Canad. Math. Bull. Vol. 62 (4), 2019 pp. 841–855
http://dx.doi.org/10.4153/S000843951900016X

The Shifted Turán Sieve Method on Tournaments

Wentang Kuo, Yu-Ru Liu, Sávio Ribas, and Kevin Zhou

Abstract. We construct a shifted version of the Turán sieve method developed by R. Murty and the second author and apply it to counting problems on tournaments. More precisely, we obtain upper bounds for the number of tournaments which contain a fixed number of restricted r-cycles. These are the first concrete results which count the number of cycles over "all tournaments".

1 Introduction

In 1934, P. Turán [20] gave a very simple proof of a celebrated result of Hardy and Ramanujan [6] that the normal order of distinct prime factors of a natural number n is $\log\log n$. If $\omega(n)$ denotes the number of distinct prime factors of n, Turán proved that

$$\sum_{n\leq x}\big(\omega(n)-\log\log x\big)^2 \ll x\log\log x;$$

from which the normal order of $\omega(n)$ is easily deduced. Turán's original derivation of the Hardy–Ramanujan Theorem was essentially probabilistic and concealed in it an elementary sieve method. In [9], R. Murty and the second author introduced the Turán sieve method and applied it to probabilistic Galois theory problems. This method was further generalized to bipartite graphs in [10] to investigate a variety of combinatorial questions, including graph colourings, Latin squares, *etc.*

Let X be a bipartite graph with finite partite sets (A,B). For $a\in A$ and $b\in B$, we write $a\sim b$ if there is an edge that joins a and b. For $b\in B$, we define the *degree* of b to be

$$\deg b := \#\{a\in A \mid a\sim b\}.$$

For $b_1, b_2\in B$, we define the *number of common neighbors* of b_1 and b_2 as

$$n(b_1,b_2) := \#\{a\in A \mid a\sim b_1 \text{ and } a\sim b_2\}.$$

For $a\in A$, we define

$$\omega(a) := \#\{b\in B \mid a\sim b\}.$$

Received by the editors September 9, 2017; revised March 24, 2019.
Published online on Cambridge Core August 20, 2019.
The research of the first and second authors were partially supported by NSERC discovery grants. The research of the third author was partially supported by CAPES and CSF/CNPQ, Brazil.
AMS subject classification: **05A05,** 11N35.
Keywords: shifted Turán sieve, tournament, cycle.

Notice that

$$\sum_{a\in A}\omega(a)=\sum_{b\in B}\deg b \quad\text{and}\quad \sum_{a\in A}\omega^2(a)=\sum_{b_1,b_2\in B}n(b_1,b_2).$$

Then R. Murty and the second author proved the following theorem [10, Theorem 1].

Theorem 1.1 *Suppose that A and B are partite sets of a bipartite graph X. Then*

$$\sum_{a\in A}\Big(\omega(a)-\frac{1}{|A|}\sum_{b\in B}\deg b\Big)^2=\sum_{b_1,b_2\in B}n(b_1,b_2)-\frac{1}{|A|}\Big(\sum_{b\in B}\deg b\Big)^2.$$

From which, they derive the Turán sieve method [10, Corollary 1] which states the following.

Corollary 1.2 (The Turán Sieve)

$$\#\{a\in A\mid \omega(a)=0\}\le |A|^2\cdot\frac{\sum_{b_1,b_2\in B}n(b_1,b_2)}{(\sum_{b\in B}\deg b)^2}-|A|.$$

The extension of sieve methods to a combinatorial setting has been attempted before. For example, R. Wilson [24] and T. Chow [2] have formulated the Selberg sieve in a combinatorial context (see also [11, Section 2]). However, due to the difficulty of computation of the Möbius function of a lattice in an abstract setting, it is not clear how one can apply the Selberg sieve to general combinatorial problems. This obstruction is eliminated by the Turán sieve, as the bound in Corollary 1.2 does not involve the Möbius function and thus can be applied to many questions in combinatorics.

For combinatorial applications, one could be interested in estimating the number of $a\in A$ with $\omega(a)>0$. Thus, to allow us more flexibility in some counting questions, we construct a "shifted" version of the Turán sieve. For a fixed integer $k\in\mathbb{N}\cup\{0\}$,

$$\#\{a\in A\mid \omega(a)=k\}\cdot\Big(k-\frac{1}{|A|}\sum_{b\in B}\deg b\Big)^2\le\sum_{a\in A}\Big(\omega(a)-\frac{1}{|A|}\sum_{b\in B}\deg b\Big)^2.$$

Combining this inequality with Theorem 1.1, we obtain the following 'shifted' version of the Turán sieve method.

Corollary 1.3 (The shifted Turán sieve) *For $k\in\mathbb{N}\cup\{0\}$, we have*

$$\#\{a\in A\mid \omega(a)=k\}\le\frac{|A|^2\sum_{b_1,b_2\in B}n(b_1,b_2)-|A|(\sum_{b\in B}\deg b)^2}{(|A|\cdot k-\sum_{b\in B}\deg b)^2}.$$

We notice that for $k=0$, Corollary 1.3 implies Corollary 1.2. In this paper, we apply Corollary 1.3 to some counting problems on tournaments.

For $t\ge 2$, let $X_1,X_2,\dots,X_t$ be t pairwise disjoint sets of points with $|X_i|=m_i$ $(1\le i\le t)$. We join each pair of points that are not in the same X_i by a line oriented towards exactly one point and thus obtain a complete oriented t-partite graph. Such a graph is called a t partite tournament with $m_1\times\cdots\times m_t$ players, and we let $T_{m_1,\dots,m_t}$ denote the set of all such tournaments. If $m_1=m_2=\cdots=m_t=1$, we denote it by T_t, which is the set of all complete oriented graphs of t elements. In all subsequent

https://doi.org/10.4153/S000843951900016X Published online by Cambridge University Press

sections, if $\{x, y\}$ is an oriented edge toward y, we write $x \to y$, meaning that y defeates x.

For a tournament T, suppose that $V = \{x_1, x_2, \dots, x_r\} \subseteq T$ is a set of players (vertices) such that $x_1 \to x_2 \to \cdots \to x_r \to x_1$. We call this subgraph an *r-cycle* on T and denote it by $(V, \tau) = (x_1, x_2, \dots, x_r, \tau)$, where τ represents the collection of games $x_1 \to x_2 \to \cdots \to x_r \to x_1$; we say τ *generates* $T|_V$, the restriction of T on V. An r-cycle (V, τ) is called a *restricted r-cycle* on a t-partite tournament T if every partite set $X_1, X_2, \dots, X_t$ contains at most one point in V. Otherwise we say that it is an *unrestricted r-cycle*.

There are many results on cycles in t-partite tournaments. For example, the paper [15] contains a study on the average number of 4-cycles on random bipartite tournaments (*i.e.*, $t = 2$) and a proof that the distribution of the 4-cycles satisfies the same conclusion as the Central Limit Theorem. In addition, the paper [25] exhibits conditions to ensure that some bipartites tournaments contains $2s$-cycles for every $2 \le s \le r$. In addition, some results valid for bipartite tournaments were extended to t-partite tournaments in [3]. For 3-partite tournaments, [21] and [22] consider certain lengths of cycles contained in tournaments. Different types of cycles contained in general t-partite tournaments are considered in [23], [14], and [5].

Given $k \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, one can ask how many tournaments have exactly k restricted, or unrestricted, r-cycles. In this paper, we work on the restricted case. We first consider the case of complete oriented graphs.

Notation For $x \in \mathbb{R}, x > 0$, let $f(x)$ and $g(x)$ be two functions of x. If $g(x)$ is a positive function and there exists a constant $C > 0$ such that $|f(x)| \le Cg(x)$, we write $f(x) \ll g(x)$ or $f(x) = O\big(g(x)\big)$; if $\lim_{x\to\infty} f(x)/g(x) = 0$, write $f(x) = o\big(g(x)\big)$. In all theorems of this paper, the O-terms mean absolute constants. ■

Theorem 1.4 *Let $3 \le r \le n$ and $0 \le k \le \binom{n}{r} r!$ be integers. We have*

$$\#\{T \in T_n \mid T \text{ contains exactly } k \text{ restricted } r\text{-cycles}\}$$
$$\le 2^{\binom{n}{2}} \cdot \binom{n}{r} r! \cdot \left\{ \frac{2n^{r-3} + O\Big(\big(\frac{6r}{e}\big)^r \cdot \frac{n^{r-4} \cdot r^{-4}}{(r-4)!} \Big)}{\Big[2^r k - (r-1)! \binom{n}{r}\Big]^2} \right\}.$$

Notice that as $n \to \infty$, the above bound is of size $2^{\binom{n}{2}} \cdot O\big(\frac{1}{n^3}\big)$. Since the size of T_n is $2^{\binom{n}{2}}$, this result gives a non-trivial upper bound of the number of tournaments in T_n containing exactly k restricted r-cycles.

The proof of Theorem 1.4 is based on the shifted Turán sieve. The main technical difficulty in applying Corollary 1.3 lies in the counting of the sum of $n(b_1, b_2)$. In the earlier applications of the Turán sieve method on combinatorial problems, such estimates were often done by considering various cases and their subcases (see the Latin square counting in [10]). However, this approach could become computationally impossible if the associated bipartite graph has a more complicated structure. For example, in our case, the partite sets A and B are respectively chosen to be all tournaments in T_n and the set of all r-cycles on $\{1, 2, \dots, n\}$. To count the number of tournaments $a \in A$ that associate to both r-cycles $b_1, b_2 \in B$, we need to first discuss how cycles b_1

https://doi.org/10.4153/S000843951900016X Published online by Cambridge University Press

and b_2 intersect each other. This involves much more case studying than the one in Latin squares, and hence increases the difficulty in computing the sum of $n(b_1, b_2)$. In this paper, we develop a new counting method for estimating the sum of $n(b_1, b_2)$. The central idea is to first "omit some existing cases" and "include some non-existing cases" to get the expected main contribution. Then we compare the "under-counting" and "over-counting" of the main contribution to get the correct estimate. Such an approach greatly simplifies many of our calculations. For example, in Section 2, for the case $l = 2$, since one can argue that the numbers of under-counting and over-counting are the same, only the estimate of the expected main term is required.

Because of the new counting method, we can also consider the restricted cycle problem on general t-partite tournaments now. For $m_1, \dots, m_t \in \mathbb{N}$ and $s \in \mathbb{Z}$ with $0 \leq s \leq t$, write $\sigma_s = \sigma_s(m_1, \dots, m_t)$ to be the s-symmetric sum of $m_1, \dots, m_t$ $(0 \leq s \leq r)$, *i.e.*,

$$\prod_{i=1}^{t}(x + m_i) = \sum_{s=0}^{t} \sigma_s x^{t-s}. \tag{1}$$

We prove the following theorem.

Theorem 1.5 *Let $3 \leq r \leq t$ and $m_1, \dots, m_t \in \mathbb{N}$. For $0 \leq s \leq t$, let $\sigma_s = \sigma_s(m_1, \dots, m_t)$ be the s-symmetric sum of $m_1, \dots, m_t$. If $0 \leq k \leq \sigma_r r!$, then we have*

$$\begin{aligned}\#\{T \in T_{m_1,\dots,m_t} \mid T \text{ contains exactly } k \text{ restricted } r\text{-cycles}\}\\ \leq 2^{\sigma_2} \cdot (r-3)!^2 \sigma_r \cdot \Big\{ \frac{12\binom{r}{3}\sigma_{r-3} + O(6^r \sigma_{r-4})}{[2^r k - (r-1)!\sigma_r]^2} \Big\}.\end{aligned}$$

Notice that as $m_1, \dots, m_t \to \infty$, the above bound is of size $2^{\sigma_2} \cdot O(\frac{\sigma_{r-3}}{\sigma_r})$. Since the size of $T_{m_1,\dots,m_t}$ is 2^{σ_2}, this result gives a non-trivial upper bound of the number of tournaments in $T_{m_1,\dots,m_t}$ containing exactly k restricted r-cycles.

We remark that the values of k are also bounded according to the other parameters. For example, in Theorem 1.4, each subset with r vertices can form at most $r!$ cycles and hence $k \leq \binom{n}{r} r!$. Analogously, in Theorem 1.5, we have $k \leq \sigma_r r!$.

We will prove the above theorems in Section 2 and Section 3. We remark here that estimates on the number of restricted 3-cycles on a tournament can be found in [1], [4], [7], [13], [16] and [18] (see also [12, Sections 5 and 6]), and estimates on the number of restricted 4-cycles on a tournament can be found in [8] and [19]. Although research on restricted 3-cycles and 4-cycles has been active, the previous results are focused on the number of restricted cycles on "one tournament". Thus, the theorems in this paper are the first to at once deal with all tournaments and cycles of any length.

From our proofs, it is possible to find the average number of cycles of certain length in the tournaments we studied (see Corollaries 2.2 and 3.2). We can compare this average with existing results. For example, in [17], the number of 6-cycles in a regular tournament of order n is bounded by $(n+1)n(n-1)(n-3)(n^2-6n+3)/384$, yet the average number of 6-cycles in Corollary 2.2 is $(n-5)n(n-1)(n-3)(n^2-6n+8)/384$. Thus, the bound in [17] is very close to the average.

The upper bounds given by Theorems 1.4 and 1.5 are not always tight. For example, for the case $k = 0$ in Theorem 1.4, the tournaments which do not contain 3-cycles are exactly the *transitive tournaments* (*i.e.*, satisfying the property "$x \to y$ and $y \to z \Longrightarrow x \to z$"). Therefore

$$\#\{T \in T_n \mid T \text{ contains no cycles of any length}\} = n!\,.$$

One might expect that

$$\#\{T \in T_{m_1,\ldots,m_t} \mid T \text{ contains no cycles of any length}\} = t!\, m_1!\, m_2! \cdots m_t!\,.$$

Although the above result and conjecture indicate that our results are not sharp in this case, for general $k \in \mathbb{N} \cup \{0\}$, since it is difficult to compute the number of tournaments containing exactly k r-cycles, our theorems do provide non-trivial upper bounds for the first time in the literature.

2 Restricted r-cycles on Tournaments

In this section, we apply the shifted Turán sieve to count the number of tournaments in T_n which contains a fixed number of restricted r-cycles. All cycles considered in this section are restricted cycles. For simplicity, we drop the word "restricted".

Proof of Theorem 1.4 Let $A = T_n$ and B be the set of all r-cycles on n vertices. An element of B can be denoted by (V, τ), where $V \subseteq \{1, 2, \ldots, n\}$, $|V| = r$, and τ is a cyclic permutation of V. Since there are $\binom{n}{r}$ choices for V and $(r-1)!$ ways to form an r-cycle with vertices on V, we have

$$|A| = 2^{\binom{n}{2}} \quad \text{and} \quad |B| = \binom{n}{r}(r-1)!\,.$$

For $a = T_a \in A$ and $b = (V_b, \tau_b) \in B$, we say $a \sim b$ if τ_b generates $T_a|_{V_b}$. Thus,

$$\begin{aligned} \omega(a) &= \#\text{ of } r\text{-cycles contained in } T_a, \\ \deg b &= \#\{a \in A \mid \tau_b \text{ generates } T_a|_{V_b}\}. \end{aligned}$$

Since τ_b generates an r-cycle on $T_a|_{V_b}$, it determines r games of T_a. Thus, $\deg b = 2^{\binom{n}{2}-r}$ and it follows that

$$\sum_{b \in B} \deg b = \binom{n}{r} 2^{\binom{n}{2}-r}(r-1)!\,.$$

For $b_1 = (V_{b_1}, \tau_{b_1}) \in B$ and $b_2 = (V_{b_2}, \tau_{b_2}) \in B$, consider

$$n(b_1, b_2) = \#\{a \in A \mid \tau_{b_1} \text{ generates } T_a|_{V_{b_1}} \text{ and } \tau_{b_2} \text{ generates } T_a|_{V_{b_2}}\}.$$

For $0 \le l \le r$, suppose that $|V_{b_1} \cap V_{b_2}| = l$. We consider the following possibilities for l.

(i) $l = 0$: in this case, there are $\binom{n}{r}$ ways to choose V_{b_1} and $\binom{n-r}{r}$ ways to choose V_{b_2}. Also, there are $(r-1)!$ ways to construct each cycle and there are $2r$ determined games. Thus,

$$\sum_{\substack{b_1, b_2 \in B \\ |V_{b_1} \cap V_{b_2}| = 0}} n(b_1, b_2) = \binom{n}{r}\binom{n-r}{r} 2^{\binom{n}{2}-2r}(r-1)!^2.$$

(ii) $l = 1$: In this case, there are $\binom{n}{r}$ ways to choose V_{b_1}, $\binom{r}{1}$ ways to choose the intersection point and $\binom{n-r}{r-1}$ ways to choose the other points of V_{b_2}. Also, there are $(r-1)!$ ways to construct each cycle and there are $2r$ determined games. Thus,

$$\sum_{\substack{b_1,b_2\in B\\ |V_{b_1}\cap V_{b_2}|=1}} n(b_1,b_2) = \binom{n}{r}\binom{n-r}{r-1}\binom{r}{1}2^{\binom{n}{2}-2r}(r-1)!^2.$$

(iii) $l = 2$: In this case, there are $\binom{n}{r}$ choices for V_{b_1} and $\binom{n-r}{r-2}\binom{r}{2}$ choices for V_{b_2}. For fixed V_{b_1} and V_{b_2}, observe that the game between their two intersection points, say x and y, may or may not belong to either cycle. We first ignore the possibility that τ_{b_1} and τ_{b_2} share the game $x \to y$ or $y \to x$, and denote by $D_2 := D_{2,b_1,b_2}$ the number of tournaments where the games in τ_{b_1} and τ_{b_2} are chosen independently. Thus, we have

$$D_2 = 2^{\binom{n}{2}-2r}(r-1)!^2.$$

This term gives the expected main contribution, and we now estimate its difference from the actual case. Let $G_{x\to y} := G_{x\to y;b_1,b_2}$ be the collection of all tournaments with fixed V_{b_1} and V_{b_2} which share the game $x \to y$. We notice that for each tournament in $G_{x\to y}$, since there are $(2r-1)$ games determined by τ_{b_1} and τ_{b_2}, there are $2^{\binom{n}{2}-(2r-1)} = 2\cdot 2^{\binom{n}{2}-2r}$ possible games aside from τ_{b_1} and τ_{b_2}. However, in the counting of D_2, for such a tournament, we only count $2^{\binom{n}{2}-2r}$ possible games aside from τ_{b_1} and τ_{b_2}. Thus we "undercount" some games for tournaments in $G_{x\to y}$. On the other hand, in the counting of D_2, we "overcount" invalid tournaments which have $x \to y$ in τ_{b_1} and $y \to x$ in τ_{b_2}. However, by reversing the direction of the cycle τ_{b_2}, these cases are in one-to-one correspondence with the cases that $x \to y$ belongs to both cycles. In other words, the undercounting of tournaments in $G_{x\to y}$ balances out its overcounting. The same conclusion holds for $G_{y\to x}$. Since the numbers of undercounting and overcounting in D_2 are the same, we have

$$\begin{aligned}\sum_{\substack{b_1,b_2\in B\\ |V_{b_1}\cap V_{b_2}|=2}} n(b_1,b_2) &= \binom{n}{r}\binom{n-r}{r-2}\binom{r}{2}\cdot D_2\\ &= \binom{n}{r}\binom{n-r}{r-2}\binom{r}{2}2^{\binom{n}{2}-2r}(r-1)!^2.\end{aligned}$$

(iv) $l = 3$: In this case, there are $\binom{n}{r}$ choices for V_{b_1} and $\binom{n-r}{r-3}\binom{r}{3}$ choices for V_{b_2}. For fixed V_{b_1} and V_{b_2}, observe that the game between their three intersection points, say x, y, and z, may or may not belong to either cycle. Similar to the case $l = 2$, we first ignore the sharing games among $\{x, y, z\}$ and denote by $D_3 := D_{3,b_1,b_2}$, the number of tournaments where the games in τ_{b_1} and τ_{b_2} are chosen independently. Thus, we have

$$D_3 = 2^{\binom{n}{2}-2r}(r-1)!^2.$$

We now consider the undercounting and overcounting in D_3. There are three cases.

https://doi.org/10.4153/S000843951900016X Published online by Cambridge University Press

(a) τ_{b_1} and τ_{b_2} have no game among $\{x, y, z\}$: In this case, there is no undercounting or overcounting in D_3.

(b) τ_{b_1} and τ_{b_2} share one game among $\{x, y, z\}$, say $x \to y$: We have seen in (iii) that in this case, the numbers of undercounting and overcounting in D_3 are the same. Thus, there is no adjustment required here.

(c) τ_{b_1} and τ_{b_2} share two or more games among $\{x, y, z\}$: We notice that if τ_{b_1} and τ_{b_2} share two games, since they are r-cycles and the games among $\{x, y, z\}$ do not form a cycle, we need to have $r \geq 4$. On the other hand, if τ_{b_1} and τ_{b_2} share three games among $\{x, y, z\}$, it means that they form a 3-cycle and so $r = 3$.

(c-1) Suppose that τ_{b_1} and τ_{b_2} share two games and $r \geq 4$. Notice that there are $3! = 6$ possible choices for two games among $\{x, y, z\}$. Fix one of such games, say $x \to y \to z$. Let $G_{x\to y\to z} = G_{x\to y\to z;b_1,b_2}$ be the collection of all tournaments with fixed V_{b_1} and V_{b_2} which contains the game $x \to y \to z$. We notice that for each tournament in $G_{x\to y\to z}$, since there are $(2r-2)$ games determined by τ_{b_1} and τ_{b_2}, there are $2^{\binom{n}{2}-(2r-2)} = 4 \cdot 2^{\binom{n}{2}-2r}$ possible games aside from τ_{b_1} and τ_{b_2}. However, in the counting of D_3, for such a tournament, we only count $2^{\binom{n}{2}-2r}$ possible games aside from τ_{b_1} and τ_{b_2}. Thus we "undercount" $3 \cdot 2^{\binom{n}{2}-2r}$ possible games for each tournament in $G_{x\to y\to z}$. On the other hand, in the counting of D_3, we "overcount" invalid tournaments which have $x \to y \to z$ in τ_{b_1} and $z \to y \to x$ in τ_{b_2}. However, by reversing the direction of the cycle τ_{b_2}, these cases are in one-to-one correspondence with the cases that $x \to y \to z$ belongs to both cycles. Thus for a tournament in $G_{x\to y\to z}$, the difference between undercounting games and overcounting games in D_3 is $2 \cdot 2^{\binom{n}{2}-2r}$. Furthermore, for such a tournament, there are $\big((r-3)!\big)^2$ choices for τ_{b_1} and τ_{b_2}. The same argument can be applied to all other five permutations of $\{x, y, z\}$. Combining this with (a) and (b), we see that if $r \geq 4$,

$$\begin{aligned}\sum_{\substack{b_1,b_2\in B\\ |V_{b_1}\cap V_{b_2}|=3}} & n(b_1, b_2)\\ &= \binom{n}{r}\binom{n-r}{r-3}\binom{r}{3}\cdot\Big(D_3 + 6\cdot 2\cdot 2^{\binom{n}{2}-2r}\big((r-3)!\big)^2\Big)\\ &= \binom{n}{r}\binom{n-r}{r-3}\binom{r}{3}2^{\binom{n}{2}-2r}[(r-1)!^2 + 12(r-3)!^2].\end{aligned}$$

(c-2) Suppose that τ_{b_1} and τ_{b_2} share three games and $r = 3$. In this case, the games among $\{x, y, z\}$ form a 3-cycle and there are 2 possible choices for such a cycle. Fix one of such cycles, say $x \to y \to z \to x$. Let $G_{x\to y\to z\to x} = G_{x\to y\to z\to x;b_1,b_2}$ be the collection of all tournaments with fixed V_{b_1} and V_{b_2} which contains the cycle $x \to y \to z \to x$. We notice that for each tournament in $G_{x\to y\to z\to x}$, since there are $3 = (2r-3)$ games determined by τ_{b_1} and τ_{b_2}, there are $2^{\binom{n}{2}-(2r-3)} = 8 \cdot 2^{\binom{n}{2}-2r}$

https://doi.org/10.4153/S000843951900016X Published online by Cambridge University Press

possible games aside from τ_{b_1} and τ_{b_2}. However, in the counting of D_3, for such a tournament, we only count $2^{\binom{n}{2}-2r}$ possible games aside from τ_{b_1} and τ_{b_2}. Thus, we "undercount" $7\cdot 2^{\binom{n}{2}-2r}$ possible games for each tournament in $G_{x\to y\to z\to x}$. On the other hand, in the counting of D_3, we "overcount" invalid tournaments which have $x\to y\to z\to x$ in τ_{b_1} and $x\to z\to y\to x$ in τ_{b_2}. However, by reversing the direction of the cycle τ_{b_2}, these cases are in one-to-one correspondence with the case that $x\to y\to z\to x$ belong to both cycles. Thus, for a tournament in $G_{x\to y\to z\to x}$, the difference between undercounting games and overcounting games in D_3 is $6\cdot 2^{\binom{n}{2}-2r}$. Furthermore, for such a tournament, there are $1^2 = \big((r-3)!\big)^2$ choices for τ_{b_1} and τ_{b_2}. Combining this with (a) and (b), we see that for $r=3$,

$$\begin{aligned}\sum_{\substack{b_1,b_2\in B\\ |V_{b_1}\cap V_{b_2}|=3}} n(b_1,b_2) &= \binom{n}{r}\binom{n-r}{r-3}\binom{r}{3}\cdot\Big(D_3 + 2\cdot 6\cdot 2^{\binom{n}{2}-2r}\big((r-3)!\big)^2\Big)\\ &= \binom{n}{r}\binom{n-r}{r-3}\binom{r}{3}2^{\binom{n}{2}-2r}[(r-1)!^2+12(r-3)!^2].\end{aligned}$$

We notice that this formula is exactly the same as the one in (c-1).

(v) $4\le v\le r$: In these cases, there are $\binom{n}{r}$ choices for V_{b_1}, $\binom{n-r}{r-v}\binom{r}{v}$ choices for V_{b_2} and the number of ways to form each cycle is at most $(r-1)!$. Since the number of determined games is at least $\big(2r-(v-1)\big)$, we have

$$\sum_{\substack{b_1,b_2\in B\\ 4\le|V_{b_1}\cap V_{b_2}|\le r}} n(b_1,b_2) \le \binom{n}{r}2^{\binom{n}{2}-2r}(r-1)!^2\sum_{4\le v\le r}\binom{n-r}{r-v}\binom{r}{v}2^{v-1}.$$

Combining all the above five possibilities, we obtain

$$\begin{aligned}\sum_{b_1,b_2\in B} n(b_1,b_2) \le 2^{\binom{n}{2}-2r}\binom{n}{r}(r-1)!^2\Bigg\{&\binom{n-r}{r}+\binom{n-r}{r-1}\binom{r}{1}\\ &+\binom{n-r}{r-2}\binom{r}{2}+\binom{n-r}{r-3}\binom{r}{3}\Big[1+\frac{12}{(r-1)^2(r-2)^2}\Big]\\ &+\sum_{4\le v\le r}\binom{n-r}{r-v}\binom{r}{v}2^{v-1}\Bigg\}.\end{aligned}$$

To estimate the sum in the error term, for $n\ge r$, $r\ge v\ge 4$, we apply the inequality

$$\binom{n-r}{r-v}\le\binom{n}{r-v}\le\binom{n}{r-4}\cdot\binom{r}{v}\cdot\binom{r}{4}^{-1}$$

to the summation in the error term, and get

$$\begin{aligned}\sum_{4\le v\le r}\binom{n-r}{r-v}\binom{r}{v}2^{v-1} &\le \binom{n}{r-4}\cdot\Big(\sum_{4\le v\le r}\binom{r}{v}^2 2^{v-1}\Big)\cdot\binom{r}{4}^{-1}\\ &\le \binom{n}{r-4}\cdot\Big(\sum_{4\le v\le r}\binom{r}{v}2^{v-1}\Big)\cdot 2^r\cdot\binom{r}{4}^{-1}\le\binom{n}{r-4}\cdot 6^r\cdot\binom{r}{4}^{-1}.\end{aligned}$$

https://doi.org/10.4153/S000843951900016X Published online by Cambridge University Press

Thus, we have

$$(2)\quad \sum_{b_1,b_2\in B} n(b_1,b_2) \le 2^{\binom{n}{2}-2r}\binom{n}{r}(r-1)!^2\left[\binom{n}{r}+\frac{12\binom{n-r}{r-3}\binom{r}{3}}{(r-1)^2(r-2)^2}+O\Big(\frac{6^r\cdot n^{r-4}}{r^4(r-4)!}\Big)\right].$$

By applying Corollary 1.3 and Stirling's approximation, Theorem 1.4 follows. ∎

We notice that in the proof of Theorem 1.4, we have calculated explicitly the cases $r = 3$ and $r = 4$. Thus, we can derive the following from the above estimates.

Corollary 2.1

$$\begin{aligned}&\#\{T\in T_n \mid T \textit{ contains exactly } k \textit{ restricted } 3\textit{-cycles}\}\\ &\qquad\le 2^{\binom{n}{2}}\cdot\left\{\frac{3}{16}\binom{n}{3}\left(k-\frac{1}{4}\binom{n}{3}\right)^2\right\},\quad \textit{and}\\ &\#\{T\in T_n \mid T \textit{ contains exactly } k \textit{ restricted } 4\textit{-cycles}\}\\ &\qquad\le 2^{\binom{n}{2}}\cdot\left\{\frac{3}{64}\binom{n}{4}(4n-11)\left(k-\frac{3}{8}\binom{n}{4}\right)^2\right\}.\end{aligned}$$

Remark We see from Theorem 1.4 that for n sufficiently large, we have

$$\#\{T\in T_n \mid T \text{ contains exactly } k \text{ restricted } r\text{-cycles}\} \ll 2^{\binom{n}{2}}\cdot\left\{\frac{1}{n^3}\right\}.$$

Thus, as $n\to\infty$, the probability that a tournament contains exactly k restricted r-cycles is 0. One can obtain various conclusions that are much stronger than the above one from Theorem 1.4. For example, let $f:\mathbb{N}\to\mathbb{R}_+$ such that $f(n) = o(n^3)$ and $3\le r\le(\log n)^{1-\epsilon}$ for any $\epsilon>0$. By Theorem 1.4, one can show that as $n\to\infty$,

$$\text{Prob}\{T\in T_n \text{ contains at most } f(n) \text{ restricted } r\text{-cycles}\}\longrightarrow 0.$$

Since the total number of r-cycles in all tournaments is $\sum_{b\in B}\deg b$ and the number of tournaments is $|A|$, the average number of r-cycles in a tournament is $(\sum_{b\in B}\deg b)/|A|$. We have the following result.

Corollary 2.2 *The average number of r-cycles in a tournament is* $\frac{\binom{n}{r}(r-1)!}{2^r}$.

3 Restricted r-cycles on t-partite Tournaments

In this section, we consider restricted cycles on t-partite tournaments. All cycles considered in this section are restricted. For simplicity, we drop the word "restricted" as before.

For $m_1,\dots,m_t\in\mathbb{N}$, we recall that $T_{m_1,m_2,\dots,m_t}$ is the set of all t-partite tournaments with $m_1\times m_2\times\dots\times m_t$ players. Also, for $s\in\mathbb{Z}$ with $0\le s\le t$, we recall the definition of s-symmetric sum $\sigma_s=\sigma_s(m_1,\dots,m_t)$ given in equation (1). In particular, we have

$$\sigma_2=\sum_{1\le i<j\le t} m_i m_j \quad\text{and}\quad \sigma_t=\prod_{i=1}^{t} m_i.$$

Let $1 \le u \le s \le t$. Since all $m_i \ge 1$, if $\{j_1, \dots, j_u\} \subseteq \{i_1, \dots, i_s\}$, then the term $m_{i_1} \cdots m_{i_s}$ in σ_s is greater or equal to the term $m_{j_1} \cdots m_{j_u}$ of σ_u. In this case, we say that $m_{i_1} \cdots m_{i_s}$ *covers* $m_{j_1} \cdots m_{j_u}$. Notice that each term $m_{i_1} \cdots m_{i_s}$ in σ_s covers all terms $m_{j_1} \cdots m_{j_u}$ in σ_s with $\{j_1, \dots, j_u\} \subseteq \{i_1, \dots, i_s\}$ and there are $\binom{s}{u}$ such terms. Thus in the sum $\binom{s}{u} \cdot \sigma_s$, each term $m_{j_1} \cdots m_{j_u}$ in σ_u is covered $\binom{t-u}{s-u}$ times. It follows that

$$\sigma_s \binom{s}{u} \ge \sigma_u \binom{t-u}{s-u},$$

which implies that

$$\binom{t}{s}^{-1} \cdot \binom{t}{u} \sigma_s \ge \sigma_u. \tag{3}$$

We make a remark here that the factor $\binom{t}{s}^{-1} \cdot \binom{t}{u}$ is a decreasing function in t, and therefore it reaches the maximum at the minimal value of t.

Next, we claim that for any positive integer u with $1 \le u \le t$, we have

$$\prod_{i=1}^{t} (m_i - 1) = \sum_{j=0}^{u-1} (-1)^j \sigma_{t-j} + O(\sigma_{t-u}). \tag{4}$$

In fact, we can establish the following stronger statement

$$\sum_{j=0}^{2\lfloor (u-1)/2 \rfloor + 1} (-1)^j \sigma_{t-j} \le \prod_{i=1}^{t} (m_i - 1) \le \sum_{j=0}^{2\lfloor u/2 \rfloor} (-1)^j \sigma_{t-j}. \tag{5}$$

We now prove the above claim by induction on t. For $t = 1$, it is clear. Assume that (5) holds for all $t < v$ and consider the case when $t = v$. We might assume that u is even and the proof for the case of odd u is similar. For an integer s with $1 \le s \le (v-1)$, let $\sigma_s^{(v-1)}$ denote the term of degree s in the expansion $(m_2 - 1) \cdots (m_v - 1)$. We have the relation

$$\sigma_s = m_1 \sigma_{s-1}^{(v-1)} + \sigma_s^{(v-1)}. \tag{6}$$

By induction,

$$\sum_{j=0}^{u-1} (-1)^j \sigma_{v-1-j}^{(v-1)} \le \prod_{i=2}^{v} (m_i - 1) \le \sum_{j=0}^{u} (-1)^j \sigma_{v-1-j}^{(v-1)}.$$

Since $m_1 \ge 1$, by multiplying $(m_1 - 1)$ on both sides of the above inequality and applying (6), we can get

$$\begin{aligned} \sum_{j=0}^{u-1} (-1)^i \sigma_{v-j} &\le \sum_{j=0}^{u-1} (-1)^j \sigma_{v-j} + \sigma_{v-2u}^{(v-1)} \\ &\le \prod_{i=1}^{v} (m_i - 1) \le \sum_{j=0}^{u} (-1)^j \sigma_{v-j} - \sigma_{v-2u-1}^{(v-1)} \le \sum_{j=0}^{u} (-1)^j \sigma_{v-j}. \end{aligned}$$

It completes the proof of inequality (5).

From now on, we apply the shifted Turán sieve to count the number of t-partite tournaments which contain a fixed number of r-cycles. In the special case when $t = r$, we can obtain the following theorem, which is sharper than Theorem 1.5. Since the

https://doi.org/10.4153/S000843951900016X Published online by Cambridge University Press

proof of the theorem is in the same spirit as Theorem 1.4, we will only provide a sketch of the proof below.

Theorem 3.1 *Let $r \geq 3$ and $m_1, \dots, m_r \in \mathbb{N}$. For $0 \leq s \leq r$, let $\sigma_s = \sigma_s(m_1, \dots, m_r)$ be the s-symmetric sum of $m_1, \dots, m_t$. If $0 \leq k \leq \sigma_r r!$, then we have*

$$\#\{T \in T_{m_1,\dots,m_r} \mid T \text{ contains exactly } k \text{ restricted } r\text{-cycles}\}$$
$$\leq 2^{\sigma_2} \cdot (r-3)!^2 \sigma_r \cdot \left\{ \frac{12\sigma_{r-3} + O(3^r \sigma_{r-4})}{[2^r k - (r-1)!\sigma_r]^2} \right\}.$$

Sketch of proof Let $A = T_{m_1,\dots,m_r}$ and B be the set of all r-cycles on $T_{m_1,\dots,m_r}$. An element of B can be denoted by $(x_1, \dots, x_r, \tau)$, where $\{x_1, \dots, x_r\}$ are taken from distinct partite sets X_i $(1 \leq i \leq r)$, and τ is a cyclic permutation of $\{x_1, \dots, x_r\}$. Since there are σ_r choices for $\{x_1, \dots, x_r\}$ and $(r-1)!$ ways to form an r-cycle with vertices $x_1, \dots, x_r$, we have

$$|A| = 2^{\sigma_2} \quad \text{and} \quad |B| = \sigma_r (r-1)!\,. \tag{7}$$

For $a = T_a \in A$ and $b = (x_{1,b}, \dots, x_{r,b}, \tau_b) \in B$, we say $a \sim b$ if τ_b generates $T_a|_{\{x_{1,b},\dots,x_{r,b}\}}$. Thus, $\omega(a)$ is the number of r-cycles contained in T_a and $\deg b$ is the number of $a \in A$ such that τ_b generates $T_a|_{\{x_{1,b},\dots,x_{r,b}\}}$. Since τ_b generates an r-cycle on $T_a|_{\{x_{1,b},\dots,x_{r,b}\}}$, it determines r games of T_a. Thus, $\deg b = 2^{\sigma_2 - r}$ and it follows that

$$\sum_{b \in B} \deg b = \sigma_r 2^{\sigma_2 - r} (r-1)!\,. \tag{8}$$ ■

For $b_1 = (x_{1,b_1}, \dots, x_{r,b_1}, \tau_{b_1}) \in B$ and $b_2 = (x_{1,b_2}, \dots, x_{r,b_2}, \tau_{b_2}) \in B$, consider

$$n(b_1, b_2) = \#\big\{a \in A \mid \tau_{b_1} \text{ generates } T_a|_{\{x_{1,b_1},\dots,x_{r,b_1}\}} \text{ and } \tau_{b_2} \text{ generates } T_a|_{\{x_{1,b_2},\dots,x_{r,b_2}\}}\big\}.$$

For $i = 1, \dots, r$, suppose

$$|\{x_{i,b_1}\} \cap \{x_{i,b_2}\}| = N_i, \tag{9}$$

where $N_i \in \{0,1\}$. Let $M(N_1, N_2, \dots, N_r)$ denote the collection of all pairs $(b_1, b_2) \in B^2$ such that (9) holds. By counting the number of 1's in N_i $(1 \leq i \leq r)$, up to symmetry, there are $(r+1)$ distinct possibilities for $(N_1, \dots, N_r)$, of which we group into five cases with the similar estimates as in the proof of Theorem 1.4.

(i) $(N_1, N_2, \dots, N_r) = (0, 0, \dots, 0)$: In this case,

$$\sum_{(b_1,b_2) \in M(0,\dots,0)} n(b_1, b_2) = \sigma_r (m_1 - 1)(m_2 - 1)(m_3 - 1) \cdots (m_r - 1) 2^{\sigma_2 - 2r} (r-1)!^2.$$

(ii) $(N_1, N_2, N_3, \dots, N_r) = (1, 0, 0, \dots, 0)$: In this case,

$$\sum_{(b_1,b_2) \in M(1,0,\dots,0)} n(b_1, b_2) = \sigma_r (m_2 - 1)(m_3 - 1) \cdots (m_r - 1) 2^{\sigma_2 - 2r} (r-1)!^2.$$

(iii) $(N_1, N_2, N_3, N_4, \dots, N_r) = (1, 1, 0, 0, \dots, 0)$: In this case,

$$\sum_{(b_1,b_2) \in M(1,1,0,\dots,0)} n(b_1, b_2) = \sigma_r (m_3 - 1)(m_4 - 1) \cdots (m_r - 1) 2^{\sigma_2 - 2r} (r-1)!^2.$$

(iv) $(N_1, N_2, N_3, N_4, N_5, \ldots, N_r) = (1, 1, 1, 0, 0, \ldots, 0)$: In this case,

$$\sum_{(b_1,b_2)\in M(1,1,1,0,\ldots,0)} n(b_1, b_2) = \sigma_r(m_4-1)(m_5-1)\cdots(m_r-1)2^{\sigma_2-2r}[(r-1)!^2 + 12(r-3)!^2].$$

(v) $(N_1, N_2, \ldots, N_v, N_{v+1}, N_{v+2}, \ldots, N_r) = (\underbrace{1, 1, \ldots, 1}_{v \text{ times}}, 0, 0, \ldots, 0)$: In these cases,

$$\sum_{\substack{4\le v\le r \\ (b_1,b_2)\in M(\underbrace{1,1,\ldots,1}_{v \text{ times}},0,0,\ldots,0)}} n(b_1, b_2) \le \sigma_r 2^{\sigma_2-2r}(r-1)!^2 \sum_{4\le v\le r}(m_{v+1}-1)\ldots(m_r-1)2^{v-1}.$$

Combining all these possibilities and their symmetrical cases, by (4), we get

$$\begin{aligned}\sum_{b_1,b_2\in B} n(b_1, b_2) = 2^{\sigma_2-2r}\sigma_r(r-1)!^2 \\ &\times\Bigg\{\underbrace{[\sigma_r - \sigma_{r-1} + \sigma_{r-2} - \sigma_{r-3} + O(\sigma_{r-4})]}_{\substack{\text{symmetrical sum of} \\ (0,\ldots,0)}} \\ &+ \underbrace{[\sigma_{r-1} - 2\sigma_{r-2} + 3\sigma_{r-3} + O(\sigma_{r-4})]}_{\substack{\text{symmetrical sum of} \\ (1,0,0,\ldots,0)}} \\ &+ \underbrace{[\sigma_{r-2} - 3\sigma_{r-3} + O(\sigma_{r-4})]}_{\substack{\text{symmetrical sum of} \\ (1,1,0,0,\ldots,0)}} \\ &+ \underbrace{\Big[\Big(1 + \frac{12}{(r-1)^2(r-2)^2}\Big)\sigma_{r-3} + O(\sigma_{r-4})\Big]}_{\substack{\text{symmetrical sum of} \\ (1,1,1,0,0,\ldots,0)}} \\ &+ \underbrace{O\Big(\sum_{4\le v\le r} 2^{v-1}\sigma_{r-v}\Big)}_{\substack{\text{symmetrical sum of} \\ \text{remainder cases}}}\Bigg\} \\ &= 2^{\sigma_2-2r}\sigma_r(r-1)!^2\Big[\sigma_r + \frac{12\sigma_{r-3}}{(r-1)^2(r-2)^2} + O(3^r\cdot r^{-4}\sigma_{r-4})\Big],\end{aligned}$$

where we use the estimate (3) and

$$\begin{aligned}\sum_{4\le v\le r} 2^{v-1}\sigma_{r-v} &\le \sum_{4\le v\le r} 2^{v-1}\cdot\sigma_{r-4}\cdot\binom{r}{r-v}\cdot\binom{r}{r-4}^{-1} \\ &\le \sigma_{r-4}\cdot\binom{r}{4}^{-1}\cdot\Big(\sum_{4\le v\le r}\binom{r}{v}2^{v-1}\Big) \le \sigma_{r-4}\cdot\binom{r}{4}^{-1}\cdot 3^r.\end{aligned}$$

By applying Corollary 1.3 with equations (7), (8) and estimate (3), Theorem 3.1 follows. ∎

We now consider r-cycles on general t-partite tournaments with $t \geq r$. Theorem 1.5 is more general, but less sharp than Theorem 3.1. Since its proof is in the same spirit of Theorem 3.1, we will only provide a sketch of proof below.

Sketch of Theorem 1.5 Let A and B be defined as in the proof of Theorem 3.1. Thus, we have

$$|A| = 2^{\sigma_2} \quad \text{and} \quad \sum_{b \in B} \deg b = \sigma_r 2^{\sigma_2 - r}(r-1)!\,. \tag{10}$$

It remains to consider the sum of $n(b_1, b_2)$. For $b_1, b_2 \in B$, write $b_i = \{x_{1,b_i} \cdots, x_{r,b_i}, \tau_{b_i}\}$ $(1 \leq i \leq 2)$. Assume x_{j,b_1} $(1 \leq j \leq r)$ are in $X_1, \dots, X_r$ and $x_{j,b_2} (1 \leq j \leq r)$ are in $X_1, \dots, X_l, X_{u_{l+1}}, \dots, X_{u_r}$ for some $0 \leq l \leq r$ and $u_j \in \{r+1, \dots, t\}$ $(l+1 \leq j \leq r)$. For $v \in \mathbb{N}$, define $\sigma_s^{[v]} = \sigma_s(m_1, \dots, m_v, 0, \dots, 0)$. Thus $\sigma_s^{[t]} = \sigma_s$. Then using the same argument as the one to prove (3), we can obtain that

$$\sum_{\substack{b_1, b_2 \in B \\ b_1 \in X_1, \dots, X_r \\ b_2 \in X_1, \dots, X_l, X_{u_{l+1}}, \dots, X_{u_r}}} n(b_1, b_2) \leq 2^{\sigma_2 - 2r} m_1 \cdots m_r (r-1)!^2$$

$$\times \Big[\sigma_l^{[l]} + \frac{12\sigma_{l-3}^{[l]}}{(r-1)^2(r-2)^2} + O\Big(\sum_{v=4}^{r} 2^{v-1} \sigma_{l-v}^{[l]} \Big) \Big] m_{u_{v+1}} \cdots m_{u_r}.$$

We now sum over the symmetric cases of b_2. Notice that the term with the factor $\sigma_{l-3}^{[l]} m_{u_{l+1}} \cdots m_{u_r}$ appears at most $\binom{r}{3}$ times. Also, the error term appears at most $\binom{r}{v}$ times for each $4 \leq v \leq r$. Thus, similar to the end of proof of Theorem 1.4, using the estimates (3), we have

$$\sum_{v=4}^{r} 2^{v-1} \binom{r}{v} \sigma_{r-v} \leq \sum_{v=4}^{r} 2^{v-1} \binom{r}{v} \cdot \binom{r}{r-v} \cdot \binom{r}{r-4}^{-1} \sigma_{r-4} \leq 6^r \cdot \binom{r}{4}^{-1} \sigma_{r-4}.$$

Finally, we get

$$\sum_{b_1, b_2 \in B} n(b_1, b_2) \leq 2^{\sigma_2 - 2r} \sigma_r (r-1)!^2 \Big[\sigma_r + \frac{12\binom{r}{3}\sigma_{r-3}}{(r-1)^2(r-2)^2} + O\left(6^r \cdot r^{-4} \sigma_{r-4}\right) \Big]. \tag{11}$$

Then, using equations in (10) and estimate (11), Theorem 1.5 follows from Corollary 1.3. ∎

Similar to Corollary 2.2, we have the following.

Corollary 3.2 *The average number of r-cycles in a t-partite tournament is* $\dfrac{\sigma_r(r-1)!}{2^r}$.

Remark

(i) Notice that when we take $m_1 = \cdots = m_t = 1$, by writing $n = t$, we have $T_{1,\ldots,1} = T_t = T_n$. Thus, in Theorem 1.5, we obtain

$$\sum_{b_1,b_2\in B} n(b_1,b_2) \leq 2^{\binom{n}{2}-2r}\binom{n}{r}(r-1)!^2\left[\binom{n}{r}+\frac{12\binom{r}{3}\binom{n}{r-3}}{(r-1)^2(r-2)^2}+O\left(6^r\cdot r^{-4}\binom{n}{r-4}\right)\right].$$

Although this upper bound is different from the one in inequality (2), the main terms in both expressions have the same order of magnitude in n. The inconsistency in these expressions comes from the coarser estimate (4) used in the proof of Theorem 1.5.

(ii) We see from Theorem 1.5 that for $m_1,\ldots,m_t$ sufficiently large, we have

$$\#\{T\in T_{m_1,\ldots,m_t} \mid T \text{ contains exactly } k \text{ restricted } r\text{-cycles}\} \ll 2^{\sigma_2}\cdot\left\{\frac{\sigma_{r-3}}{\sigma_r}\right\}.$$

Thus, as one of $m_1,\ldots,m_t\to\infty$, the probability that a tournament contains exactly k restricted r-cycles is 0. Similar to the remark at the end of Section 2, one can obtain various conclusions that are stronger than the above one from Theorem 1.5. In addition, the same conclusion is valid for fixed $m_1 = m_2 = \cdots = m_t$ with $t\to\infty$.

We see in this paper that the setting of the shifted Turán sieve is rather flexible. Also, when the structure of the partite set B is more complicated, the new counting method helps in estimating the sum of $n(b_1,b_2)$. Thus, the combination of these two methods will allow us to investigate more combinatorial problems, and we will report further applications of these methods in our future papers. In particular, we plan to study problems about unrestricted cycles in tournaments, which are more difficult to approach than the restricted ones, due to the repetitions on the sets X_i.

Acknowledgments This project started when the second author visited Queen's University as a graduate student. She would like to thank the Mathematics and Statistics department for its hospitality. She would also like to thank Professor R. Murty for his valuable comments on the first version of the paper. This project was completed when the third author, as a Ph.D. student at Universidade Federal de Minas Gerais, visited the University of Waterloo in 2015–2016. He would like to thank the Pure Mathematics department for its hospitality. He would also like to thank *Ciência sem Fronteiras* for the Exchange Student Fellowship. Finally, all authors would like to thank the referee for the detailed comments and many helpful suggestions that substantially improved the paper.

References

[1] L. W. Beineke and F. Harary, *The maximum number of strongly connected subtournaments*. Canad. Math. Bull. 8(1965), 491–498.
[2] T. Y. Chow, *The combinatorics behind number-theoretic sieves*. Adv. Math. 138(1998), 293–305.
[3] P. Erdös and G. Szekeres, *A combinatorial problem in geometry*. Compositio Math. 2(1965), 463–470.
[4] D. Gale, *On the number of faces of a convex polygon*. Canad. J. Math. 16(1964), 12–17.

[5] G. Gutin, A. Rafiey, and A. Yeo, *On n-partite tournaments with unique n-cycle*. Graphs Combin. 22(2006), 241–249.
[6] G. H. Hardy and S. Ramanujan, *The normal number of prime factors of a number n*. Quart. J. Pure Appl. Math. 48(1917), 76–97.
[7] M. G. Kendall and B. Babington Smith, *On the method of paired comparisons*. Biometrika 33(1940), 239–251.
[8] A. Kotzig, *Sur le nombre des 4-cycles dans un tournoi*. Mat. Časopis Sloven. Akad. Vied 18(1968), 247–254.
[9] Y.-R. Liu and M. R. Murty, *The Turán sieve method and some of its applications*. J. Ramanujan Math. Soc. 14(1999), 21–35.
[10] Y.-R. Liu and M. R. Murty, *Sieve Methods in combinatorics*. J. Combin. Theory Ser. A 111(2005), 1–23.
[11] L. Lovász, *Combinatorial Problems and Exercises*. North-Holland Pub. Co., Amsterdam, 1993.
[12] J. W. Moon, *Topics on Tournaments*. Holt, Rinehart and Winston Inc., New York, 1968.
[13] J. W. Moon, *Uncovered nodes and 3-cycles in tournaments*. Australas. J. Combin. 7(1993), 157–173.
[14] J. W. Moon, *On the score sequence of an n-partite tournament*. Canad. Math. Bull. 5(1962), 51–58.
[15] J. W. Moon and L. Moser, *On the distribution of 4-cycles in random bipartite tournaments*. Canad. Math. Bull. 5(1962), 5–12.
[16] P. A. P. Moran, *On the method of paired comparisons*. Biometrika 34(1947), 363–365.
[17] S. V. Savchenko, *On 5-cycles and 6-cycles in regular n-tournaments*. J. Graph Theory 83(2016), 44–77.
[18] T. Szele, *Kombinatorikai vizsgálatok az irányított teljes gráffal kapcsolatban*. Mat. Fiz. Lapok 50(1943), 223–256. For a German translation, see *Kombinatorische Untersuchungen über gerichtete vollständige Graphen*, Publ. Math. Debrecen **13**(1966), 145–168.
[19] C. Tabib, *The number of 4-cycles in regular tournaments*. Utilitas Math. 22(1982), 315–322.
[20] P. Turán, *On a theorem of Hardy and Ramanujan*. J. London Math. Soc. 9(1934), 274–276.
[21] L. Volkmann, *On cycles in regular 3-partite tournaments*. Discrete Math. 306(2006), 1198–1206.
[22] L. Volkmann, *Longest cycles in almost regular 3-partite tournaments*. Discrete Math. 306(2006), 2931–2942.
[23] L. Volkmann and S. Winzen, *Cycles through a given arc and certain partite sets in almost regular multipartite tournaments*. Discrete Math. 283(2004), 217–229.
[24] R. J. Wilson, *The Selberg sieve for a lattice. In: 'Combinatorial Theory and Its Applications, Balatofüred (Hungary)'*. Colloq. Math. Soc. János Bolyai 4(1969), 1141–1149.
[25] K. M. Zhang, *Vertex even-pancyclicity in bipartite tournaments*. Nanjing Daxue Xuebao Shuxue Bannian Kan 1(1984), 85–88.

Department of Pure Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada
e-mail: wtkuo@uwaterloo.ca yrliu@uwaterloo.ca kkqzhou@gmail.com

Departamento de Matemática, Universidade Federal de Ouro Preto, Ouro Preto, MG, 35400-000, Brazil
e-mail: savio.ribas@ufop.edu.br