnothing$ by convention), and for $0 \leqslant j \leqslant k$, $t|_j = (u_i)_{i=1}^j$. Given $t = (u_i)_{i=1}^k \in \{\varnothing\} \cup D^{\leqslant n-1}$ and $u \in D$, we let $t \frown u = (u_1, \ldots, u_k, u) \in D^{\leqslant n}$. For $s, t \in D^{\leqslant n}$, we let $s \frown t$ be the concatenation of s and t. If X is a Banach space, we say a collection $(x_t)_{t \in D^{\leqslant n}} \subset X$ is weakly null provided that for every $t \in \{\varnothing\} \cup D^{\leqslant n-1}$, $(x_{t \frown u})_{u \in D}$ is a weakly null net. We say a map $\phi : D^{\leqslant n} \to D^{\leqslant n}$ is a pruning provided that $|\phi(t)| = |t|$ and $\phi(t)^- = \phi(t^-)$ for each $t \in D^{\leqslant n}$ and such that the collection $(x'_t)_{t \in D^{\leqslant n}}$ is weakly null, where $x'_t = x_{\phi(t)}$. The following can be easily proved by induction on n. We will use this result frequently.

PROPOSITION 2.1. *Let D be a directed set, $n \in \mathbb{N}$, X a Banach space, and $(x_t)_{t \in D^{\leqslant n}}$ a weakly null collection. Let (M, d) be a compact metric space and suppose $F : D^n \to M$ is any function. Then, for any $\delta > 0$, there exist a pruning $\phi : D^{\leqslant n} \to D^{\leqslant n}$ and $\varpi \in M$ such that $d(\varpi, F(\phi(t))) \leqslant \delta$ for any $t \in D^n$.*

For a Banach space X and $n \in \mathbb{N}$, we let $\{X\}_n$ denote the set of all norms $|\cdot|$ on $\mathbb{K}^n$ such that for any $b > 1$, there exists a directed set D, a weakly null collection $(x_t)_{t \in D^{\leqslant n}} \subset S_X$ such that for any $(a_i)_{i=1}^n \in S_{\ell_\infty^n}$ and any $t \in D^n$,

$$b^{-1} \left\| \sum_{i=1}^n a_i x_{t|_i} \right\| \leqslant \left| \sum_{i=1}^n a_i e_i \right| \leqslant b \left\| \sum_{i=1}^n a_i x_{t|_i} \right\|.$$

A standard compactness argument yields that $\{X\}_n \neq \varnothing$ whenever $\dim X = \infty$. In keeping with the terminology in [11], we say that X is Asymptotic c_0 if $\dim X = \infty$ and there exists a constant $C \geqslant 1$ such that

$$C^{-1} \left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_\infty^n} \leqslant \left| \sum_{i=1}^n a_i e_i \right| \leqslant C \left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_\infty^n}$$

for each $n \in \mathbb{N}$, each $|\cdot| \in \{X\}_n$, and each $(a_i)_{i=1}^n \in \mathbb{K}^n$. We remark that since the canonical $\mathbb{K}^n$ basis is normalized and monotone in $(\mathbb{K}^n, |\cdot|)$ for each $|\cdot| \in \{X\}_n$, we always have

$$\left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_\infty^n} \leqslant 2 \left| \sum_{i=1}^n a_i e_i \right|,$$

so the upper inequality is the only one we need to check in order to establish that a given infinite dimensional space is Asymptotic c_0.

Let us also note that in the previous paragraph, the definition of $\{X\}_n$ involves weakly null trees indexed by $D^{\leqslant n}$ for some directed set D. However, it is equivalent to include only the definition of $\{X\}_n$ trees indexed by $D^{\leqslant n}$, where D is a fixed weak neighborhood basis at 0 in X. Moreover, if X^* is separable, it is sufficient to include in the definition only those trees indexed by $\mathbb{N}^{\leqslant n}$.

We recall that a sequence $E = (E_n)_{n=1}^\infty$ of finite dimensional subspaces of the Banach space X is called an *FDD* for X provided that for each $x \in X$, there exists a unique sequence $(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty E_n$ such that $x = \sum_{n=1}^\infty x_n$. In this case, for each $m \in \mathbb{N}$, the projection $P_m^E \sum_{n=1}^\infty x_n = \sum_{n=1}^m x_n$ is continuous. We let $P_0^E = 0$. By the Principle of Uniform Boundedness, $\sup_{0 \leqslant m < n} \|P_n^E - P_m^E\| < \infty$. If $0 = m_0 < m_1 < \cdots$ and $F_n = \oplus_{i=m_{n-1}+1}^{m_n} E_i$, then $F = (F_n)_{n=1}^\infty$ is also an FDD for X. In this case, we say F is a blocking of E. We say E is shrinking if $\{(P_n^E)^*(X^*) : n \in \mathbb{N}\}$ is dense in X^*, which occurs if and only if $(E_n^*)_{n=1}^\infty$ is an FDD for X^*. Here, E_n^* is identified with $(P_n^E)^*(X^*) \cap \ker((P_{n-1}^E)^*)$. We say E is asymptotic c_0 in X if there exists $C \geqslant 1$ such that for any $n \leqslant k_0 < \cdots < k_n$ and any $x_i \in \oplus_{j=k_{i-1}}^{k_i - 1} E_j$,

$$C^{-1} \max_{1 \leqslant i \leqslant n} \|x_i\| \leqslant \left\| \sum_{i=1}^n x_i \right\| \leqslant C \max_{1 \leqslant i \leqslant n} \|x_i\|.$$

We remark that if $b = \sup_{0 \leqslant m < n} \|P_n^E - P_m^E\|$ and if $(x_i)_{i=1}^n$ is any block sequence with respect to E,

$$\max_{1 \leqslant i \leqslant n} \|x_i\| \leqslant b \left\| \sum_{i=1}^n x_i \right\|,$$

so the upper inequality is the only one we need to check in order to establish that E is asymptotic c_0 in X.

We next define the Szlenk derivations and the Szlenk index. The definition goes back to Szlenk [19] in a somewhat different form which is equivalent for separable spaces not containing ℓ_1. Given a Banach space X, a weak*-compact subset K of X^*, and $\varepsilon > 0$, we let $s_\varepsilon(K)$ denote the subset of K consisting of those $x^* \in K$ such that for each weak*-neighborhood V of x^*, $\mathrm{diam}(V \cap K) > \varepsilon$. For convenience, we let $s_\varepsilon(K) = K$ whenever $\varepsilon \leqslant 0$. We then define by transfinite induction

$$s_\varepsilon^0(K) = K,$$
$$s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K)),$$

and if ξ is a limit ordinal, we let

$$s_\varepsilon^\xi(K) = \bigcap_{\zeta < \xi} s_\varepsilon^\zeta(K).$$

If there exists an ordinal ξ such that $s_\varepsilon^\xi(K) = \varnothing$, we let $Sz(K, \varepsilon)$ be the minimum such ordinal. If no such ordinal exists, we write $Sz(K, \varepsilon) = \infty$. We define $Sz(K) = \sup_{\varepsilon > 0} Sz(K, \varepsilon)$. If $A : X \to Y$ is an operator, we let $Sz(A, \varepsilon) = Sz(A^* B_{Y^*}, \varepsilon)$ and $Sz(A) = Sz(A^* B_{Y^*})$. If X is a Banach space, we let $Sz(X, \varepsilon) = Sz(B_{X^*}, \varepsilon)$ and $Sz(X) = Sz(B_{X^*})$. For $M \geqslant 0$, we say K has M-summable Szlenk index provided that if $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{R}$ (equivalently, if $\varepsilon_1, \ldots, \varepsilon_n > 0$) are such that $s_{\varepsilon_1} \ldots s_{\varepsilon_n}(K) \neq \varnothing$, $\sum_{i=1}^n \varepsilon_i \leqslant M$. This implies that $Sz(K, \varepsilon) \leqslant M/\varepsilon + 1$ for all $\varepsilon > 0$, and in particular, $Sz(K) \leqslant \omega$. We say K has summable Szlenk index if it has M-summable Szlenk index for some $M \geqslant 0$.

We let **Ban** denote the class of all Banach spaces over $\mathbb{K}$. We let $\mathfrak{L}$ denote the class of all operators between Banach spaces and for $X, Y \in$ **Ban**, we let $\mathfrak{L}(X, Y)$ denote the set of operators from X into Y. For $\mathfrak{I} \subset \mathfrak{L}$ and $X, Y \in$ **Ban**, we let $\mathfrak{I}(X, Y) = \mathfrak{I} \cap \mathfrak{L}(X, Y)$. We recall that a class $\mathfrak{I}$ is called an ideal if

(i) For any $W, X, Y, Z \in$ **Ban**, any $C \in \mathfrak{L}(W, X)$, $B \in \mathfrak{I}(X, Y)$, and $A \in \mathfrak{L}(Y, Z)$, $ABC \in \mathfrak{I}$,

(ii) $I_{\mathbb{K}} \in \mathfrak{I}$,

(iii) for each $X, Y \in$ **Ban**, $\mathfrak{I}(X, Y)$ is a vector subspace of $\mathfrak{L}(X, Y)$.

We recall that an ideal $\mathfrak{I}$ is said to be closed provided that for any $X, Y \in$ **Ban**, $\mathfrak{I}(X, Y)$ is closed in $\mathfrak{L}(X, Y)$ with its norm topology.

If $\mathfrak{I}$ is an ideal and ι assigns to each member of $\mathfrak{I}$ a non-negative real value, then we say ι is an ideal norm provided that

(i) for each $X, Y \in$ **Ban**, ι is a norm on $\mathfrak{I}(X, Y)$,

(ii) for any $W, X, Y, Z \in$ **Ban** and any $C \in \mathfrak{L}(W, X), B \in \mathfrak{I}(X, Y), A \in \mathfrak{I}(Y, Z), \iota(ABC) \leqslant \|A\| \iota(B) \|C\|$,

(iii) for any $X, Y \in$ **Ban**, any $x \in X$, and any $y \in Y$, $\iota(x \otimes y) = \|x\| \|y\|$.

If $\mathfrak{I}$ is an ideal and ι is an ideal norm on $\mathfrak{I}$, we say $(\mathfrak{I}, \iota)$ is a Banach ideal provided that for every $X, Y \in$ **Ban**, $(\mathfrak{I}(X, Y), \iota)$ is a Banach space.

3. An ideal seminorm. Given a Banach space X and a weak*-compact subset K of X^*, for $x \in X$, we let $r_K(x) = 0$ if $K = \varnothing$, and otherwise we let $r_K(x) = \max_{x^* \in K} \mathrm{Re}\, x^*(x)$. We note that $r_{A^* B_{Y^*}}(x) = \|Ax\|$, $r_{B_{X^*}}(x) = \|x\|$, and $r_K = r_{\overline{\mathrm{co}}^{\mathrm{weak}^*}(K)}$ for any weak*-compact K. Note also that r_K is a sublinear functional, and it is a seminorm if K is balanced. Given $n \in \mathbb{N}$, we let $\Sigma_n(K)$ be the infimum of those $s > 0$ such that for every directed set D, every weakly null $(x_t)_{t \in D^{\leqslant n}} \subset B_X$,

$$\inf_{t \in D^n} r_K \left(\sum_{i=1}^{n} x_{t|_i} \right) \leqslant s.$$

We let $\Sigma(K) = \sup_n \Sigma_n(K)$. If $A : X \to Y$ is an operator, we let $\Sigma_n(A) = \Sigma_n(A^* B_{Y^*})$, $\Sigma(A) = \Sigma(A^* B_{Y^*})$. If X is a Banach space, we let $\Sigma_n(X) = \Sigma_n(I_X)$, $\Sigma(X) = \Sigma(I_X)$.

REMARK 3.1. We note that it is convenient to allow any directed set in the definition of Σ_n. However, we obtain the same value of $\Sigma_n(K)$ if in the definition we only consider weakly null collections indexed by $D_1^{\leqslant n}$, where D_1 is a fixed weak neighborhood basis at 0 in X. Indeed, if for some $s \in \mathbb{R}$, $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ is a weakly null collection such that $\inf_{t \in D^n} r_K \left(\sum_{i=1}^{n} x_{t|_i} \right) > s$, one can define by induction some map $\phi : D_1^{\leqslant n} \to D^{\leqslant n}$ such that $|\phi(t)| = |t|$ and $\phi(t^-) = \phi(t)^-$ for all $t \in D^{\leqslant n}$, and $(x_{\phi(t)})_{t \in D_1^{\leqslant n}}$ is also weakly null. From this it follows that with $x_t' = x_{\phi(t)}$, $(x_t')_{t \in D_1^{\leqslant n}} \subset B_X$ is weakly null and

$$\inf_{t \in D_1^n} r_K \left(\sum_{i=1}^{n} x_{t|_i}' \right) \geqslant \inf_{t \in D^n} r_K \left(\sum_{i=1}^{n} x_{t|_i} \right) > s.$$

In what follows, $\mathbb{S}$ denotes the set of unimodular scalars. We let $\mathbb{S}K = \{\varepsilon x^* : \varepsilon \in \mathbb{S}, x^* \in K\}$.

PROPOSITION 3.2. Let X be a Banach space, $L, K, K_1, K_2, \ldots, K_l \subset X^*$ weak*-compact, and $n \in \mathbb{N}$.

  (i) K is norm compact if and only if $\Sigma_1(K) = 0$ if and only if $\Sigma(K) = 0$.
  (ii) If $R \geqslant 0$ is such that $K \subset R B_{X^*}$, $\Sigma_n(K) \leqslant Rn$.
  (iii) $\Sigma_n(K + L) \leqslant \Sigma_n(K) + \Sigma_n(L)$.
  (iv) If ε is a unimodular scalar, $\Sigma_n(\varepsilon K) = \Sigma_n(K)$.
  (v) $\Sigma_n\left(\cup_{i=1}^{l} K_i \right) = \max_{1 \leqslant i \leqslant l} \Sigma_n(K_i)$.
  (vi) $\Sigma_n(K) = \Sigma_n(\mathbb{S}K)$.
  (vii) $\Sigma_n\left(\overline{\text{abs co}}^{\text{weak}^*}(K) \right) = \Sigma_n(K)$.
  (viii) For $s > 0$, $\Sigma_n(K) < s$ if and only if for every $(x_t)_{t \in D^{\leqslant n}} \subset B_X$, there exists a pruning $\phi : D^{\leqslant n} \to D^{\leqslant n}$ such that

$$\sup_{t \in D^n} \sup_{(a_i)_{i=1}^{n} \in B_{\ell_\infty^n}} r_K \left(\sum_{i=1}^{n} a_i x_{\phi(t)|_i} \right) < s.$$

  (ix) If $\dim X = \infty$, $\Sigma_n(K)$ is the infimum of those $s > 0$ such that for every directed set D and every weakly null $(x_t)_{t \in D^{\leqslant n}} \subset S_X$,

$$\inf_{t \in D^n} r_K \left(\sum_{i=1}^{n} x_{t|_i} \right) \leqslant s.$$

Proof.

  (i) Since r_K is a sublinear functional, it follows that $\Sigma_n(K) \leqslant n \Sigma_1(K)$, so $\Sigma(K) = 0$ if and only if $\Sigma_1(K) = 0$ is clear. The fact that K is norm compact if and only if $\Sigma_1(K) = 0$ follows from the fact that K is norm compact if and only if for any bounded, weakly null net $(x_\lambda) \subset B_X$, $\lim_\lambda r_K(x_\lambda) = 0$.
  (ii) This follows from the fact that $r_K \leqslant R \| \cdot \|$, so $\Sigma_n(K) \leqslant n \Sigma_1(K) \leqslant Rn$.

(iii) Fix $a > \Sigma_n(K)$ and $b > \Sigma_n(L)$. Fix a weakly null $(x_t)_{t \in D^{\leqslant n}} \subset B_X$. By applying Proposition 2.1 twice, we may fix a pruning $\phi : D^{\leqslant n} \to D^{\leqslant n}$ such that either $r_K\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) \leqslant a$ or $r_K\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) > a$ for all $t \in D^n$, and such that either $r_L\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) \leqslant b$ or $r_L\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) > b$ for all $t \in D^n$. Since $a > \Sigma_n(K)$, $r_K\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) \leqslant a$ for all $t \in D^n$. Similarly, since $b > \Sigma_n(L)$, $r_L\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) \leqslant b$ for all $t \in D^n$. Then, fix any $t \in D^n$ and note that

$$r_{K+L}\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) = r_K\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) + r_L\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) \leqslant a + b.$$

From this it follows that

$$\inf_{t \in D^n} r_{K+L}\left(\sum_{i=1}^n x_{t|_i}\right) \leqslant \Sigma_n(K) + \Sigma_n(L).$$

Since $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ was an arbitrary weakly null collection, $\Sigma_n(K+L) \leqslant \Sigma_n(K) + \Sigma_n(L)$.

(iv) This follows from the fact that $r_{\varepsilon K}\left(\sum_{i=1}^n x_{t|_i}\right) = r_K\left(\sum_{i=1}^n \varepsilon x_{t|_i}\right)$ and $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ is weakly null if and only if $(\varepsilon x_t)_{t \in D^{\leqslant n}} \subset B_X$ is.

(v) Obviously, $\Sigma_n(\cup_{i=1}^l K_i) \geqslant \max_{1 \leqslant i \leqslant l} \Sigma_n(K_i)$. Now fix $a < \Sigma_n(\cup_{i=1}^l K_i)$ and a weakly null collection $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ such that

$$\inf_{t \in D^n} r_{\cup_{i=1}^l K_i}\left(\sum_{i=1}^n x_{t|_i}\right) > a.$$

Now for each $t \in D^n$, fix $i_t \in \{1, \ldots, l\}$ and $x_t^* \in K_{i_t}$ such that

$$x_t^*\left(\sum_{i=1}^n x_{t|_i}\right) = r_{\cup_{i=1}^l K_i}\left(\sum_{i=1}^n x_{t|_i}\right).$$

Define $f : D^n \to \{1, \ldots, l\}$ by $f(t) = i_t$ and fix a pruning $\phi : D^{\leqslant n} \to D^{\leqslant n}$ and $i \in \{1, \ldots, l\}$ such that $f \circ \phi|_{D^n} \equiv i$. We may do this by Proposition 2.1. Then,

$$a < \inf_{t \in D^n} \operatorname{Re} x_{\phi(t)}^*\left(\sum_{i=1}^n x_{\phi(t)|_i}\right) \leqslant \Sigma_n(K_i).$$

(vi) Obviously, $\Sigma_n(K) \leqslant \Sigma_n(\mathbb{S}K)$. For any $\delta > 0$, we may fix a finite subset T of $\mathbb{S}$ such that $\mathbb{S}K \subset (\cup_{\varepsilon \in T} \varepsilon K) + \delta B_{X^*}$. We now combine (ii)–(v) to deduce that

$$\Sigma_n(\mathbb{S}K) \leqslant \Sigma_n(\cup_{\varepsilon \in T} \varepsilon K) + \Sigma_n(\delta B_{X^*}) \leqslant \Sigma_n(K) + \delta n.$$

Since this holds for any $\delta > 0$, we deduce (vi).

(vii) Since $r_{\mathbb{S}K} = r_{\overline{\operatorname{co}}^{\text{weak}^*}(\mathbb{S}K)}$,

$$\Sigma_n(\mathbb{S}K) = \Sigma_n(\overline{\operatorname{co}}^{\text{weak}^*}(\mathbb{S}K)) = \Sigma_n\left(\overline{\operatorname{abs co}}^{\text{weak}^*}(K)\right).$$

By (vi), $\Sigma_n(K) = \Sigma_n(\mathbb{S}K)$.

(viii) Assume $\Sigma_n(K) < s' < s$. Fix $R > 0$ such that $K \subset R B_{X^*}$ and $\delta > 0$ such that $R\delta n + s' < s$. Fix a finite δ-net F of $B_{\ell_\infty^n}$ and $(x_t)_{t \in D^{\leqslant n}} \subset B_X$. By applying Proposition 2.1 repeatedly, once for each $(a_i)_{i=1}^n \in F$, we may fix a pruning $\phi : D^{\leqslant n} \to D^{\leqslant n}$ such that for each $(a_i)_{i=1}^n \in F$, either

$$r_K\left(\sum_{i=1}^{n} a_i x_{\phi(t)|_i}\right) \leqslant s'$$

or

$$r_K\left(\sum_{i=1}^{n} a_i x_{\phi(t)|_i}\right) > s'$$

for all $t \in D^n$. Since $(a_{|t|}x_t)_{t \in D^{\leqslant n}} \subset B_X$ is weakly null, the latter is impossible. By our choice of R and δ, we deduce that

$$r_K\left(\sum_{i=1}^{n} a_i x_{\phi(t)|_i}\right) \leqslant s$$

for all $(a_i)_{i=1}^{n} \in B_{\ell_{\infty}^n}$ and $t \in D^n$.

The converse is clear.

(ix) Assume $\dim X = \infty$. Let $\Sigma'_n(K)$ be the infimum of those $s > 0$ such that for every directed set D and every weakly null $(x_t)_{t \in D^{\leqslant n}} \subset S_X$, $\inf_{t \in D^n} r_K\left(\sum_{i=1}^{n} x_{t|_i}\right) \leqslant s$. It is clear that $\Sigma'_n(K) \leqslant \Sigma_n(K)$. Seeking a contradiction, assume $s, s' > 0$ are such that $\Sigma'_n(K) < s' < s < \Sigma_n(K)$. Fix $R > 0$ such that $K \subset RB_{X^*}$ and fix $\delta > 0$ such that $2Rn\delta < s - s'$. Fix $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ such that

$$\inf_{t \in D^n} r_K\left(\sum_{i=1}^{n} x_{t|_i}\right) > s.$$

By applying Proposition 2.1 and relabeling, we may assume there exist numbers $a_1, \ldots, a_n \in [0, 1]$ such that for each $t \in D^{\leqslant n}$, $|\|x_t\| - a_{|t|}| < \delta/2$. Let $I = \{i \leqslant n : a_i \geqslant \delta\}$ and note that

$$\inf_{t \in D^n} r_K\left(\sum_{i \in I} x_{t|_i}\right) > s - R\delta n.$$

Let M be a weak neighborhood basis at 0 in X and note that there exists a map $\phi : M^{\leqslant |I|} \to D^{\leqslant n}$ such that $(x_{\phi(t)}/\|x_{\phi(t)}\|)_{t \in M^{\leqslant |I|}}$ is weakly null (see [5, Proposition 7.2]). Note that $\Sigma_{|I|}(K) \leqslant \Sigma_n(K) < s'$, since $\dim X = \infty$. Then, with $x'_t = x_{\phi(t)}/\|x_{\phi(t)}\|$, applying Proposition 2.1 as usual to $(\varepsilon_i x'_t)_{t \in M^{\leqslant |I|}}$ for each $(\varepsilon_i)_{i=1}^{|I|} \in \{\pm 1\}^{|I|}$, we may relabel one more time and assume that for each $t \in M^{|I|}$ and $(\varepsilon_i)_{i=1}^{|I|} \in \{\pm 1\}^{|I|}$,

$$r_K\left(\sum_{i=1}^{|I|} a_i x'_{t|_i}\right) > s - 2R\delta n \quad \text{and} \quad r_K\left(\sum_{i=1}^{|I|} \varepsilon_i x'_{t|_i}\right) < s'.$$

But these conditions are in contradiction, since r_K is sublinear, $s' < s - 2R\delta n$, and $\sum_{i=1}^{|I|} a_i x'_{t|_i}$ lies in the convex hull of $\left\{\sum_{i=1}^{|I|} \varepsilon_i x'_{t|_i} : (\varepsilon_i)_{i=1}^{|I|} \in \{\pm 1\}^{|I|}\right\}$. $\square$

The following lemma uses standard techniques. It is a generalization of results from [10] (specifically, Proposition 3.4 and Lemmas 3.1–3.3, 4.3) to arbitrary, weak*-compact sets in the duals of possibly non-separable spaces. We note that these techniques for arbitrary weak*-compact sets and non-separable spaces have appeared for example in [3, Theorem 2.2]. For these reasons, we only sketch the proof.

LEMMA 3.3. *Let X be a Banach space and let $K \subset X^*$ be weak*-compact.*

(i) *If K has M-summable Szlenk index, $\Sigma(K) \leqslant M$.*
(ii) *If $\Sigma(K) \leqslant M/4$, then K has M-summable Szlenk index.*

Proof.

(i) Assume $\Sigma(K) > M' > M$ and fix $n \in \mathbb{N}$, $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ weakly null, and $(x_t^*)_{t \in D^n} \subset K$ such that

$$M' < \inf_{t \in D^n} \operatorname{Re} x_t^* \left(\sum_{i=1}^n x_{t|_i} \right).$$

Fix $R > 0$ such that $K \subset RB_{X^*}$ and define $f : D^n \to RB_{(\ell_\infty^n)^{\mathbb{R}}}$ by $f(t) = (\operatorname{Re} x_t^* (x_{t|_i}))_{i=1}^n$. Fix $\delta > 0$ such that $M + 3\delta n < M'$ and apply Proposition 2.1 and relabel to assume there exists a sequence $(a_i)_{i=1}^n \in RB_{(\ell_\infty^n)^{\mathbb{R}}}$ such that

$$|a_i - \operatorname{Re} x_t^*(x_{t|_i})| < \delta$$

for all $t \in D^n$ and $1 \leqslant i \leqslant n$. Then,

$$M' < n\delta + \sum_{i=1}^n a_i.$$

Now an easy induction proof yields that for any $0 \leqslant i \leqslant n$ and any $t \in \{\varnothing\} \cup D^{\leqslant i}$, there exists $x_t^* \in s_{a_{i+1}-2\delta} \dots s_{a_n-2\delta}(K)$ such that if $\varnothing < s \leqslant t$, $\operatorname{Re} x_t^*(x_s) \geqslant a_{|s|} - \delta$. In particular, $x_\varnothing^* \in s_{a_1-2\delta} \dots s_{a_n-2\delta}(K)$. Since

$$\sum_{i=1}^n (a_i - 2\delta) > M' - 3\delta n > M,$$

this shows that K does not have M-summable Szlenk index.

(ii) Assume that K does not have M-summable Szlenk index. Then, there exist $\varepsilon_1, \dots, \varepsilon_n > 0$ such that $s_{\varepsilon_1} \dots s_{\varepsilon_n}(K) \neq \varnothing$ and $\sum_{i=1}^n \varepsilon_i = M' > M$. Fix $\delta > 0$ such that $M' - \delta n > M$. Let D be a weak neighborhood basis at 0 in X and let N be a weak*-neighborhood basis at 0 in X^*. Then, by standard techniques, we may fix $(x_t^*)_{t \in \{\varnothing\} \cup N^{\leqslant n}} \subset K$ such that for each $t \in \{\varnothing\} \cup D^{\leqslant n-1}$, weak*-$\lim_{v \in N} x_{t \frown v}^* = x_t^*$ and for each $v \in N$, $\|x_{t \frown v}^* - x_t^*\| > \varepsilon_{|t|+1}/2$. Now we may define a map $\phi : D^{\leqslant n} \to N^{\leqslant n}$ and a weakly null collection $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ such that $\operatorname{Re} x_{\phi(t)}^*(x_s) \geqslant (\varepsilon_{|s|} - \delta)/4$ for any $\varnothing < s \leqslant t$. In particular,

$$\inf_{t \in D^n} r_K \left(\sum_{i=1}^n x_{t|_i} \right) \geqslant \inf_{t \in D^n} \operatorname{Re} x_{\phi(t)} \left(\sum_{i=1}^n x_{t|_i} \right) \geqslant \frac{1}{4} \left(\sum_{i=1}^n \varepsilon_i - n\delta \right) > M/4.$$

This shows that $\Sigma(K) > M/4$. $\square$

COROLLARY 3.4. *Let X be a Banach space and let $K \subset X^*$ be weak*-compact. Then, K has summable Szlenk index if and only if $\Sigma(K) < \infty$ if and only if $\overline{\operatorname{abs co}}^{\text{weak}^*}(K)$ has summable Szlenk index.*

For each operator A, let $\mathfrak{s}(A) = \|A\| + \Sigma(A)$ and let $\mathfrak{S}$ denote the class of all operators with summable Szlenk index. Note that by Corollary 3.4, $\mathfrak{S}$ is the class of all operators A such that $\mathfrak{s}(A) < \infty$.

THEOREM 3.5. *The class $(\mathfrak{S}, \mathfrak{s})$ is a Banach ideal.*

Proof. Fix $X, Y \in \mathbf{Ban}$ and note that by Proposition 3.2 and the positive homogeneity of Σ, Σ is a seminorm on $\mathfrak{S}(X, Y)$. From this we can deduce that $(\mathfrak{S}(X, Y), \mathfrak{s})$ is a normed space.

Now fix $W, Z \in \mathbf{Ban}$, $C : W \to X$, $B : X \to Y$, and $A : Y \to Z$ with $\|A\| = \|C\| = 1$. Fix $n \in \mathbb{N}$ and a weakly null $(x_t)_{t \in D^{\leqslant n}} \subset B_W$. Then, $(Cw_t)_{t \in D^{\leqslant n}} \subset B_X$ is weakly null, and

$$\inf_{t \in D^n} \left\| ABC \sum_{i=1}^{n} w_{t|_i} \right\| \leqslant \inf_{t \in D^n} \left\| B \sum_{i=1}^{n} Cw_{t|_i} \right\| \leqslant \Sigma_n(B).$$

Thus, $\Sigma_n(ABC) \leqslant \Sigma_n(B)$. By homogeneity, we deduce that $\Sigma_n(ABC) \leqslant \|A\| \Sigma_n(B) \|C\|$ and $\mathfrak{s}(ABC) \leqslant \|A\| \mathfrak{s}(B) \|C\|$ for any $C : W \to X$ and $A : Y \to Z$.

Next, since $\Sigma(A) = 0$ for any compact operator, $\mathfrak{S}$ contains all finite rank operators and $\mathfrak{s}(x \otimes y) = \|x \otimes y\| = \|x\| \|y\|$ for each $x \in X$ and $y \in Y$.

It remains to show that $(\mathfrak{S}(X, Y), \mathfrak{s})$ is complete. To that end, fix a $\mathfrak{s}$-Cauchy sequence $(A_k)_{k=1}^{\infty}$ in $\mathfrak{S}(X, Y)$. Since $(A_k)_{k=1}^{\infty}$ is also norm Cauchy, it is norm convergent to some A. Since $\Sigma_n(A - A_k) \leqslant n\|A - A_k\|$ for any $n, k \in \mathbb{N}$, it follows that

$$\Sigma(A) = \sup_n \Sigma_n(A) \leqslant \sup_n \limsup_k \Sigma_n(A_k) \leqslant \limsup_k \Sigma(A_k) < \infty$$

and

$$\limsup_n \Sigma(A - A_n) \leqslant \limsup_n \limsup_k \Sigma(A_k - A_n) = 0. \qquad \square$$

REMARK 3.6. The class $\mathfrak{S}$ is not a closed ideal. Indeed, let X_n be the completion of c_{00} with respect to the norm

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{X_n} = \max \left\{ \sum_{i \in T} |a_i| : |T| = n \right\}.$$

It is quite clear that $\Sigma(X_n) = n$, so that $A : (\oplus_{n=1}^{\infty} X_n)_{c_0} \to (\oplus_{n=1}^{\infty} X_n)_{c_0}$ given by $A|_{X_n} = n^{-1/2} I_{X_n}$ quite obviously fails to have summable Szlenk index, but is the norm limit of operators which have summable Szlenk index.

4. Embedding. The equivalence of (*i*) and (*iii*) of the next theorem is no doubt known to specialists. We are unaware of any mention of this fact in the literature, and we will need it for later results, so we include it here. We mention that in the case that X has an FDD, the equivalence of (*ii*)$-$(*iv*) was shown in a dual form in [13, Proposition 6.7].

THEOREM 4.1. *Let $A : X \to Y$ be an operator. The following are equivalent:*

(i) $\Sigma(A) < \infty$.

(ii) *A has summable Szlenk index.*
Furthermore, if $A = I_X$ and $\dim X = \infty$, each of the above is equivalent to the condition

(iii) *X is Asymptotic c_0.*
Finally, if $A = I_X$, $\dim X = \infty$, and X has a shrinking FDD E, each of the above is equivalent to the condition

(iv) *There exists a blocking F of E which is asymptotic c_0 in X.*

Proof of Theorem 4.1. The equivalence of (*i*) and (*ii*) comes from Corollary 3.4. The equivalence (*ii*) $\Rightarrow$ (*iii*) follows from Proposition 3.2 (*viii*) and (*ix*).

Assume $A = I_X$, $\dim X = \infty$, and E is a shrinking FDD for X. Fix $C \geqslant 1$ such that $|\sum_{i=1}^n e_i| \leqslant C$ for every $n \in \mathbb{N}$ and $|\cdot| \in \{X\}_n$. Fix $C_1 > C$. For an infinite subset M of $\mathbb{N}$, if $M = \{m_1, m_2, \ldots\}$ with $m_1 < m_2 < \ldots$ and $m_0 = 0$, let F^M be the blocking of E given by $F_n^M = \oplus_{j=m_{n-1}+1}^{m_n} F_j$. Let $\mathcal{V}$ denote the set of those infinite subsets M of $\mathbb{N}$ such that there exists $(x_i)_{i=1}^n \subset B_X$ such that $x_i \in F_{2i}^M$ for each $1 \leqslant i \leqslant n$ and $\|\sum_{i=1}^n x_i\| \geqslant C_1$. Arguing as in [15, Theorem 3.3], we deduce the existence of some infinite subset M of $\mathbb{N}$ such that for any infinite subset N of M, $N \notin \mathcal{V}$. From the definition of $\mathcal{V}$, if $(x_i)_{i=1}^{2n} \subset B_X$ is any block sequence of F^M, then

$$\left\| \sum_{i=1}^{2n} x_i \right\| \leqslant \left\| \sum_{i=1}^n x_{2i-1} \right\| + \left\| \sum_{i=1}^n x_{2i} \right\| \leqslant 2C_1.$$

Since we may do this for any n, a standard diagonalization procedure yields that (*iii*) $\Rightarrow$ (*iv*).

Last, (*iv*) $\Rightarrow$ (*iii*) is obvious. $\qquad\square$

The following result provides a negative solution to a conjecture from [10].

COROLLARY 4.2. *There exists an ℓ_1 predual which has summable Szlenk index but contains no isomorph of c_0.*

Proof. By [1, Proposition 5.7], there exists an $\mathcal{L}_\infty$ Banach space X with FDD E such that E is asymptotic c_0 in X and such that X contains no isomorph of c_0 and X^* is isomorphic to ℓ_1. This space X has summable Szlenk index. $\qquad\square$

The following result generalizes a theorem from [16], where it was shown that any separable, reflexive, Asymptotic c_0 space embeds into a Banach space with Z with FDD E such that E is asymptotic c_0 in Z.

THEOREM 4.3. *Let X be a separable Banach space. Then X is Asymptotic c_0 if and only if there exists a Banach space Z with FDD E such that E is asymptotic c_0 in Z and Z is isometric to a subspace of Z. Moreover, if X is reflexive, Z can be taken to be reflexive.*

Proof. By [18], there exists a weak*-compact set $B \subset B_{X^*}$ and a Banach space Z with shrinking FDD E such that X embeds isomorphically into Z and such that Z is reflexive if X is. Furthermore, there exist a subset $\mathbb{B} \subset B_{Z^*}$ such that $\overline{\text{abs co}}^{\text{weak}^*}(\mathbb{B}) = B_{Z^*}$, a constant $c > 0$, and a map $I^* : Z^* \to X^*$ such that

$$I^*(s_{\varepsilon_1} \ldots s_{\varepsilon_n}(\mathbb{B})) \subset s_{\varepsilon_1/c} \ldots s_{\varepsilon_n/c}(B).$$

Each of these properties except the last comes from the construction of the space Z. The last property follows from an inessential modification of [18, Lemma 5.5]. If X has summable Szlenk index, so does B, and therefore so does $\mathbb{B}$. By Corollary 3.4, $B_{Z^*} = \overline{\text{abs co}}^{\text{weak}^*}(\mathbb{B})$ has summable Szlenk index as well. This means Z is Asymptotic c_0, and therefore some blocking of E is asymptotic c_0 in Z. $\qquad\square$

5. Injective tensor products.
Let us recall that the injective tensor product is the closed span in $\mathfrak{L}(Y^*, X)$ of the operators $x \otimes y : Y^* \to X$, where $x \otimes y(y^*) = y^*(y)x$. For $i = 0, 1$, if $A_i : X_i \to Y_i$ is an operator, we may define the operator $A_0 \otimes A_1 : X_0 \hat{\otimes}_\varepsilon X_1 \to$

$Y_0 \hat{\otimes}_\varepsilon Y_1$. This operator is given by $A_0 \otimes A_1(u) = A_0 u A_1^* : Y_1^* \to Y_0$. A convenient and more common description of $A_0 \otimes A_1$ is given by its action on simple tensors, which is given by $(A_0 \otimes A_1)(x_0 \otimes x_1) = A_0 x_0 \otimes A_1 x_1$.

Given subsets $K_0 \subset X_0^*$, $K_1 \subset X_1^*$, we let

$$[K_0, K_1] = \{x_0^* \otimes x_1^* : x_0^* \in K_0, x_1^* \in K_1\} \subset (X_0 \hat{\otimes}_\varepsilon X_1)^*.$$

PROPOSITION 5.1. *Let J be a finite set. Suppose that $R > 0$ and for each $i = 0, 1$ and $j \in J$, $K_{i,j} \subset RB_{X_i^*}$ is a weak*-compact set. Then, for any $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{R}$ and any $n \in \{0\} \cup \mathbb{N}$,*

$$s_{\varepsilon_1} \ldots s_{\varepsilon_n} \left(\bigcup_{j \in J} [K_{0,j}, K_{1,j}] \right) \subset \bigcup_{j \in J, (k_i)_{i=1}^n \in \{0,1\}^n} \left[s_{\varepsilon_1/4R}^{k_1} \ldots s_{\varepsilon_n/4R}^{k_n}(K_{0,j}), s_{\varepsilon_1/4R}^{1-k_1} \ldots s_{\varepsilon_n/4R}^{1-k_n}(K_{1,j}) \right].$$

Proof. We induct on n with the $n = 0$ case true by definition.

It is easy to see that if $R > 0$, $x_0^*, z_0^* \in RB_{X_0^*}$, $x_1^*, z_1^* \in RB_{X_1^*}$, and $\|x_0^* \otimes x_1^* - z_0^* \otimes z_1^*\| > \varepsilon$, then

$$\max \left\{ \|x_0^* - z_0^*\|, \|x_1^* - z_1^*\| \right\} > \varepsilon/2R.$$

Now assume the result holds for n and

$$u^* \in s_{\varepsilon_1} \ldots s_{\varepsilon_{n+1}} \left(\cup_{j \in J} [K_{0,j}, K_{1,j}] \right) = s_{\varepsilon_1} \left(s_{\varepsilon_2} \ldots s_{\varepsilon_{n+1}} \left(\cup_{j \in J} [K_{0,j}, K_{1,j}] \right) \right).$$

This means there exists a net $(u_\lambda^*) \subset s_{\varepsilon_2} \ldots s_{\varepsilon_{n+1}} \left(\cup_{j \in J} [K_{0,j}, K_{1,j}] \right)$ converging weak* to u^* such that $\|u^* - u_\lambda^*\| > \varepsilon_1/2$ for all λ. By the inductive hypothesis, for each λ there exists $j_\lambda \in J$ and $(k_i^\lambda)_{i=2}^{n+1} \in \{0, 1\}^n$ such that

$$u_\lambda^* \in \left[s_{\varepsilon_2/4R}^{k_2^\lambda} \ldots s_{\varepsilon_{n+1}/4R}^{k_{n+1}^\lambda}(K_{0,j_\lambda}), s_{\varepsilon_2/4R}^{1-k_2^\lambda} \ldots s_{\varepsilon_{n+1}/4R}^{1-k_{n+1}^\lambda}(K_{1,j_\lambda}) \right].$$

By passing to a subnet, we may assume there exist $j \in J$ and $(k_i)_{i=2}^{n+1} \in \{0, 1\}^n$ such that $j = j_\lambda$ for all λ, $k_i = k_i^\lambda$ for all λ and $2 \leqslant i \leqslant n + 1$. For each λ, write

$$u_\lambda^* = x_{0,\lambda}^* \otimes x_{1,\lambda}^* \in \left[s_{\varepsilon_2/4R}^{k_2} \ldots s_{\varepsilon_{n+1}/4R}^{k_{n+1}}(K_{0,j}), s_{\varepsilon_2/4R}^{1-k_2} \ldots s_{\varepsilon_{n+1}/4R}^{1-k_{n+1}}(K_{1,j}) \right].$$

By passing to a subnet again, we may assume $x_{0,\lambda}^* \underset{\text{weak}^*}{\to} x_0^* \in s_{\varepsilon_2/4R}^{k_2} \ldots s_{\varepsilon_{n+1}/4R}^{k_{n+1}}(K_{0,j})$, $x_{1,\lambda}^* \underset{\text{weak}^*}{\to} x_1^* \in s_{\varepsilon_2/4R}^{1-k_2} \ldots s_{\varepsilon_{n+1}/4R}^{1-k_{n+1}}(K_{1,j})$, and either

$$\|x_0^* - x_{0,\lambda}^*\| > \varepsilon_1/4R$$

for all λ or

$$\|x_1^* - x_{1,\lambda}^*\| > \varepsilon_1/4R$$

for all λ. For this we are using the fact that $u^* = x_0^* \otimes x_1^*$. If $\|x_0^* - x_{0,\lambda}^*\| > \varepsilon_1/4R$ for all λ, let $k_1 = 1$, and otherwise let $k_1 = 0$. Then,

$$u^* = x_0^* \otimes x_1^* \in \left[s_{\varepsilon_1/4R}^{k_1} \ldots s_{\varepsilon_{n+1}/4R}^{k_{n+1}}(K_{0,j}), s_{\varepsilon_1/4R}^{1-k_1} \ldots s_{\varepsilon_{n+1}/4R}^{1-k_{n+1}}(K_{1,j}) \right]. \qquad \square$$

COROLLARY 5.2. *Let $A_0 : X_0 \to Y_0$, $A_1 : X_1 \to Y_1$ be non-zero operators. Then, A_0, A_1 have summable Szlenk index if and only if A_0, A_1 do.*

Proof. If $A_0 \otimes A_1$ has summable Szlenk index, by the ideal property, A_0, A_1 do.

Let $K = [A_0^* B_{Y_0^*}, A_1^* B_{Y_1^*}] \subset (X_0 \hat{\otimes}_\varepsilon X_1)^*$ and note that $\overline{\text{abs co}}^{\text{weak}*}(K) = (A_0 \otimes A_1)^* B_{(Y_0 \hat{\otimes}_\varepsilon Y_1)^*}$ by the Hahn–Banach theorem. By Corollary 3.4, it is sufficient to show that K has summable Szlenk index. Assume A_0 has M_0-summable Szlenk index and A_1 has M_1-summable Szlenk index. Let $R = \max\{\|A_0\|, \|A_1\|\}$. Fix $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $\sum_{i=1}^n \varepsilon_i > 4R(M_0 + M_1)$. Then, for any $(k_i)_{i=1}^n \in \{0, 1\}^n$,

$$M_0 + M_1 < \sum_{i=1}^n k_i \varepsilon_i / 4R + \sum_{i=1}^n (1 - k_i) \varepsilon_i / 4R,$$

so that either $\sum_{i=1}^n k_i \varepsilon_i / 4R > M_0$ or $\sum_{i=1}^n (1 - k_i) \varepsilon_i / 4R > M_1$. In either case,

$$\left[s_{\varepsilon_1/4R}^{k_1} \ldots s_{\varepsilon_n/4R}^{k_n} (A_0^* B_{Y_0^*}), s_{\varepsilon_1/4R}^{1-k_1} \ldots s_{\varepsilon_n/4R}^{1-k_n} (A_1^* B_{Y_1^*}) \right] = \varnothing.$$

By Proposition 5.1,

$$s_{\varepsilon_1} \ldots s_{\varepsilon_n}(K) \subset \bigcup_{(k_i)_{i=1}^n \in \{0,1\}^n} \left[s_{\varepsilon_1/4R}^{k_1} \ldots s_{\varepsilon_n/4R}^{k_n} (A_0^* B_{Y_0^*}), s_{\varepsilon_1/4R}^{1-k_1} \ldots s_{\varepsilon_n/4R}^{1-k_n} (A_1^* B_{Y_1^*}) \right] = \varnothing,$$

whence K has $4R(M_0 + M_1)$-summable Szlenk index.   □

The following answers a question from [8].

COROLLARY 5.3. *Let X_0, X_1 be non-zero Banach spaces. Then, $X_0 \hat{\otimes}_\varepsilon X_1$ is Asymptotic c_0 if and only if X_0, X_1 are. Equivalently, $X_0 \hat{\otimes}_\varepsilon X_1$ has summable Szlenk index if and only if X_0, X_1 do.*

6. Direct sums. The first result of this section is an operator version of a result from [7]. However, we will use Corollary 3.4 to give a new proof.

THEOREM 6.1. *Suppose that Λ is a non-empty set and for each $\lambda \in \Lambda$, $A_\lambda : X_\lambda \to Y_\lambda$ is an operator. Assume also that $\sup_{\lambda \in \Lambda} \|A_\lambda\| < \infty$ and let $A : (\oplus_{\lambda \in \Lambda} X_\lambda)_{c_0(\Lambda)} \to (\oplus_{\lambda \in \Lambda} Y_\lambda)_{c_0(\Lambda)}$ be the operator such that $A|_{X_\lambda} = A_\lambda$.*

Then, A has summable Szlenk index if and only if there exists M such that for each $\lambda \in \Lambda$, A_λ has M-summable Szlenk index.

Proof. Throughout the proof, we will identify each X_λ in the natural way with a subspace of $(\oplus_{\lambda \in \Lambda} X_\lambda)_{c_0(\Lambda)}$, and the same with Y_λ and X_λ^*, Y_λ^*. This does not change the hypotheses or the conclusion, since the identification of the dual of X_λ^* with the natural subspace of the direct sum is an isometric, weak*–weak* homeomorphism.

It is clear that if A has M-summable Szlenk index, A_λ has M-summable Szlenk index for each $\lambda \in \Lambda$, which gives one direction.

Now suppose there exists M such that A_λ has M-summable Szlenk index for each $\lambda \in \Lambda$. Let $K = \bigcup_{\lambda \in \Lambda} A_\lambda^* B_{Y_\lambda^*}$. It is clear that for any $n \in \mathbb{N}$ and $\varepsilon_1, \ldots, \varepsilon_n$,

$$s_{\varepsilon_1} \ldots s_{\varepsilon_n}(K) \subset \{0\} \cup \bigcup_{\lambda \in \Lambda} s_{\varepsilon_1} \ldots s_{\varepsilon_n} (A_\lambda^* B_{X_\lambda^*}).$$

From this it follows that with $M' = M + 2\sup_{\lambda \in \Lambda} \|A_\lambda\|$, K has M'-summable Szlenk index. Indeed, suppose $\varepsilon_1, \ldots, \varepsilon_n > 0$ are such that $\sum_{i=1}^n \varepsilon_i > M'$. Note that $\varepsilon_i \leqslant 2\sup_{\lambda \in \Lambda} \|A_\lambda\|$ for each $1 \leqslant i \leqslant n$. This means that $\sum_{i=2}^n \varepsilon_i > M$, whence

$$s_{\varepsilon_1} \ldots s_{\varepsilon_n}(K) \subset s_{\varepsilon_1}\left(\{0\} \cup \bigcup_{\lambda \in \Lambda} s_{\varepsilon_2} \ldots s_{\varepsilon_n}\left(A_\lambda^* B_{Y_\lambda^*}\right)\right) = s_{\varepsilon_1}(\{0\}) = \varnothing.$$

We now conclude by Corollary 3.4, since by the geometric Hahn–Banach theorem,

$$A^* B_{(\oplus_{\lambda \in \Lambda} Y_\lambda^*)^*_{c_0(\Lambda)}} = \overline{\text{abs co}}^{\text{weak}^*}(K). \qquad \square$$

We next turn to a facet of this problem which is of interest for operators, but not for spaces. Above we considered c_0 direct sums, while below we wish to consider ℓ_p direct sums, $1 \leqslant p \leqslant \infty$. However, if $(X_\lambda)_{\lambda \in \Lambda}$ is a collection of non-zero Banach spaces, $(\oplus_{\lambda \in \Lambda} X_\lambda)_{\ell_p(\Lambda)}$ contains a copy of ℓ_p and therefore cannot have summable Szlenk index except in the case that Λ is finite. Our final goal is to elucidate the situation for operators.

PROPOSITION 6.2. *Fix $1 \leqslant p \leqslant \infty$. For any operators $A_i : X_i \to Y_i$, $1 \leqslant i \leqslant k$ and $n \in \mathbb{N}$,*

$$\Sigma_n\left(A : \left(\oplus_{i=1}^k X_i\right)_{\ell_p^k} \to \left(\oplus_{i=1}^k Y_i\right)_{\ell_p^k}\right) \leqslant \left\|(\Sigma_n(A_i))_{i=1}^k\right\|_{\ell_p^k}$$

and

$$\Sigma\left(A : \left(\oplus_{i=1}^k X_i\right)_{\ell_p^k} \to \left(\oplus_{i=1}^k Y_i\right)_{\ell_p^k}\right) = \left\|(\Sigma(A_i))_{i=1}^k\right\|_{\ell_p^k}.$$

Proof. In the proof, we identify X_i and Y_i with subspaces of $X = \left(\oplus_{i=1}^k X_i\right)_{\ell_p^k}$ and $Y = \left(\oplus_{i=1}^k Y_i\right)_{\ell_p^k}$, respectively. Let $A : X \to Y$ denote the operator with $A|_{X_i} = A_j$. Let $P_j : X \to X$ denote the projection from X onto X_j. Then, $\Sigma_n(A_i) = \Sigma_n(AP_i)$.

Fix $n \in \mathbb{N}$ and a weakly null collection $(x_t)_{t \in D^{\leqslant n}} \subset B_X$. For each $i \in I$, fix $a_i > \Sigma_n(A_i)$. By applying Proposition 2.1 to $(P_j x_t)_{t \in D^{\leqslant n}}$ for each $j = 1, \ldots, k$ and relabeling, we may assume

$$\left\|AP_j \sum_{m=1}^n x_{t|_m}\right\| \leqslant a_j,$$

and

$$\left\|A \sum_{m=1}^n x_{t|_m}\right\| = \left\|\left(\left\|AP_j \sum_{m=1}^n x_{t|_m}\right\|\right)_{j=1}^k\right\|_{\ell_p^k} \leqslant \left\|(a_i)_{i=1}^k\right\|_{\ell_p^k}.$$

Since $a_i > \Sigma_n(A_i)$ was arbitrary, we conclude

$$\Sigma_n(A) \leqslant \left\|(\Sigma_n(A_i))_{i=1}^k\right\|_{\ell_p^k}.$$

Now for each $1 \leqslant i \leqslant k$, fix $0 < b_i < \Sigma(A_i)$ if $\Sigma(A_i) > 0$ and otherwise let $b_i = 0$. If $b_i > 0$, fix $n_i \in \mathbb{N}$ such that $b_i < \Sigma_{n_i}(K_i)$, and otherwise let $n_i = 1$. Let $n = \max_{1 \leqslant i \leqslant k} n_i$. Let D be a weak neighborhood basis at 0 in X. By Remark 3.1, we may fix for each $1 \leqslant j \leqslant k$ some weakly null collection $(x_t^j)_{t \in D^{\leqslant n}} \subset B_{X_j} \subset B_X$ such that

$$\inf_{t \in D^n}\left\|A \sum_{m=1}^n x_{t|_m}^j\right\| \geqslant b_j.$$

Now fix $t \in D^{\leqslant kn}$ and assume $|t| = (j-1)n + r$, $j, r \in \mathbb{N}$, $0 \leqslant r < n$. We may write $t = s \frown s'$, where $|s| = (j-1)n$ and $|s'| = r$, and let $x_t = x^j_{s'}$. Then, $(x_t)_{t \in D^{\leqslant kn}} \subset B_X$ is weakly null and

$$\inf_{t \in D^{kn}} \left\| A \sum_{m=1}^{kn} x_{t|m} \right\| \geqslant \left\| (b_i)_{i=1}^k \right\|_{\ell_p^k}.$$

This shows that $\Sigma(A) \geqslant \|(\Sigma(A_i))_{i=1}^k\|_{\ell_p^k}$. The reverse inequality follows from the previous paragraph. $\qquad\square$

COROLLARY 6.3. *Fix* $1 \leqslant p \leqslant \infty$. *Assume that* Λ *is a non-empty set and for each* $\lambda \in \Lambda$, $A_\lambda : X_\lambda \to Y_\lambda$ *is an operator. Assume also that* $\sup_{\lambda \in \Lambda} \|A_\lambda\| < \infty$ *and let* $A : \left(\oplus_{\lambda \in \Lambda} X_\lambda\right)_{\ell_p(\Lambda)} \to (\oplus_{\lambda \in \Lambda} Y_\lambda)_{\ell_p(\Lambda)}$ *be the operator such that* $A|_{X_\lambda} = A_\lambda$. *Then, A has summable Szlenk index if and only if* $(\|A_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$ *and* $(\Sigma(A_\lambda))_{\lambda \in \Lambda} \in \ell_p(\Lambda)$. *Moreover, in this case,*

$$\Sigma(A) = \|(\Sigma(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_p(\Lambda)}.$$

Proof. Throughout the proof, for a finite subset Υ of Λ, let $P_\Upsilon A$ denote the map given by $P_\Upsilon A|_{X_\lambda} = A_\lambda$ if $\lambda \in \Upsilon$ and $P_\Upsilon A|_{X_\lambda} = 0$ if $\lambda \in \Lambda \setminus \Upsilon$.

If $(\|A_\lambda\|)_{\lambda \in \Lambda} \in \ell_\infty(\Lambda) \setminus c_0(\Lambda)$, then A preserves an isomorphic copy of ℓ_p and cannot have summable Szlenk index. By Proposition 6.2,

$$\Sigma(A) \geqslant \sup\{\Sigma(P_\Upsilon A) : \Upsilon \subset \Lambda \text{ finite}\} = \sup\{\|(\Sigma(A_\lambda))_{\lambda \in \Upsilon}\|_{\ell_p(\Upsilon)} : \Upsilon \subset \Lambda \text{ finite}\}$$
$$= \|(\Sigma(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_p(\Lambda)}.$$

Therefore, if A has summable Szlenk index, $(\|A_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$ and $\|(\Sigma(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_p(\Lambda)} \leqslant \Sigma(A) < \infty$.

Now if $(\|A_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$ and $\|(\Sigma(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_p(\Lambda)} < \infty$, $A \in \overline{\{P_\Upsilon A : \Upsilon \subset \Lambda \text{ finite}\}}$. Arguing as in the proof of Theorem 3.5,

$$\Sigma(A) \leqslant \sup\{\Sigma(P_\Upsilon A) : \Upsilon \subset \Lambda \text{ finite}\} = \|(\Sigma(A_\lambda))_{\lambda \in \Lambda}\|_{\ell_p(\Lambda)} < \infty. \qquad\square$$

REFERENCES

1. S. A. Argyros, I. Gasparis and P. Motakis, On the structure of separable $\mathcal{L}_\infty$-spaces, *Mathematika* **62**(3) (2016), 685–700.

2. F. Baudier, N. Kalton and G. Lancien, A new metric invariant for Banach spaces, *Studia Math.* **199** (2010), 73–94.

3. R. M. Causey, An alternate description of the Szlenk index with applications, *Illinois J. Math.* **59**(2) (2015), 359–390.

4. R. M. Causey, Power type ξ-asymptotically uniformly smooth norms, to appear in *Trans. Amer. Math. Soc.*

5. R. M. Causey, Power type ξ-asymptotically uniformly smooth and ξ-asymptotically uniformly flat norms, submitted.

6. R. M. Causey and S. J. Dilworth, ξ-asymptotically uniformly smooth, ξ-asymptotically uniformly convex, and (β) operators, submitted.

7. S. Draga and T. Kochanek, Direct sums and summability of the Szlenk index, *J. Funct. Anal.* **271** (2016), 642–671.

8. S. Draga and T. Kochanek, The Szlenk power type and injective tensor products of Banach spaces, *Proc. Amer. Math. Soc.* **145** (2017), 1685–1698.

9. G. Godefroy, N. Kalton and G. Lancien, Subspaces of $c_0(\mathbb{N})$ and Lipschitz isomorphisms, *Geom. Funct. Anal.* **10** (2000), 798–820.

10. G. Godefroy, N. Kalton and G. Lancien, Szlenk indices and uniform homeomorphisms, *Trans. Amer. Math. Soc.* **353**(1) (2001), 3895–3918.

11. M. Junge, D. Kutzarova and E. Odell, On asymptotically symmetric Banach spaces, *Studia Math.* **173** (2006), 203–231.

12. N. Kalton and L. Randrianarivony, The coarse Lipschitz structure of $\ell_p \oplus \ell_q$, *Math. Ann.* **341** (2008), 223–237.

13. H. Knaust, Th. Schlumprecht and E. Odell, On Asymptotic structure, the Szlenk index and UKK properties in Banach spaces, *Positivity* **3** (1999), 173–199.

14. G. Lancien and M. Raja, Asymptotic and coarse Lipschitz structures of quasi-reflexive Banach spaces, *Houston J. of Math.* **44**(3) (2018), 927–940.

15. E. Odell and Th. Schlumprecht, Trees and branches in Banach spaces, *Trans. Amer. Math. Soc.* **354**(10) (2002), 4085–4108.

16. E. Odell, Th. Schlumprecht and A. Zsak, On the structure of asymptotic ℓ_p spaces, *Q. J. Math.* **59**, (2008), 85–122.

17. R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics (Springer-Verlag London, Ltd., London, 2002).

18. Th. Schlumprecht, On Zippin's embedding theorem, *Adv. Math.* **274** (2015), 833–880.

19. W. Szlenk, The non existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces, *Studia Math.* **30** (1968), 53–61.