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OPERATOR APPROXIMATIONS WITH STABLE EIGENVALUES

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Abstract

Suppose λ is an isolated eigenvalue of the (bounded linear) operator T on the Banach space X and the algebraic multiplicity of λ is finite. Let T_n be a sequence of operators on X that converge to T pointwise, that is, $T_n x \to Tx$ for every $x \in X$. If $\|(T - T_n)T_n\|$ and $\|T_n(T - T_n)\|$ converge to 0 then T_n is strongly stable at λ.

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1. Introduction

The main theorem in this note was motivated by the theory of collectively compact sequences of operators in numerical analysis. The applications and methods given here are from functional analysis. Let X be a fixed complex Banach space and let T be a fixed (bounded linear) operator on X. Let T_n be a sequence of operators on X that converge to T pointwise, that is, $T_n x \to Tx$ for every $x \in X$. If T is compact and T_n is collectively compact then P. M. Anselone showed that $\|(T - T_n)T_n\| \to 0$ and $\|(T - T_n)T\| \to 0$ [2, Corollary 1.9]. These limits are used to show the stability of isolated eigenvalues of finite algebraic multiplicity. (See [2, Theorem 4.16].) Recently M. Ahues [1] showed that the stability of the isolated eigenvalues of finite algebraic multiplicity followed from the assumptions that T is compact and

$\|(T-T_n)T_n\| \to 0$, without assuming that T_n is collectively compact. In this note we prove that the isolated eigenvalues of finite algebraic multiplicity are stable provided that $\|(T-T_n)T_n\| \to 0$ and $\|T_n(T-T_n)\| \to 0$.

2. Main results

NOTATION. Suppose λ is a nonzero eigenvalue of the operator T on the Banach space X and suppose λ is isolated from the remainder of the spectrum of T. Let P be the spectral projection associated with T, that is,

$$P = \frac{-1}{2\pi i}\int_\Gamma R(z)\,dz,$$

where $R(z) = (T-zI)^{-1}$ and Γ is a simple closed Jordan curve that isolates λ and has the origin outside. Let Δ be the union of Γ and the domain inside Γ, so that Δ is compact. The dimension of PX is said to be the algebraic multiplicity of λ for T.

The first lemma and its corollary permit us to consider $(T_n - zI)^{-1}$ on Γ.

LEMMA 1. *Suppose that $\|(T-T_n)T_n\| \to 0$, $\|T_n(T-T_n)\| \to 0$ and $T_n x \to Tx$ for every $x \in X$. Then there exists an n_0 such that Γ is contained in the resolvent set of T_n for $n \geq n_0$.*

PROOF. Since $R(z) = (T-zI)^{-1}$ is continuous on the compact set Γ, $\sup\{\|R(z)\| : z \in \Gamma\}$ is finite. Choose n_0 such that

$$\|z^{-1}R(z)(T-T_n)T_n\| \leq |z|^{-1}\|R(z)\|\|(T-T_n)T_n\| \leq \tfrac{1}{2}$$

and

$$\|z^{-1}T_n(T-T_n)R(z)\| \leq \tfrac{1}{2}$$

for every $z \in \Gamma$ and $n \geq n_0$. It follows that $I - z^{-1}R(z)(T-T_n)T_n$ and $I - z^{-1}T_n(T-T_n)R(z)$ are invertible for every $z \in \Gamma$ provided $n \geq n_0$. The easily verified equations

$$(*) \qquad I - z^{-1}R(z)(T-T_n)T_n = R(z)[I - z^{-1}(T-T_n)](T_n - zI),$$

$$(**) \qquad I - z^{-1}T_n(T-T_n)R(z) = (T_n - zI)[I - z^{-1}(T-T_n)]R(z)$$

imply that $(T_n - zI)$ is one-to-one and onto for every $z \in \Gamma$ and $n \geq n_0$. Thus, $(T_n - zI)$ is invertible by the Open Mapping Theorem and Γ is contained in the resolvent set of T_n for $n \geq n_0$.

Now we obtain some conclusions from the proof of Lemma 1 that are more significant.

Corollary. *Assume the hypotheses of Lemma* 1 *and let* $R_n(z)$ *denote* $(T_n - zI)^{-1}$. *Then for every* $x \in X$, $R_n(z)x \to R(z)x$ *uniformly for* $z \in \Gamma$ *and there are constants* M *and* n_0 *such that*

$$\sup\{\|R_n(z)\| : z \in \Gamma,\, n \geq n_0\} \leq M.$$

Proof. From the proof of Lemma 1 and the equation $(*)$ we obtain

$$\begin{aligned} R_n(z) &= [I - z^{-1}R(z)(T - T_n)T_n]^{-1}R(z)[I - z^{-1}(T - T_n)] \\ &= \left[I + \sum_{k=1}^{\infty}(z^{-1}R(z)(T - T_n)T_n)^k]R(z)[I - z^{-1}(T - T_n)\right]. \end{aligned}$$

It follows that

$$\begin{aligned} \|R_n(z)\| &\leq \left[1 + \sum_{k=1}^{\infty}\|z^{-1}R(z)(T - T_n)T_n\|^k\right]\|R(z)[I - z^{-1}(T - T_n)]\| \\ &\leq 2\sup\{\|R(z)\|[1 + |z|^{-1}\|T - T_n\|] : z \in \Gamma,\, n \geq n_0\}. \end{aligned}$$

The last supremum is finite because $\|R(z)\|$ is continuous on the compact set Γ, 0 is not an element of Γ, and $\|T - T_n\|$ is bounded by the Uniform Boundedness Theorem. This proves that

$$\sup\{\|R_n(z)\| : z \in \Gamma,\, n \geq n_0\} \leq M < \infty.$$

For $z \in \Gamma$ and $n \geq n_0$ we have $R(z) - R_n(z) = R_n(z)(T_n - T)R(z)$. Thus, for every $x \in X$ we get

$$\|[R(z) - R_n(z)]x\| \leq \|R_n(z)\|\|(T_n - T)R(z)x\| \leq M\|(T_n - T)R(z)x\|.$$

It is elementary that $\|(T_n - T)R(z)x\| \to 0$ uniformly for $z \in \Gamma$ [3, Theorem 3.2]. This proves the corollary.

Now we know that Γ separates the spectrum of T_n, denoted $\sigma(T_n)$, and we may consider the spectral projections of T_n associated with Γ.

Notation. Define P_n, X_n and X_0 by

$$P_n = \frac{-1}{2\pi i}\int_{\Gamma} R_n(z)\,dz, \quad X_n = P_nX,\, X_0 = PX.$$

It is well known that P commutes with T (P_n commutes with T_n) and the spectrum of T restricted to the invariant subspace X_0 lies inside Δ, that is, $\sigma(T|X_0) \subset \Delta$. (See [4, Theorem 20], for example.)

The following technical lemma plays a key role in the proof of the main theorem.

LEMMA 2. *Suppose that* $\|(T-T_n)T_n\| \to 0$, $\|T_n(T-T_n)\| \to 0$ *and* $T_n x \to Tx$ *for every* $x \in X$. *Then we have*

$$\|P_n - T_n[(T|X_0)^{-1} \oplus 0]P\| \to 0.$$

PROOF. We use the Taylor-Dunford operational calculus. (See [4, Theorem 10], for example.) Note that

$$\begin{aligned} P_n - T_n[(T|X_0)^{-1} \oplus 0]P &= T_n([(T_n|X_n)^{-1} \oplus 0]P_n - [(T|X_0)^{-1} \oplus 0]P) \\ &= T_n\left(\frac{-1}{2\pi i}\right)\int_\Gamma z^{-1}(R_n(z) - R(z))\,dz \\ &= T_n\left(\frac{-1}{2\pi i}\right)\int_\Gamma z^{-1}R_n(z)(T - T_n)R(z)\,dx \\ &= \left(\frac{-1}{2\pi i}\right)\int_\Gamma z^{-1}R_n(z)T_n(T - T_n)R(z)\,dz. \end{aligned}$$

Thus,

$$\|P_n - T_n[(T|X_0)^{-1} \oplus 0]P\| \le \frac{1}{2\pi}\|T_n(T - T_n)\| \sup_{z\in\Gamma}\|z^{-1}R_n(z)\|\|R(z)\|.$$

The last supremum is finite according to the corollary to Lemma 1. Thus, Lemma 2 is proved.

In the next corollary we state a consequence of Lemma 2 that is easier to understand than the lemma itself.

COROLLARY. *If the hypotheses of Lemma* 2 *hold then* $\|(P - P_n)P_n\| \to 0$.

PROOF. Note that

$$\begin{aligned} (P - P_n)P_n &= \frac{-1}{2\pi i}\int_\Gamma (R(z) - R_n(z))P_n\,dz \\ &= \frac{-1}{2\pi i}\int_\Gamma (R(z)(T_n - T)R_n(z)P_n)\,dz \\ &= \frac{-1}{2\pi i}\int_\Gamma (R(z)[(T_n - T)P_n]R_n(z))\,dz. \end{aligned}$$

Thus,

$$\|(P - P_n)P_n\| \le \frac{1}{2\pi}\|(T_n - T)P_n\| \sup_{z\in\Gamma}\|R(z)\|\|R_n(z)\|$$

and it clearly suffices to show that $\|(T_n - T)P_n\| \to 0$. In view of Lemma 2 it suffices to show that

$$\|(T_n - T)T_n[(T|X_0)^{-1} \oplus 0]P\| \to 0$$

and that is clear.

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Now we make the notion of "stable eigenvalue" precise.

DEFINITION. We say that T_n is strongly stable at the nonzero isolated eigenvalue λ provided there is a neighborhood of λ such that the following hold whenever Γ is a simple closed Jordan curve in that neighborhood.

(i) $R_n(z)x \to R(z)x$ for every $z \in \Gamma$ and $x \in X$.

(ii) $\dim X_n = \dim X_0$ for $n \geq n_1$.

With the terms and notation that have been given, we can state and prove the main theorem.

THEOREM. *Suppose that* $\|(T-T_n)T_n\| \to 0$, $\|T_n(T-T_n)\| \to 0$ *and* $T_n x \to Tx$ *for every* $x \in X$. *Then* T_n *is strongly stable at every isolated nonzero eigenvalue* λ *of* T *that has finite algebraic multiplicity.*

PROOF. Recall that it was proved in the corollary to Lemma 1 that $R_n(z)x \to R(z)x$ for every $z \in \Gamma$ and $x \in X$. We shall show that $P_n x \to Px$ for every $x \in X$. Choose n_0 according to the corollary to Lemma 1; for $n \geq n_0$ and $x \in X$ we have

$$\begin{aligned}\|(P_n - P)x\| &= \frac{1}{2\pi}\left\|\int_\Gamma (R_n(z) - R(z))x\,dz\right\| \\ &= \frac{1}{2\pi}\left\|\int_\Gamma (R_n(z)(T - T_n)R(z))x\,dz\right\| \\ &\leq \frac{1}{2\pi} M \sup\{\|(T - T_n)R(z)x\| : z \in \Gamma,\ n \geq n_0\}\end{aligned}$$

It is elementary that the supremum above converges to 0 as n_0 increases [3, Theorem 3.2].

Since $P_n x \to Px$ for every $x \in X$ and PX is finite dimensional, we conclude that $\|(P - P_n)P\| \to 0$. It is easy to see that the gap between X_n and X_0 (see [6, page 197] for the definition of gap), denoted $\gamma(X_n, X_0)$, does not exceed

$$\max\{\|(P - P_n)P\|, \|(P - P_n)P_n\|\}.$$

The corollary to Lemma 2 and the preceding limit show that $\gamma(X_n, X_0) < 1$ for all n sufficiently large. According to [6, Corollary 2.6], this implies that $\dim X_n = \dim X_0$ and the theorem is proved.

3. Applications and remarks

It is easy to see that if $T_n x \to Tx$ for every $x \in X$ and T_n is collectively compact then T is compact. We observed in the introduction that if T_n is collectively compact then $\|(T - T_n)T_n\| \to 0$. It is interesting to note that if

T_n is collectively compact then $\|T_n(T - T_n)\|$ may fail to converge to 0. We construct an example to illustrate this.

Let $\{e_1, e_2, \dots\}$ be an orthonormal basis for the Hilbert space H and define T and A_n by

$$T = \sum_{k=1}^{\infty} \langle \cdot , e_k \rangle \frac{1}{k} e_k , \quad A_n = \langle \cdot , e_n \rangle e_1 ,$$

and let $T_n = T + A_n$. It is easy to see that A_n is collectively compact, T is compact, and $T_n x \to Tx$ for every $x \in X$. Thus, T_n is collectively compact and

$$T_n(T_n - T) = (T + A_n)A_n = TA_n = A_n \quad \text{for } n \geq 2.$$

Since $\|A_n\| = 1$ for every n, we have $\|T_n(T_n - T)\| = 1$ for $n \geq 2$.

Thus, the main theorem in this note does not generalize the theory of collectively compact sequences. Rather this note offers an alternative route to the same conclusion. Our next proposition will make it clear that the hypotheses of our main theorem do not require T_n to be collectively compact or T to be compact.

Proposition. *Let T be an operator on the Hilbert space H with orthonormal basis $\{e_0, e_1, \dots\}$. Suppose T is specified by giving its infinite matrix, that is,*

$$T = \sum_{i,j=0}^{\infty} \langle \cdot , e_j \rangle a_{ij} e_i .$$

Define T_n by

$$T_n = \sum_{i,j=0}^{n} \langle \cdot , e_j \rangle a_{ij} e_i .$$

Then T_n is strongly stable at every isolated eigenvalue of T with finite algebraic multiplicity.

Proof. It is routine to see that $T_n x \to Tx$ for every $x \in X$ and

$$T_n(T - T_n) = 0 = (T - T_n)T_n .$$

So this proposition follows from our main theorem.

There are many consequences of the preceding proposition; the next corollary is illustrative. Let L^2 be the Hilbert space of "square integrable functions" with respect to normalized Lebesgue measure on the interval $[0, 2\pi]$. If ϕ is measurable and essentially bounded then the operator that multiplies by ϕ is the Laurent operator denoted by L_ϕ. Let H^2 denote the Hilbert space in L^2 spanned by $\{e^{int} : 0 \leq t \leq 2\pi,\ n = 0, 1, 2, \dots\}$ and let P

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denote the orthogonal projection of L^2 onto H^2. Recall that the Toeplitz operator T_ϕ associated with ϕ is defined by restricting PL_ϕ to H^2, that is, $T_\phi = PL_\phi|H^2$. If ϕ belongs to H^2 then T_ϕ is said to be analytic Toeplitz operator.

COROLLARY. *Let T be an analytic Toeplitz operator on the Hilbert space H^2 and let $\{e_0, e_1, e_2, \ldots\}$ be the orthonormal basis indicated above. Then T and its adjoint T^* have no nonzero eigenvalues of finite algebraic multiplicity.*

PROOF. Define a_{ij} by

$$T = \sum_{i,j=0}^{\infty} \langle \cdot, e_j \rangle a_{ij} e_i$$

and define T_n by

$$T_n = \sum_{i,j=0}^{n} \langle \cdot, e_j \rangle a_{ij} e_i.$$

Recall that the matrix for T is constant along the diagonals parallel to the main diagonal and that the matrix for T is lower triangular when T is analytic. (See [5, pages 135–140], for example.) Thus, the $n \times n$ matrix for each T_n is lower triangular with constant diagonals. It follows that the spectrum of T_n is $\{\lambda, 0\}$ where λ is the constant enumerated on the main diagonal. Thus, λ is the only possible isolated eigenvalue of T. Since the algebraic multiplicity of λ for T_n is n, λ cannot be a strongly stable eigenvalue of T. Thus, T has no nonzero eigenvalues of finite algebraic multiplicity.

The preceding argument applies equally well to T^* except that its matrix is upper triangular.

Clearly there is a result for Laurent operators analogous to the preceding corollary. This application does not follow from the theory of collectively compact sequences since the only compact Toeplitz operator is the zero operator [5, page 137].

References

[1] M. Ahues, 'A class of strongly stable operator approximations,' *J. Austral. Math. Soc. Ser. B* **28** (1987), 435–442.

[2] P. M. Anselone, *Collectively compact operator approximation theory*, (Prentice-Hall, Englewood Cliffs, N. J., 1971).

[3] F. Chatelin, *Spectral approximation of linear operators*, (Academic Press, New York, 1983).
[4] N. Dunford and J. T. Schwartz, *Linear operators*, Vol. 1, (John Wiley, New York, 1964).
[5] P. R. Halmos, *A Hilbert space problem book*, (Van Nostrand, Princeton, N. J., 1967).
[6] T. Kato, *Perturbation theory for linear operators*, (Springer-Verlag, New York, 1966).

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