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A LATTICE ISOMORPHISM THEOREM FOR NONSINGULAR RETRACTABLE MODULES

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ABSTRACT. Let ${}_RM$ be a nonsingular module such that $B = \mathrm{End}_R(M)$ is left nonsingular and has $A = \mathrm{End}_R(\bar{M})$ as its maximal left quotient ring, where $\bar{M}$ is the injective hull of ${}_RM$. Then it is shown that there is a lattice isomorphism between the lattice $C(M)$ of all complement submodules of ${}_RM$ and the lattice $C(B)$ of all complement left ideals of B, and that ${}_RM$ is a CS module if and only if B is a left CS ring. In particular, this is the case if ${}_RM$ is nonsingular and retractable.

1. **Introduction.** Let ${}_RM$ be a left module over the associative ring R with identity. M is said to be *retractable* if $\mathrm{Hom}_R(M, U) \neq 0$ for every nonzero submodule U of M. M is said to be *e-retractable* if $\mathrm{Hom}_R(M, U) \neq 0$ for every nonzero complement submodule U of M. M is said to be *nondegenerate* if $Tm \neq 0$ for every nonzero $m \in M$, where T is the trace of M in R. M is called a CS *module* if every complement submodule of M is a direct summand of M. A ring B is called a *left* CS *ring* if ${}_BB$ is a CS module [6].

In [5], 1989, S. M. Khuri showed that if M is nonsingular and nondegenerate, then the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^e$ determine a projectivity (that is, an order-preserving bijection) between $C(M)$ and $C(B)$ [5, Theorem 3.10], where $I_B(U) = \{b \in B, Mb \subseteq U\}$ and $(MH)^e$ is the essential closure (*cf.* [1, p. 61 Proposition 7]) of MH in M, and therefore that B is a left CS ring if and only if M is a CS module [5, Corollary 3.11]. It is also known that any nondegenerate module is retractable [5, Proposition 3.2], but not conversely (for example, let M be the Z-module Z/p^nZ).

As the main result in [6], 1991, Khuri successfully generalized the second result above to the case where M is nonsingular and retractable [6, Theorem 3.2], and gave a necessary and sufficient condition so that the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^e$ determine a projectivity between $C(M)$ and $C(B)$ under this weaker condition [6, Theorem 3.1].

In this paper, more generally, let ${}_RM$ be a nonsingular module such that $B = \mathrm{End}_R(M)$ is left nonsingular and has $A = \mathrm{End}_R(\bar{M})$ as its maximal left quotient ring, where $\bar{M}$ is the injective hull of ${}_RM$, then the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^e$ determine a lattice isomorphism between the lattice $C(M)$ and the lattice $C(B)$ (Theorem 2.4), and that ${}_RM$ is a CS module if and only if B is a left CS ring (Theorem 2.5). In particular, if ${}_RM$ is nonsingular and retractable, the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^e$ determine a lattice isomorphism between $C(M)$ and $C(B)$ already (Theorem 2.6), which contains [5, Theorem 3.10] as a special case, and we get the result of [6, Theorem 3.2] again in a simpler and more explicit way (Corollary 2.7).

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2. **A Lattice isomorphism theorem and applications.** Throughout this paper, modules, unless otherwise specified, are consistently left modules, $U \subseteq_e V$ will mean that U is an essential submodule of V, that is, U has nonzero intersection with every nonzero submodule of V, while $U \subseteq V$ will mean that U is a submodule or subset of V when V is a module or just a set. $\bar{U}$ denotes the injective hull of U, U^e the essential closure of U. Finally, $B = \mathrm{End}(M)$, $A = \mathrm{End}(\bar{M})$.

Recall that M is nonsingular if for any $I \subseteq_e R$, $m \in M$, $\mathrm{Im} = 0$ implies $m = 0$. A submodule U of M is called a *complement* in M if U has no proper essential extension in M.

LEMMA 2.1. *If M is nonsingular, then the maps $U \longmapsto I_A(U)$, $H \longmapsto \bar{M}H$ determine a lattice isomorphism between $C(\bar{M})$ and $C({}_AA)$.*

PROOF. Since M is nonsingular, so is $\bar{M}$; then $C(\bar{M})$ is a complete modular lattice [8, p. 251 Corollary 4.4]. Since $\bar{M}$ is nonsingular, $A = \mathrm{Hom}(\bar{M}, \bar{M})$ is regular and left self-injective (*cf.* [3] or [1, p. 44 Theorem 1]); therefore ${}_AA$ is also nonsingular as left A-module, and $C({}_AA)$ is a complete modular lattice. So it remains to show that the maps determine a projectivity between $C(\bar{M})$ and $C({}_AA)$. First let $U \in C(\bar{M})$; then U is a direct summand of $\bar{M}$ since U is a complement submodule of $\bar{M}$ and $\bar{M}$ is injective. Therefore $U = \bar{M}e$ for some $e^2 = e \in A$, and $I_A(U) = I_A(\bar{M}e) = \{a \in A, \bar{M}a \subseteq \bar{M}e\} = Ae \in C({}_AA)$. Similarly, since ${}_AA$ is nonsingular and injective, $C({}_AA) = \{Ae, e^2 = e \in A\}$, then $\bar{M}Ae = \bar{M}e \in C(\bar{M})$. Secondly, if $\bar{M}e \in C(\bar{M})$, then $\bar{M}I_A(\bar{M}e) = \bar{M}Ae = \bar{M}e$; if $Ae \in C({}_AA)$, $I_A(\bar{M}Ae) = I_A(\bar{M}e) = Ae$, *i.e.* the two maps are inverses of each other. Finally, they are clearly order-preserving maps.

LEMMA 2.2. *Let B be a left nonsingular ring with the maximal left quotient ring A. Then $C({}_AA) = C({}_BA)$.*

PROOF. By [8, p. 247 Proposition 2.1(i)], ${}_AA$ is regular and left self-injective, and therefore $C({}_AA) = \{Ae, e^2 = e \in A\}$. Since A is the maximal left quotient ring of B, any A-submodule of ${}_AA$ is clearly a B-submodule of ${}_BA$ and hence $C({}_AA) \subseteq C({}_BA)$. On the other hand, ${}_BA$ is also a nonsingular injective B-module; in fact, ${}_BA$ is the injective hull of ${}_BB$. So $C({}_BA)$ consists of the nonsingular injective submodules of ${}_BA$, which are actually injective A-modules again by [8, p. 247 Proposition 2.1(ii)]. Hence they are all direct summands of A. Therefore $C({}_BA) \subseteq C({}_AA)$, *i.e.* $C({}_BA) = C({}_AA)$.

We also need the following known result from [1].

LEMMA 2.3 [1, P. 61 COROLLARY 8]. *If M is nonsingular and $M \subseteq_e M'$ then the maps $U' \longmapsto U' \cap M$ and $U \longmapsto U^e$ form a lattice isomorphism between $C(M')$ and $C(M)$, where $U' \in C(M')$ and U^e is the unique essential closure of U in M'.*

Now we are able to show our isomorphism theorem.

THEOREM 2.4. *Let M be a nonsingular module such that $B = \mathrm{Hom}(M, M)$ is left nonsingular and has $A = \mathrm{Hom}(\bar{M}, \bar{M})$ as its maximal left quotient ring. Then the maps*

$$F\colon U \longmapsto I_A(\bar{U}) \cap B, \quad F^{-1}\colon H \longmapsto (\bar{M}\bar{H}) \cap M$$

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form a lattice isomorphism between $C(M)$ *and* $C(B)$. *Moreover* $I_A(\bar{U}) \cap B = I_B(U)$ *for* $U \in C(M)$, *and* $(\bar{M}\bar{H}) \cap M = (MH)^e$ *for* $H \in C(B)$.

PROOF. The desired isomorphism follows immediately from Lemmas 2.1, 2.2, and 2.3. Now we show that $I_A(\bar{U}) \cap B = I_B(U)$. We identify $I_B(U)$ with $\mathrm{Hom}(M, U)$, $I_A(\bar{U})$ with $\mathrm{Hom}(\bar{M}, \bar{U})$ and $B = \mathrm{Hom}(M, M)$ with $\{f \in A, f(M) \subseteq M\}$. Then it is clear that $\mathrm{Hom}(M, U) \subseteq B$. Let $f \in A, f(M) \subseteq U$. Then $f(M) \subseteq \bar{U}$. Notice that $\bar{U}$ is injective, so there exists an extension f' of $f|_M$ such that $f'(\bar{M}) \subseteq \bar{U}$. Therefore $f = f' \in \mathrm{Hom}(\bar{M}, \bar{U})$ since $f(M) = f'(M)$, $M \subseteq_e \bar{M}$ and $\bar{M}$ is nonsingular. This shows that $\mathrm{Hom}(M, U) \subseteq \mathrm{Hom}(\bar{M}, \bar{U})$, also. Hence $\mathrm{Hom}(M, U) \subseteq B \cap \mathrm{Hom}(\bar{M}, \bar{H})$. On the other hand, if $f \in B \cap \mathrm{Hom}(\bar{M}, \bar{U})$, then $f(M) \subseteq M \cap \bar{U}$, which is exactly U, *i.e.* $f \in \mathrm{Hom}(M, U)$. So $B \cap \mathrm{Hom}(\bar{M}, \bar{U}) = \mathrm{Hom}(M, U)$. That is, $I_A(\bar{U}) \cap B = I_B(U)$.

Next we show that $(\bar{M}\bar{H}) \cap M = (MH)^e$. It suffices to show that $\big(MF(U)\big)^e = U$. Since F is a lattice isomorphism, if $0 \neq U \in C(M)$, then $F(U) = I_B(U) = \mathrm{Hom}(M, U) \neq 0$, *i.e.* M is e-retractable. Hence by [6, Theorem 2.4], $MI_B(U) \subseteq_e U$ for any $U \in C(M)$. So $\big(MF(U)\big)^e = \big(MI_B(U)\big)^e = U$, *i.e.* $F^{-1}(H) = (MH)^e$ for any $H \in C(B)$.

THEOREM 2.5. *Under the assumptions above, M is a* CS *module if and only if B is a left* CS *ring.*

PROOF. Let M be a CS module. Then for any $U \in C(M)$, $U = Me$ for some $e^2 = e \in B$, and $F(U) = I_B(U) = I_B(Me) = Be$, which is a direct summand of B and in $C(B)$. But by Theorem 2.4, F is a lattice isomorphism between $C(M)$ and $C(B)$. This implies B is a left CS ring. Conversely if B is a left CS ring, then for any $H \in C(B)$, $H = Be$ for some $e^2 = e \in B \subseteq A$. So $F^{-1}(Be) = (MBe)^e = (Me)^e = Me$, which is a direct summand of M and in $C(M)$. F^{-1} is also a lattice isomorphism between $C(B)$ and $C(M)$. Therefore M is a CS module.

In [6], it is shown that for a nonsingular and retractable module M, the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^e$ determine a projectivity between $C(M)$ and $C(B)$ if and only if $H \subseteq_e I_B(MH)$ for every $H \subseteq B$ [6, Theorem 3.1]. Here we have, as a consequence of Theorem 2.4, that the maps above determine a projectivity (in fact, a lattice isomorphism) already, provided M is nonsingular and retractable.

THEOREM 2.6. *If M is nonsingular and retractable, then the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^e$ determine a lattice isomorphism between $C(M)$ and $C(B)$.*

PROOF. Under this assumption, we have, by [4, Theorem 3.1] that B is left nonsingular, $B \subseteq_e {}_BA$ and A is the maximal left quotient ring of B. The conclusion follows directly from Theorem 2.4.

COROLLARY 2.7 [6, THEOREM 3.2]. *If M is nonsingular and retractable, then M is a* CS *module if and only if B is a left* CS *ring.*

COROLLARY 2.8 [5, THEOREM 3.10]. *Let M be nonsingular and nondegenerate. Then the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^e$ determine a lattice isomorphism between $C(M)$ and $C(B)$.*

Combining our Theorem 2.6 with Theorem 3.1 in [6], we immediately have

COROLLARY 2.9. *If M is nonsingular and retractable, then $H \subseteq_e I_B(MH)$ for any left ideal H of B.*

Consider the following two properties:

(I) For $U \subseteq V \subseteq M$, $U \subseteq_e V$ if and only if $I_B(U) \subseteq_e I_B(V)$.

(II) For $H \subseteq J \subseteq B$, $H \subseteq_e J$ if and only if $MH \subseteq_e MJ$.

In [5, Proposition 3.2], it was shown that when M is nondegenerate, M is retractable and has the properties (I) and (II). If M is nonsingular and retractable, then Khuri showed further that M has the property (I) [6, Theorem 2.2], and that M has the property (II) if and only if $H \subseteq_e I_B(MH)$ for any left ideal H of B [6, Corollary 2.6]. So it follows immediately from Corollary 2.9.

COROLLARY 2.10. *If M is nonsingular and retractable, then M has the properties (I) and (II) above.*

Let $d(M)$ be the Goldie dimension of a module M. Then it is known that $d(M) < \infty$ if and only if $C(M)$ satisfies the a. c. c. (the ascending chain condition) or the d. c. c. (the descending chain condition) [5], [2, p. 83]. Therefore another immediate consequence of Theorem 2.4 is

COROLLARY 2.11. *If M satisfies the assumptions in Theorem 2.4, then*

(1) $C(M)$ satisfies the a. c. c. *or the* d. c. c. *if and only if $C(B)$ does.*

(2) $d(M) < \infty$ if and only if $d(B) < \infty$, and in this case $d(M) = d(B)$.

In particular, this is the case when M is nonsingular and retractable.

PROOF. (1) It is obvious from Theorem 2.4. (2) follows directly from part (v) of the corollary on page 52 in [7].

A submodule U of M is called *a-closed* if $U = \mathrm{Ann}_M(H) = \{m \in M, mH = 0, H$ is a subset of $B\}$ [5]. Let $L(M)$ denote the set of all a-closed submodules of M, $L(B)$ the set of all left annihilator ideals of B. It is known that $L(M) \subseteq C(M)$ when M is nonsingular (*cf.* the proof of [5, Lemma 3.12]), and, in addition, if M is e-retractable, the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^a$ determine a lattice isomorphism between $L(M)$ and $L(B)$ [5, Lemma 3.12, Theorem 2.5], where $(MH)^a$ means the a-closure of MH [5, Definition 1]. But from Theorem 2.4, we know that if M satisfies the assumptions in Theorem 2.4, then for any $U \in C(M)$, $U = F^{-1}F(U)$, and $F^{-1}\big(F(U)\big) = \big(MF(U)\big)^e$. Therefore if $0 \neq U \in C(M)$, then $F(U) \neq 0$, that is, M is e-retractable, also. Consequently we have from Theorem 2.4

COROLLARY 2.12. *If M satisfies the assumptions in Theorem 2.4, then the maps $U \longmapsto I_B(U)$ and $H \longmapsto (MH)^a$ determine a lattice isomorphism between $L(M)$ and $L(B)$, and hence $L(B) \subseteq C(B)$.*

In particular, this is the case when M is nonsingular and retractable.

A ring B is a left Goldie ring if it satisfies the a. c. c. on $L(B)$ and on $C(B)$ [5]. So the last application we get is

COROLLARY 2.13. *If M satisfies the assumptions in Theorem 2.4, then B is a left Goldie ring if and only if M satisfies the* a. c. c. *on $C(M)$, and if and only if $d(M) < \infty$.*

In particular, this is the case when M is nonsingular and retractable.

This result contains [5, Corollary 3.14] as a special case.

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