Can. J. Math.Vol. 46 (1), 1994 pp. 81–119

KERNEL GENERATED TWO-TIME PARAMETER GAUSSIAN PROCESSES AND SOME OF THEIR PATH PROPERTIES

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ABSTRACT. We study path properties of kernel generated two-time parameter, not necessarily stationary, Gaussian processes. We establish large deviation results for some increments of these processes and use these results to prove some of their moduli of continuity and other path properties.

1. **Introduction and statement of results.** This exposition was inspired by some recent studies of infinite dimensional stochastic evolution equations which, in turn, were initiated by D. A. Dawson [17, 18], where the solution was used to model continuous space-time population processes. In particular, the roots of this paper are related to the infinite dimensional Gaussian process $Y(t) = (X_1(t), \ldots, X_i(t), \ldots)$, where the $\{X_i(t), -\infty < t < \infty\}$, are independent Ornstein-Uhlenbeck processes with coeffiecients $\gamma_i \geq 0$ and $\lambda_i > 0$, *i.e.*, $EX_i(t) = 0$ and $EX_i(s)X_i(t) = (\gamma_i/\lambda_i)\exp(-\lambda_i|s-t|), i = 1, 2, \ldots$. In Dawson [17] the process $Y(\cdot)$ is the stationary solution of the infinite array of stochastic differential equations

$$dX_i(t) = -\lambda_i X_i(t)\, dt + (2\gamma_i)^{1/2}\, dW_i(t) \quad (i = 1, 2, \ldots), \tag{1.1}$$

where $\{W_i(t), -\infty < t < \infty\}$ are independent Wiener processes (*cf.* also Dawson [18], Walsh [38, 39] and Antoniadis and Carmona [2]). Such processes have been used as a model for neuronal behaviour (Kallianpur and Wolpert [28], Walsh [38]); they have proved important in quantum field theory (Carmona [4], Röckner [32]); and they also arise as fluctuation limits in infinite particle systems (Holley and Strook [24]). Owing to the variety of applications of these processes, they have been looked at from a number of different mathematical angles. For instance, they have been considered as examples of stochastic evolution equations (DaPrato, Kwapien and Zabczyk [16], Kotelenez [29], Miyahara [31]); or as reversible Markov processes which may be studied by using the

The first author was supported by an NSERC Canada grant at Carleton University.

The second author was supported by the National Natural Science Foundation of China and by an NSERC Canada grant of M. Csörgő at Carleton University.

The third author was supported by the Fok Yingtung Education Foundation and by an NSERC Canada Scientific Exchange Award at Carleton University.

Received by the editors June 19, 1992; revised June 11, 1993.

AMS subject classification: Primary: 60G15, 60G17, 60F10, 60F15; secondary: 60H05.

Key words and phrases: two-time parameter Gaussian processes, large deviations, path properties, kernel function, Wiener, Kiefer and Ornstein-Uhlenbeck processes.

associated theory of Dirichlet forms (Schmuland [34, 35]); or as solutions to stochastic p.d.e.'s (Walsh [38, 39]). They have also been studied directly as Gaussian processes (Antoniadis and Carmona [2], Csáki and Csörgő [5, 6], Csáki, Csörgő, Lin and Révész [7], Csáki, Csörgő and Shao [8, 9], Csörgő and Lin [10, 11, 12], Csörgő, Lin and Shao [14], Fernique [21,22], Iscoe, Marcus, McDonald, Talagrand and Zinn [25], Schmuland [36]), and it is the latter lines of research which have led us to our present deliberations which we now outline.

Throughout this paper the following notation conventions will be used: $\log x = \log\max(x, e)$ and $\tilde{x} = x + 1/x$.

Concerning the infinite dimensional Ornstein-Uhlenbeck process $Y(\cdot)$ as above, a special case of Théorème in Fernique [19] reads as follows: For each $x \in \mathbb{R}^+$, let $K(x) = \{k \in \mathbf{N} : \gamma_k > \lambda_k x\}$ and $\lambda(x) = \sup\{\lambda_K : k \in K(x)\}$. Then $Y(\cdot) \in \ell^2$ is a.s. continuous if and only if we have $\sum_{k=1}^{\infty}(\gamma_k/\lambda_k) < \infty$ and $\int(\log[\lambda(x)] \vee 0)\,dx < \infty$ as well. Consequently, (*cf.* Corollary 1 of Fernique [21]), for $Y(\cdot) \in \ell^2$ *to be a.s. continuous, it is sufficient that we have*

$$\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)\Big(1 + \big((\log\lambda_k) \vee 0\big)\Big) < \infty. \tag{1.2}$$

For moduli of continuity results $Y(\cdot) \in \ell^2$ and for its closely related ℓ^2-norm squared process $\chi^2(t) = ||Y(t)||^2 = \sum_{k=1}^{\infty} X_k^2(t)$, we refer to Csörgő and Lin [11, 12], Schmuland [33], Csáki and Csörgő [5], and Csáki, Csörgő and Shao [8]. Fernique [22] gives also necessary and sufficient conditions for $Y(\cdot) \in \ell^p$, $2 \leq p < \infty$, to be a.s. continuous. Csáki and Csörgő [6], Csáki, Csörgő and Shao [9], and Schmuland [36] study continuity and moduli of continuity sample path properties of $Y(\cdot) \in \ell^p$, $1 \leq p \leq 2$, and those of $Y(\cdot) \in \ell^\infty$ are studied by Csörgő, Lin and Shao [14].

Another real valued process which is also closely related to $Y(\cdot) \in \ell^2$ is the stationary mean zero Gaussian process $X(\cdot)$ defined by

$$\{X(t), -\infty < t < \infty\} = \Big\{\sum_{k=1}^{\infty} X_k(t), -\infty < t < \infty\Big\},$$

where $X_k(\cdot)$ are the Ornstein-Uhlenbeck components of $Y(\cdot)$. Let

$$\{X(t,n), -\infty < t < \infty, n = 1, 2, \dots\} = \Big\{\sum_{k=1}^{n} X_k(t), -\infty < t < \infty, n = 1, 2, \dots\Big\}. \tag{1.3}$$

Csáki, Csörgő, Lin and Révész [7] show that $X(t,n) \longrightarrow X(t)$ uniformly in t over finite intervals with probability one as $n \longrightarrow \infty$ if for some $\delta > 0$ we have

$$\sum_{k=1}^{\infty} \gamma_k\big(\log(\lambda_k \vee e)\big)^{1+\delta}/\lambda_k < \infty, \tag{1.4}$$

and hence, under the latter condition, $X(\cdot)$ is continuous with probability one. Csáki *et al.* [7] prove also exact moduli of continuity results for $X(\cdot)$. The relationship of $X(\cdot)$ and

$Y(\cdot) \in \ell^2$ is illustrated by conditions (1.2) and (1.4). Indeed, the process $X(\cdot)$ is continuous if and only if it satisfies Fernique's necessary and sufficient condition for the continuity of a stationary Gaussian process (*cf.* [19] and Corollary 2.5 of Section IV.2 in Jain and Marcus [27]), *i.e.*, if and only if in this case $E|X(t) - X(s)|^2 = \phi^2(|t-s|)$, where $\phi(u)$ is an increasing function in $u > 0$, we have that $\phi(u)/\left(u\big(\log(1/u)\big)^{1/2}\right)$ is integrable at zero (*cf.* also Theorem 2.2 in Csáki *et al.* [7]). Since

$$E\|Y(t) - Y(s)\|^2 = E|X(t) - X(s)|^2,$$

hence, in general, checking Fernique's necessary and sufficient condition for the a.s. continuity of the real valued, stationary, mean zero Gaussian process $X(\cdot)$ should be also sufficient for that of the stationary, mean zero Gaussian process $Y(\cdot)$ in ℓ^2. The conditions (1.2) and (1.4) illustrate this point. More importantly along these lines, it was pointed out by a referee of Csörgő and Lin [13] (*cf.* p. 425 of [13]) that a more precise analysis and study of the results of Fernique [21] yield in effect that the following four properties are equivalent:

(i) $P\{\forall t, Y(t) \in \ell^2\} = 1, \quad P\{t \longrightarrow Y(t) \text{ continuous}\} = 1,$

(ii) $\sum_{k=1}^{\infty} \gamma_k/\lambda_k < \infty, \quad \int \log^+ \lambda(x)\, dx < \infty,$

(iii) $\forall t,\ P\{X(t,n) \longrightarrow X(t)\} = 1, \quad P\{t \longrightarrow X(t) \text{ continuous}\} = 1,$

(iv) $P\{X(\cdot,n) \longrightarrow X(\cdot) \text{ uniformly over every finite interval}\} = 1.$

Path properties of the two-parameter Gaussian process $X(t,n)$ of (1.3) were studied by Csörgő and Lin [10]. Integrating the equations in (1.1) from $-\infty$ to t we obtain

$$X_i(t) = \int_{-\infty}^{t} \exp(-\lambda_i|t-s|)(2\gamma_i)^{1/2}\, dW_i(s) \quad (i = 1, 2, \ldots), \tag{1.5}$$

and hence we have also

$$X(t,n) = \sum_{k=1}^{n} X_k(t) = \sum_{k=1}^{n} \int_{-\infty}^{t} \exp(-\lambda_k|t-s|)(2\gamma_k)^{1/2}\, dW_k(s). \tag{1.6}$$

The latter, in turn, has led us to study, in the same paper, also the two-parameter Gaussian process

$$X(t,v) = \int_0^v \int_{-\infty}^t \exp\big(-\lambda(y)(t-x)\big)\big(2\gamma(y)\big)^{1/2}\, dW(x,y), \tag{1.7}$$

where $\gamma(y)$ and $\lambda(y)$ are assumed to be positive continuous functions on $[0,\infty)$, and $\{W(x,y), -\infty < x < \infty, 0 \le y < \infty\}$ is a standard two-parameter Wiener process (*cf.* Sections 1.10–1.15 and the Supplementary remarks of Chapter 1 in Csörgő and Révész [15]).

This brings us to the main topic of our present exposition. Here we study two-parameter Gaussian processes $\{X(t,v), t \in \mathbb{R}, v \in \mathbb{R}^+\}$ of the form

$$X(t,v) = \int_0^\infty \int_{-\infty}^\infty \Gamma(t,v,x,y)\, dW(x,y), \tag{1.8}$$

where the kernel function $\Gamma(t,v,x,y)$ is assumed to be square integrable in (x,y) on $\mathbb{R}^+ \times \mathbb{R}$, and $W(x,y)$ is a standard two-parameter Wiener process. Thus $X(t,v)$ is a Gaussian process with mean zero and covariance function

$$\operatorname{Cov}\big(X(t,v),X(s,u)\big) = \int_0^\infty \int_{-\infty}^\infty \Gamma(t,v,x,y)\Gamma(s,u,x,y)\,dx\,dy. \tag{1.9}$$

We note that we do not assume stationarity properties for $X(t,v)$ and that we can assume $X(t,v)$ to be separable. For an earlier study of such processes we refer to Csörgő and Lin [13] and Lin [30]. Here we improve the results of [13] and [30] to a great extent and under much weakened conditions. In particular, the large deviation results of Section 2 are much sharper than those of [13] and [30]. The ones we have here are more like those of Fernique (*cf.*, *e.g.*, [19], [20], [21] and [22]). However, we pay more attention to details in the case of non-stationary Gaussian processes, especially when they are kernel generated as in (1.8). Also, the new and weaker conditions here are easier to verify than the stronger ones in [13] and [30]. Before stating and commenting on our main new theorems concerning path properties of processes $X(t,v)$ as in (1.8), we give five examples, illustrating our results in some special cases. Right after stating our four main theorems, we spell out nine corollaries which deal with path properties of the processes in Examples 1–5. These corollaries well illustrate also the main features of our theorems, which can be summarized by saying that they succeed in treating moduli of continuity and large increment path properties respectively, simultaneously in the two time parameters. These features of two-time parameter fluctuation theory are believed to be new, and the results of the nine corollaries themselves are also new, even for the well known two-time parameter Wiener and Kiefer processes.

We let

$$H_1^2(t,s,v) = E\{X(t+s,v) - X(t,v)\}^2, \tag{1.10}$$

$$X\big(R(t,s,v,u)\big) = X(t+s,v+u) - X(t,v+u) - X(t+s,v) + X(t,v), \tag{1.11}$$

$$H_2^2(t,s,v,u) = EX^2\big(R(t,s,v,u)\big). \tag{1.12}$$

It is easy to see that

$$H_1^2(t,s,v) = \int_0^\infty \int_{-\infty}^\infty \big(\Gamma(t+s,v,x,y) - \Gamma(t,v,x,y)\big)^2\,dx\,dy \tag{1.13}$$

$$\begin{aligned} H_2^2(t,s,v,u) = \int_0^\infty \int_{-\infty}^\infty \big(&\Gamma(t+s,v+u,x,y) - \Gamma(t,v+u,x,y) \\ &- \Gamma(t+s,v,x,y) + \Gamma(t,v,x,y)\big)^2\,dx\,dy. \end{aligned} \tag{1.14}$$

The following examples are immediate.

EXAMPLE 1. If $\Gamma(t,v,x,y) = \mathbf{1}_{(-\infty,t]\times[0,v]}(x,y)$, $-\infty < t < \infty$, $0 \le v < \infty$, then

$$X(t,v) = W(t,v),$$
$$H_1^2(t,s,v) = sv, \quad 0 \le s < \infty,$$
$$H_2^2(t,s,v,u) = su, \quad 0 \le s,u < \infty.$$

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EXAMPLE 2. If $\Gamma(t,v,x,y) = \mathbf{1}_{[0,t]\times[0,v]}(x,y) - t\mathbf{1}_{[0,1]\times[0,v]}(x,y)$, $0 \le t \le 1$, $0 \le v < \infty$, then $X(t,v) = W(t,v) - tW(1,v)$, a Kiefer process (*cf.* Section 1.15 in Csörgő and Révész [15]),

$$H_1^2(t,s,v) = s(1-s)v, \quad 0 \le s \le 1,\ 0 \le v < \infty,$$
$$H_2^2(t,s,v,u) = s(1-s)u, \quad 0 \le s \le 1,\ 0 \le u < \infty.$$

EXAMPLE 3. If, with $-\infty < t < \infty$, $0 < v < \infty$,

$$\Gamma(t,v,x,y) = \mathbf{1}_{(-\infty,t]\times(0,v]}(x,y)\exp\bigl(-\lambda(y)(t-x)\bigr)\bigl(2\gamma(y)\bigr)^{1/2},$$

where $\lambda(y)$ and $\gamma(y)$ are positive continuous functions on $(0,\infty)$, then $X(t,v)$ is the two parameter Gaussian process of (1.7) with

$$H_1^2(t,s,v) = 2\int_0^v \frac{\gamma(x)}{\lambda(x)}\Bigl(1-\exp\bigl(-\lambda(x)s\bigr)\Bigr)\,dx,$$
$$H_2^2(t,s,v,u) = 2\int_v^{v+u} \frac{\gamma(x)}{\lambda(x)}\Bigl(1-\exp\bigl(-\lambda(x)s\bigr)\Bigr)\,dx.$$

EXAMPLE 4. If, with $-\infty < t < \infty$, $0 < v < \infty$,

$$\Gamma(t,v,x,y) = \sum_{k=0}^{\infty}\phi_k(v)\mathbf{1}_{(-\infty,t]\times(k,k+1]}(x,y)\exp\bigl(-\lambda_k(t-x)\bigr)(2\gamma_k)^{1/2},$$

then

$$H_1^2(t,s,v) = 2\sum_{k=0}^{\infty}\phi_k^2(v)(1-e^{-\lambda_k s})\Bigl(\frac{\gamma_k}{\lambda_k}\Bigr),$$
$$H_2^2(t,s,v,u) = 2\sum_{k=0}^{\infty}\bigl(\phi_k(v+u)-\phi_k(v)\bigr)^2(1-e^{-\lambda_k s})\frac{\gamma_k}{\lambda_k},$$
$$X(t,v) = \sum_{k=0}^{\infty}\phi_k(v)X_k(t),$$

where $\{X_k(t), -\infty < t < \infty\}$ are independent Ornstein-Uhlenbeck processes with coefficients $\gamma_k \ge 0$ and $\lambda_k > 0$.

EXAMPLE 5. If $\Gamma(t,v,x,y) = f(x,y)g_1(t)g_2(v)$, then

$$H_1^2(t,s,v) = g_2^2(v)\bigl(g_1(t+s)-g_1(t)\bigr)^2\cdot c,$$
$$H_2^2(t,s,v,u) = \bigl(g_2(v+u)-g_2(v)\bigr)^2\bigl(g_1(t+s)-g_1(t)\bigr)^2\cdot c,$$

where $c = \int_0^\infty\int_{-\infty}^\infty f^2(x,y)\,dx\,dy < \infty$.

This paper is organized as follows. In Section 2 we establish new large deviation results for some increments of the process $X(t,v)$ as in (1.8). These are more general and sharper than the ones proved in [12]. We use these results to prove our main theorems concerning path properties of the $X(t,v)$ process, which we now proceed to state and comment on. The proofs of these results are given in Section 3.

Put

$$H^2(t,s,u,v) = E\Big(X(t+s, v+u) - X(t,v)\Big)^2$$
$$= \int_0^\infty \int_{-\infty}^\infty \Big(\Gamma(t+s, v+u, x, y) - \Gamma(t,v,x,y)\Big)^2\, dx\, dy,$$

and let

$$\phi(h,Z) = \sup_{0\le s,u\le h, |t|\le A, |v|\le A} H(t,s,u,v).$$

We note that, by a classical result of Fernique [20] (*cf.* Berman [3], p. 197), if we have

$$\int_0^\infty \phi(e^{-y^2}, A)\, dy < \infty \quad \text{for every } A > 0, \tag{1.15}$$

then $X(t,v)$ is almost surely continuous. Since we are mainly interested in studying moduli of continuity and other path properties of increments of $X(t,v)$, for the convenience of the statements, and for that of the proofs later on, *we assume throughout the whole paper* that $X(t,v)$ *is a.s. continuous*. Also, further in this section we assume that $H_1(t,s,v)$ is non-decreasing in s, $H_2(t,s,v,u)$ is non-decreasing in s and u, and that a_T, b_T, c_T and D_T, $H_1(t,s,T)$ and $H_2(t,s,v,u)$ are continuous functions of T, s, u. For the conditions (2.12) and (2.38), we refer to Section 2. Our main results are as follows.

THEOREM 1.1. *Assume that (2.12) is satisfied and that there are positive numbers c and α such that*

$$\frac{H_1(t,s,T)}{s^\alpha} \le c\frac{H_1(t,s_1,T)}{s_1^\alpha} \tag{1.16}$$

for each $|t| \le b_T + a_T$, $0 \le s \le s_1 \le a_T$. Moreover, assume

$$\lim_{\delta\to 0} \limsup_{T\to\infty} \sup_{|t|\le b_T + a_T} \sup_{0\le s\le a_T} \frac{H_1(t+s, \delta a_T, T)}{H_1(t, a_T, T)} = 0 \tag{1.17}$$

$$\log\log\Big(a_T + \frac{1}{b_T}\Big) = o\Big(\log\frac{b_T}{a_T}\Big) \quad \textit{as } T\to\infty. \tag{1.18}$$

Then we have

(1.19)

$$\limsup_{T\to\infty} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} |X(t+s,T) - X(t,T)| \Big/ \Big\{H_1(t,a_T,T)\cdot\Big(2\Big(\log\frac{b_T}{a_T} + \log\log\tilde H_1(t,a_T,T)\Big)\Big)^{1/2}\Big\} \le 1 \textit{ a.s}$$

If, in addition,

$$\log\log \tilde H_1(t,a_T,T) = o\Big(\log\frac{b_T}{a_T}\Big) \tag{1.20}$$

uniformly in $|t|\le b_T$, as $T\to\infty$,

$$E\Big(X\big((j+1)s, v\big) - X(js,v)\Big)\Big(X\big((l+1)s,u\big) - X(ls,u)\Big) \le 0 \tag{1.21a}$$

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for each $j \neq l$, $s > 0$, *or*

$$H_1(t,s,T) = H_1(0,s,T) \text{ and} \quad (1.21b)$$
$$E\Big(X\big((j+1)s,v\big) - X(js,v)\Big)\Big(X\big((l+1)s,v\big) - X(ls,v)\Big) \leq 0$$

for each $j \neq l$, $s > 0$, $|t| \leq b_T$, *then, we have*

$$\lim_{T\to\infty} \sup_{0\leq t\leq b_T} \frac{|X(t+a_T,T) - X(t,T)|}{H_1(t,a_T,T)(2\log\frac{b_T}{a_T})^{1/2}} = 1 \quad a.s. \quad (1.22)$$

$$\lim_{T\to\infty} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \frac{|X(t+s,T) - X(t,T)|}{H_1(t,a_T,T)(2\log\frac{b_T}{a_T})^{1/2}} = 1 \text{ a.s.} \quad (1.23)$$

THEOREM 1.2. *Let* a_T *be non-decreasing,* $H_1(t,s,T)$ *non-decreasing in* t *on* $(0,\infty)$ *and non-increasing in* t *on* $(-\infty,0]$. *Assume that (2.12), (1.16), (1.17) and*

$$\frac{b_T}{a_T} + \tilde{H}_1(t,a_T,T) \longrightarrow \infty \quad \text{uniformly in } |t| \leq b_T \quad \text{as } T \longrightarrow \infty \quad (1.24)$$

are satisfied. Then (1.19) holds true. If, in addition, $b_T \geq c_0 > 0$, *(1.20), (1.21a) or (1.21b) are satisfied, then (1.22) and (1.23) also hold true.*

THEOREM 1.3. *Assume that (2.38) is satisfied and that there are positive numbers* c *and* α *such that*

$$\frac{H_2(t,s,v,u)}{s^\alpha} \leq c\frac{H_2(t,s_1,v,u)}{s_1^\alpha} \quad (1.25)$$

for each $0 \leq s \leq s_1 \leq a_T$, $0 \leq v \leq D_T + c_T$, $0 \leq u \leq 2c_T$, $|t| \leq b_T + a_T$. *Moreover, assume*

$$\lim_{\delta\to 0} \limsup_{|t|\leq b_T+a_T} \sup_{0\leq s\leq a_T} \sup_{0\leq v\leq D_T+c_T} \sup_{-\delta c_T\leq u\leq c_T} \frac{H_2(t+s,a_T,v+u,\delta c_T) + H_2(t+s,\delta a_T,v+u,c_T)}{H_2(t,a_T,v,c_T)} = 0, \quad (1.26)$$

$$\log\log\Big(a_T + c_T + \frac{1}{D_T} + \frac{1}{b_T}\Big) = o\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(1+\frac{D_T}{c_T}\Big)\Big) \text{ as } T \longrightarrow \infty. \quad (1.27)$$

Then

(1.28)

$$\limsup_{T\to\infty} \sup_{0\leq v\leq D_T} \sup_{0\leq u\leq c_T} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \Big|X\big(R(t,s,v,u)\big)\Big| \Big/$$
$$H_2(t,a_T,v,c_T)\Bigg(2\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(1+\frac{D_T}{c_T}\Big) + \log\log\tilde{H}_2(t,a_T,v,c_T)\Big)\Bigg)^{1/2} \leq 1 \quad a.s.$$

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If, in addition, the following conditions are satisfied

$$\log\log \tilde{H}_2(t, a_T, v, c_T) = o\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(1+\frac{D_T}{c_T}\Big)\Big) \tag{1.29}$$

uniformly in $|t| \le b_T$ *and* $0 \le v \le D_T$ *as* $T \to \infty$,

$$EX\big(R(js, s, ku, u)\big)\, X\big(R(ms, s, lu, u)\big) \le 0 \tag{1.30}$$

for each $s > 0$, $u > 0$, $j + k \ne m + l$, *then*

$$\lim_{T\to\infty} \sup_{0\le v\le D_T} \sup_{0\le t\le b_T} \frac{\big|X\big(R(t, a_T, v, c_T)\big)\big|}{H_2(t, a_T, v, c_T)\big(2\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)\big)^{1/2}} = 1 \quad a.s. \tag{1.31}$$

(1.32)

$$\lim_{T\to\infty} \sup_{0\le v\le D_T} \sup_{0\le u\le c_T} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{\big|X\big(R(t, s, v, u)\big)\big|}{H_2(t, a_T, v, c_T)\big(2\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)\big)^{1/2}} = 1 \quad a.s.$$

THEOREM 1.4. *Assume that* c_T *and* a_T *are non-decreasing and* $H_2(t, s, v, u)$ *are non-decreasing in t and v. Suppose that (2.38), (1.25), (1.26) and*

$$\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{D_T}{c_T}+1\Big) + \tilde{H}_2(t, a_T, v, c_T) \longrightarrow \infty \tag{1.33}$$

uniformly in $|t| \le b_T$ *and* $0 \le v \le D_T$ *as* $T \to \infty$ *are satisfied. Then (1.28) holds true.*

REMARK 1.1. If $H_1(t, s, T)$ does not depend on t, that is $H_1(t, s, T) = H_1(0, s, T)$ for all t, then (1.16) implies (1.17).

REMARK 1.2. If $a_T \to 0$ and $b_T \ge c_0 > 0$, then (1.18) is satisfied.

The following corollaries deal with the five examples given earlier in this section.

COROLLARY 1.1. *Let* $\{W(x, y), -\infty < x < \infty, 0 \le y < \infty\}$ *be a standard two-parameter Wiener process. Assume that*

$$\log\log\Big(Ta_T + \frac{1}{Ta_T} + \frac{1}{b_T}\Big) = o\Big(\log\frac{b_T}{a_T}\Big) \quad as\ T \to \infty. \tag{1.34}$$

Then

$$\lim_{T\to\infty} \sup_{0\le t\le b_T} \frac{|W(t + a_T, T) - W(t, T)|}{(2Ta_T \log\frac{b_T}{a_T})^{1/2}} = 1 \quad a.s.,$$

$$\lim_{T\to\infty} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{|W(t + s, T) - W(t, T)|}{(2Ta_T \log\frac{b_T}{a_T})^{1/2}} = 1 \quad a.s.$$

COROLLARY 1.2. *Let* $\{W(x, y), -\infty < x < \infty, 0 \le y < \infty\}$ *be a standard two-parameter Wiener process. Assume that*

(1.35)

$$\log\log\Big(a_T + c_T + a_Tc_T + \frac{1}{a_Tc_T} + \frac{1}{b_T} + \frac{1}{D_T}\Big) = o\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{D_T}{c_T}+1\Big)\Big) \quad as\ T \to \infty.$$

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Then

$$\lim_{T\to\infty}\sup_{0\le v\le D_T}\sup_{0\le t\le b_T}\frac{\left|W\left(R(t,a_T,v,c_T)\right)\right|}{\left(2a_Tc_T\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)\right)^{1/2}}=1\quad a.s.,$$

$$\lim_{T\to\infty}\sup_{0\le v\le D_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{\left|W\left(R(t,s,v,u)\right)\right|}{\left(2a_Tc_T\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)\right)^{1/2}}=1\quad a.s.$$

COROLLARY 1.3. *Let $\{K(x,y),0\le x\le 1,0\le y<\infty\}$ be a Kiefer process, a_T and b_T be continuous functions with $0\le a_T+b_T\le 1$. Assume that*

$$\log\Big(\frac{1}{b_T}+Ta_T+\frac{1}{Ta_T}\Big)=o\Big(\log\frac{b_T}{a_T}\Big)\quad as\ T\to\infty. \tag{1.36}$$

Then

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\frac{|K(t+a_T,T)-K(t,T)|}{\left(2Ta_T(1-a_T)\log\frac{b_T}{a_T}\right)^{1/2}}=1\quad a.s.,$$

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{|K(t+s,T)-K(t,T)|}{\left(2Ta_T(1-a_T)\log\frac{b_T}{a_T}\right)^{1/2}}=1\quad a.s.$$

COROLLARY 1.4. *Let $\{K(x,y),0\le x\le 1,0\le y<\infty\}$ be a Kiefer process, a_T, b_T, c_T, D_T be continuous functions with $0\le a_T+b_T\le 1$ and $0\le a_T\le\frac{1}{2}$. Assume that*

$$\log\Big(\frac{1}{b_T}+\frac{1}{D_T}+c_T+\frac{1}{a_Tc_T}\Big)=o\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{D_T}{c_T}+1\Big)\Big)\quad as\ T\to\infty. \tag{1.37}$$

Then

$$\lim_{T\to\infty}\sup_{0\le v\le D_T}\sup_{0\le t\le b_T}\frac{\left|K\left(R(t,a_T,v,c_T)\right)\right|}{\left(2a_T(1-a_T)c_T\log(\frac{D_T}{c_T}+1)(\frac{b_T}{a_T}+1)\right)^{1/2}}=1\quad a.s.,$$

$$\lim_{T\to\infty}\sup_{0\le v\le D_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{\left|K\left(R(t,s,v,u)\right)\right|}{\left(2a_T(1-a_T)c_T\log(\frac{D_T}{c_T}+1)(\frac{b_T}{a_T}+1)\right)^{1/2}}=1\quad a.s.$$

COROLLARY 1.5. *Let $\{X(t,v),-\infty<t<\infty,0<v<\infty\}$ be a two-parameter Ornstein-Uhlenbeck process as in Example 3. Put*

$$H^2(a_T,T)=H_1^2(t,a_T,T)=2\int_0^T\frac{\gamma(x)}{\lambda(x)}\Big(1-\exp\big(-\lambda(x)a_T\big)\Big)\,dx.$$

Assume that there exists $c>0$ such that

$$\int_{0<x\le T:\lambda(x)\ge\frac{1}{s}}\frac{\gamma(x)}{\lambda(x)}\,dx\le c\,s\int_{0<x\le T:\lambda(x)\le\frac{1}{s}}\gamma(x)\,dx \tag{1.38}$$

for each $0 < s \le a_T$,

$$\log\log\Big(a_T + \frac{1}{b_T} + \tilde{H}(a_T, T)\Big) = o\Big(\log \frac{b_T}{a_T}\Big) \quad \textit{as } T \to \infty. \tag{1.39}$$

Then

$$\lim_{T\to\infty} \sup_{0\le t\le b_T} \frac{|X(t+a_T, T) - X(t,T)|}{H(a_T, T)(2\log \frac{b_T}{a_T})^{1/2}} = 1 \quad a.s.$$

$$\lim_{T\to\infty} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{|X(t+s, T) - X(t,T)|}{H(a_T, T)(2\log \frac{b_T}{a_T})^{1/2}} = 1 \quad a.s.$$

COROLLARY 1.6. *Let* $d > 0$, $b > 0$, $\{X(t, v), -\infty < t < \infty, 0 < v < \infty\}$ *be a two-parameter Ornstein-Uhlenbeck process as in Example 3. Assume that* $a_T \to 0$ *and* $c_T \to 0$ *as* $T \to \infty$ *and that*

$$\sup_{0<x\le d+b} \lambda(x) < \infty, \tag{1.40}$$

$$x^{1-\alpha}\gamma(x) \le cy^{1-\alpha}\gamma(y) \textit{ and } \frac{\gamma(y)}{y^{1/\alpha}} \le c\frac{\gamma(x)}{x^{1/\alpha}} \tag{1.41}$$

for each $0 < x < y \le 1$, *where* $0 < \alpha < 1$, $c > 0$. *Then*

$$\lim_{T\to\infty} \sup_{0\le v\le d} \sup_{0\le t\le b} \frac{\big|X\big(R(t, a_T, v, c_T)\big)\big|}{H(a_T, v, c_T)(2\log \frac{1}{c_T a_T})^{1/2}} = 1 \quad a.s.,$$

$$\lim_{T\to\infty} \sup_{0\le v\le d} \sup_{0\le u\le c_T} \sup_{|t|\le b} \sup_{0\le s\le a_T} \frac{\big|X\big(R(t, s, v, u)\big)\big|}{H(a_T, v, c_T)(2\log \frac{1}{c_T a_T})^{1/2}} = 1 \quad a.s.,$$

where $H^2(a_T, v, c_T) = 2\int_v^{v+c_T} \frac{\gamma(x)}{\lambda(x)}\Big(1 - \exp\big(-\lambda(x)a_T\big)\Big)\,dx$.

COROLLARY 1.7. *Let* $\{X(t, v), -\infty < t < \infty, 0 \le v < \infty\}$ *be a two-parameter Gaussian process as in Example 4. Assume that* $\phi_k(v)$ *is non-decreasing in* v *for each* k *and that there exists* $c > 0$ *such that*

$$\sum_{\lambda_k\ge\frac{1}{s}} \phi_k^2(v)\frac{\gamma_k}{\lambda_k} \le cs \sum_{\lambda_k\le\frac{1}{s}} \phi_k^2(v)\cdot\gamma_k \quad \textit{for } 0 < s \le 1, v > 0, \tag{1.42}$$

and

$$\log\log\Big(\sum_{k=0}^{\infty} \phi_k^2(T)\frac{\gamma_k}{\lambda_k}\Big) = o\Big(\log\frac{1}{a_T}\Big) \quad \textit{as } T \to \infty. \tag{1.43}$$

Then

$$\lim_{T\to\infty} \sup_{0\le t\le 1} \frac{|X(t+a_T, T) - X(t,T)|}{H_1(a_T, T)(2\log \frac{1}{a_T})^{1/2}} = 1 \quad a.s.,$$

$$\lim_{T\to\infty} \sup_{|t|\le 1} \sup_{0\le s\le a_T} \frac{|X(t+s, T) - X(t,T)|}{H_1(a_T, T)(2\log \frac{1}{a_T})^{1/2}} = 1 \quad a.s.,$$

where $H_1^2(a_T, T) = 2\sum_{k=1}^{\infty} \phi_k^2(T)(1 - e^{-\lambda_k a_T})\frac{\gamma_k}{\lambda_k}$.

COROLLARY 1.8. *Let* $\{X(t, v), 0 \le t < \infty, 0 \le v < \infty\}$ *be a two-parameter Gaussian process as in Example 5 with* $g_1(s) = s^{\alpha_1}$ *and* $g_2(v) = v^{\alpha_2}$ $(\alpha_1 > 0, \alpha_2 > 0)$*. Assume that as* $T \longrightarrow \infty$

$$\log\log\Big(T + a_T + \frac{1}{a_T} + \frac{1}{b_T}\Big) = o\Big(\log\frac{b_T}{a_T}\Big). \tag{1.44}$$

Then

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\frac{|X(t + a_T, T) - X(t, T)|}{H_1(t, a_T, T)(2\log\frac{b_T}{a_T})^{1/2}} = 1 \quad a.s.,$$

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{|X(t + s, T) - X(t, T)|}{H_1(t, a_T, T)(2\log\frac{b_T}{a_T})^{1/2}} = 1 \quad a.s.,$$

where $H_1^2(t, a_T, T) = c\cdot T^{2\alpha_2}\big((t + a_T)^{\alpha_1} - t^{\alpha_1}\big)^2$.

COROLLARY 1.9. *Let* $\{X(t, v), 0 \le t < \infty, 0 \le v < \infty\}$ *be a two-parameter Gaussian process as in Corollary 1.8. Assume that as* $T \longrightarrow \infty$

$$\log\log\Big(D_T + c_T + a_T + \frac{1}{a_T} + \frac{1}{b_T}\Big) = o\Big(\log\Big(\frac{b_T}{a_T} + 1\Big)\Big(\frac{D_T}{c_T} + 1\Big)\Big). \tag{1.45}$$

Then

$$\limsup_{T\to\infty}\sup_{0\le v\le D_T}\sup_{0\le u\le c_T}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{|X(R(t, s, v, u)|}{H_2(t, a_T, v, c_T)\big(2\log\big(\frac{b_T}{a_T} + 1\big)\big(\frac{D_T}{c_T} + 1\big)\big)^{1/2}} \le 1 \quad a.s.$$

where $H_2^2(t, a_T, v, c_T) = c\big((v + c_T)^{\alpha_2} - v^{\alpha_2}\big)^2\big((t + a_T)^{\alpha_1} - t^{\alpha_1}\big)^2$.

For results which are similar to those of Corollaries 1.1–1.4 on the two-parameter Wiener and Kiefer processes we refer to Chapter 1 of Csörgő and Révész [15]. The conclusions of Corollaries 1.5–1.9 are believed to be brand-new. Corollary 1.7 is related to some results of Walsh [38].

2. **Large deviations.** Let a_T, b_T, c_T and D_T be non-negative continuous functions of T. For positive k, put $K = 2^{2^k}$ and

$$H_{11}(t, s, T, K, a_T) = H_1\Big(t, \frac{a_T}{K}, T\Big) + H_1\Big(t + s, \frac{a_T}{K}, T\Big), \tag{2.1}$$

$$H_{12}(t, s, T, K, a_T) = 4\int_{2^{k-2}}^{\infty}\frac{H_1(t, a_T e^{-z}, T) + H_1(t + s, a_T e^{-z}, T)}{z}\,dz, \tag{2.2}$$

$$H_{13}(t, s, T, K, a_T) = 20\int_{2^{\frac{k}{2}-1}}^{\infty}\Big(H_1(t, a_T e^{-z^2}, T) + H_1(t + s, a_T e^{-z^2}, T)\Big)\,dz. \tag{2.3}$$

In what follows we always assume that for each $|t| \le b_T$, $0 \le s \le a_T$, $0 \le v \le D_T$ and $0 \le u \le c_T$

$$H_{13}(t, s, T, K, a_T) < \infty. \tag{2.4}$$

The first two lemmas are well known and will be useful in the sequel.

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LEMMA 2.1 (SLEPIAN [37]). *If X and Y are separable centered Gaussian processes on A such that $EX_t{}^2 = EY_t{}^2$ for all $t \in A$ and*

$$EX_tX_s \geq EY_tY_s \quad \text{for all } s,t \in A, \tag{2.5}$$

then for all $x_t > 0$

$$P\Big\{\sup_{t\in A}\frac{X_t}{x_t} \geq 1\Big\} \leq P\Big\{\sup_{t\in A}\frac{Y_t}{x_t} \geq 1\Big\}. \tag{2.6}$$

LEMMA 2.2 (GORDON [23]). *Let $\{X_{ij}\}_I$, $\{Y_{ij}\}_I$, $I = \{(i,j) : 1 \leq i \leq n\,,\, 1 \leq j \leq m\}$ be two collections of centered Gaussian variables satisfying the following three conditions*

$$EX_{ij}{}^2 = EY_{ij}{}^2, \quad (i,j) \in I \tag{2.7}$$

$$EX_{ij}X_{ik} \leq EY_{ij}Y_{ik}, \quad (i,j),(i,k) \in I \tag{2.8}$$

$$EX_{ij}X_{lk} \geq EY_{ij}Y_{lk}, \quad (i,j),(l,k) \in I, \quad i \neq l. \tag{2.9}$$

Then, for all real $\lambda_{ij} > 0$

$$P\Big\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(X_{ij} > \lambda_{ij})\Big\} \geq P\Big\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(Y_{ij} > \lambda_{ij})\Big\}. \tag{2.10}$$

LEMMA 2.3. *Assume that $H_1(t,s,T)$ is non decreasing in s. Then, for each $x > 0$, we have*

(2.11)

$$\begin{aligned} P\Big\{\sup_{|t|\leq b_T}\sup_{0\leq s\leq a_T}&|X(t+s,T)-X(t,T)| \Big/ \Big\{x\Big(H_1(t,s,T) \\ &+ H_{11}(t,s,T,K,a_T) + H_{12}(t,s,T,K,a_T)\Big) + H_{13}(t,s,T,K,a_T)\Big\} \geq 1\Big\} \\ &\leq 16\cdot 2^{2^{k+1}}\Big(\frac{b_T}{a_T}+1\Big)e^{-\frac{x^2}{2}}. \end{aligned}$$

If, in addition, we have that

$$E\Big(X(t+s,v)-X(t,v)\Big)\Big(X(t+s,u)-X(t,u)\Big) \geq E\Big(X(t+s,u)-X(t,u)\Big)^2 \tag{2.12}$$

for each $v \geq u$ and each t, s and that $H_1(t,s,T)$ is a continuous function of T for each fixed t, s then for every $0 < \epsilon < 1$, there exists a constant c, depending only on ϵ, such

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that for any subset A of $\mathbb{R}^+$, $a_0 > 0$, $-\infty < b_{1,T} \le b_{2,T} < \infty$, *we have*

(2.13)

$$P\Big\{\sup_{T\in A}\sup_{b_{1,T}\le t\le b_{2,T}}\sup_{0\le s\le a_0}|X(t+s,T)-X(t,T)| \Big/ \Big\{\Big(H_1(t,a_0,T) + 2H_{11}(t,s,T,K,a_0)+2H_{12}(t,s,T,K,a_0)\Big) \cdot \Big(x^2+(2+\epsilon)\log\log\big(\tilde{H}_1(t,a_0,T)+H_{11}(t,s,T,K,a_0)\big)\Big)^{1/2} + 2H_{13}(t,s,T,K,a_0)+H_{14}(t,s,T,K,a_0)\Big\} \ge 1\Big\}$$
$$\le c(\epsilon)2^{2^{k+1}}\sup_{T\in A}\Big(\frac{b_{2,T}-b_{1,T}}{a_0}+1\Big)\exp\Big(-\frac{x^2}{2+\epsilon}\Big),$$

where

$$H_{14}(t,s,T,K,a_0) = 16\int_{2^{k-2}}^{\infty}\frac{H_1(t,a_0e^{-z},T)}{z}\big(\log\log\tilde{H}_1(t,a_0e^{-z},T)\big)^{1/2}dz + 16\int_{2^{k-2}}^{\infty}\frac{H_1(t+s,a_0e^{-z},T)}{z}\big(\log\log\tilde{H}_1(t+s,a_0e^{-z},T)\big)^{1/2}dz,$$

and

(2.14)

$$P\Big\{\sup_{T\in A}\sup_{b_{1,T}\le t\le b_{2,T}}\sup_{0\le s\le a_0}|X(t+s,T)-X(t,T)| \Big/ \Big\{x\Big(H_1(t,s,T^*) + H_{11}(t,s,T^*,K,a_0)+H_{12}(t,s,T^*,K,a_0)\Big)+H_{13}(t,s,T^*,K,a_0)\Big\}\ge 1\Big\}$$
$$\le 8\cdot 2^{2^{k+1}}\sup_{T\in A}\Big(\frac{b_{2,T}-b_{1,T}}{a_0}+1\Big)e^{-x^2/2}$$

or each $x > 0$, *where* $T^* = \sup\{T : T\in A\}$.

PROOF. We first prove (2.11). For fixed $T > 0$, let

$$t_{k+j} = \Big(\Big[\frac{t\cdot 2^{2^{j+k}}}{a_T}\Big]+1\Big)a_T/2^{2^{j+k}}, \quad j=0,1,2,\dots,$$

where $[x]$ denotes the integer part of x. Since we assumed $X(\cdot,\cdot)$ to be almost surely continuous, we can write

$$|X(t+s,T)-X(s,T)| \le \big|X\big((t+s)_k,T\big)-X(t_k,T)\big| + \sum_{j=0}^{\infty}\big|X\big((t+s)_{k+j+1},T\big)-X\big((t+s)_{k+j},T\big)\big| + \sum_{j=0}^{\infty}|X(t_{k+j+1},T)-X(t_{k+j},T)|. \tag{2.15}$$

By the definitions of $H_1(t,s,T)$ and t_{k+j}, it is clear that

$$H_1\big(t_k,(t+s)_k-t_k,T\big) \le H_1(t,t_k-t,T)+H_1\big(t,(t+s)_k-t,T\big) \le H_1(t,s,T)+H_{11}(t,s,T,K,a_T), \tag{2.16}$$

(2.17) $$H_1((t+s)_{k+j+1}, (t+s)_{k+j} - (t+s)_{k+j+1}, T) \le 2H_1(t+s, a_T/2^{2^{k+j}}, T)$$

and

(2.18) $$H_1(t_{k+j+1}, t_{k+j} - t_{k+j+1}, T) \le 2H_1(t, a_T/2^{2^{k+j}}, T).$$

From (2.16)–(2.18) we get that for every $x_j > 0$

(2.19) $$P\left\{\sup_{|t|\le b_T}\sup_{0\le s\le a_T} \frac{\left|X\left((t+s)_k, T\right) - X(t_k, T)\right|}{x\left(H_1(t,s,T) + H_{11}(t,s,T,K,a_T)\right)} \ge 1\right\} \le 4\cdot 2^{2^{k+1}}\left(\frac{b_T}{a_T}+1\right)e^{-x^2/2},$$

(2.20) $$\begin{aligned} &P\left\{\sup_{|t|\le b_T}\sup_{0\le s\le a_T} \frac{\sum_{j=0}^{\infty}\left|X\left((t+s)_{k+j+1}, T\right) - X\left((t+s)_{k+j}, T\right)\right|}{\sum_{j=0}^{\infty} 2x_j H_1(t+s, a_T/2^{2^{k+j}}, T)} \ge 1\right\} \\ &\quad\le 4\left(\frac{b_T}{a_T}+1\right)\sum_{j=0}^{\infty} 2^{2^{k+j+1}} e^{-x_j^2/2}, \end{aligned}$$

(2.21)
$$P\left\{\sup_{|t|\le b_T}\sup_{0\le s\le a_T} \frac{\sum_{j=0}^{\infty}|X(t_{k+j+1}, T) - X(t_{k+j}, T)|}{\sum_{j=0}^{\infty} 2x_j H_1(t, a_T/2^{2^{k+j}}, T)} \ge 1\right\} \le 4\left(\frac{b_T}{a_T}+1\right)\sum_{j=0}^{\infty} 2^{2^{k+j+1}} e^{-x_j^2/2}.$$

Let ${x_j}^2 = x^2 + 2^{k+j+2}$. It follows that

(2.22) $$\begin{aligned} \sum_{j=0}^{\infty} 2^{2^{k+j+1}} e^{-x_j^2/2} &= e^{-x^2/2}\sum_{j=0}^{\infty}\left(\frac{2}{e}\right)^{2^{k+j+1}} \\ &\le 2e^{-x^2/2}, \end{aligned}$$

(2.23) $$\begin{aligned} &\sum_{j=0}^{\infty} 2x_j H_1(t, a_T/2^{2^{k+j}}, T) \\ &\quad\le 2x\sum_{j=0}^{\infty} H_1(t, a_T/2^{2^{k+j}}, T) + 2\sum_{j=0}^{\infty} 2^{(k+j+2)/2} H_1(t, a_T/2^{2^{k+j}}, T) \\ &\quad\le 4x\sum_{j=0}^{\infty}\int_{2^{k+j-1}}^{2^{k+j}} \frac{H_1(t, a_T/2^z, T)}{z}\,dz + 8\sum_{j=0}^{\infty}\int_{2^{k+j-1}}^{2^{k+j}} \frac{H_1(t, a_T/2^z, T)}{z^{1/2}}\,dz \\ &\quad= 4x\int_{2^{k-1}}^{\infty} \frac{H_1(t, a_T/2^z, T)}{z}\,dz + 8\int_{2^{k-1}}^{\infty} \frac{H_1(t, a_T/2^z, T)}{z^{1/2}}\,dz \\ &\quad\le 4x\int_{2^{k-2}}^{\infty} \frac{H_1(t, a_T e^{-z}, T)}{z}\,dz + 20\int_{2^{\frac{k}{2}-1}}^{\infty} H_1(t, a_T e^{-z^2}, T)\,dz, \end{aligned}$$

as well as
(2.24)
$$\begin{aligned} &\sum_{j=0}^{\infty} 2x_j H_1(t+s, a_T/2^{2^{k+j}}, T) \\ &\quad\le 4x\int_{2^{k-2}}^{\infty} \frac{H_1(t+s, a_T e^{-z}, T)}{z}\,dz + 20\int_{2^{\frac{k}{2}-1}}^{\infty} H_1(t+s, a_T e^{-z^2}, T)\,dz. \end{aligned}$$

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Combining the above inequalities we get (2.11).

Before proving (2.13), we show that for every $0 < \epsilon < 1$ there exists $c(\epsilon)$ such that for every fixed t, s

$$(2.25)\quad P\left\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{H_1^*(t,s,T)\big(y+(2+\epsilon)\log\log\tilde{H}_1^*(t,s,T)\big)^{1/2}}\geq 1\right\}\leq c(\epsilon)\exp\Big(-\frac{y}{2+\epsilon}\Big)$$

for each $y > 0$ and any continuous function $H_1^*(t,s,T)$ of T which is such that $H_1^*(t,s,T)\geq H_1(t,s,T)$.

Let

$$\theta=\Big(1+\frac{\epsilon}{4}\Big)^{\frac{1}{2}},$$

$$A_k=\{T:\theta^k<H_1^*(t,s,T)\leq\theta^{k+1}\},\quad -\infty<k<\infty,$$

$$T_k=\sup\{T:T\in A_kA\},\quad T_k'=\inf\{T:T\in A_kA\}.$$

Since $H_1^*(t,s,T_k)$ is assumed to be a continuous function of T, we have

$$(2.26)\qquad \theta^k\leq H_1^*(t,s,T_k)\leq\theta^{k+1},\quad \theta^k\leq H_1^*(t,s,T_k')\leq\theta^{k+1}.$$

Let $Z(T)$ be a process of independent increments with $Z(T)\overset{\mathcal{D}}{=}X(t+s,T)-X(t,T)$. Then

$$EZ^2(T)=H_1^2(t,s,T),$$

$$EZ(T)Z(T')=H_1^2(t,s,T')\leq E\big(X(t+s,T)-X(t,T)\big)\big(X(t+s),T')-X(t,T')\big)$$

for each $T > T'$ by (2.12). Applying the Slepian lemma, we have

$$P\left\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{H_1^*(t,s,T)\big(y+(2+\epsilon)\log\log\tilde{H}_1^*(t,s,T)\big)^{1/2}}\geq 1\right\}$$

$$\leq\sum_k P\left\{\sup_{T\in A_kA}\frac{X(t+s,T)-X(t,T)}{H_1^*(t,s,T)\big(y+(2+\epsilon)\log\log\tilde{H}_1^*(t,s,T)\big)^{1/2}}\geq 1\right\}$$

$$+\sum_k P\left\{\sup_{T\in A_kA}-\frac{X(t+s,T)-X(t,T)}{H_1^*(t,s,T)\big(y+(2+\epsilon)\log\log\tilde{H}_1^*(t,s,T)\big)^{1/2}}\geq 1\right\}$$

$$\leq\sum_k P\left\{\sup_{T\in A_kA}\frac{Z(T)}{H_1^*(t,s,T)\big(y+(2+\epsilon)\log\log\tilde{H}_1^*(t,s,T)\big)^{1/2}}\geq 1\right\}$$

$$+\sum_k P\left\{\sup_{T\in A_kA}\frac{-Z(T)}{H_1^*(t,s,T)\big(y+(2+\epsilon)\log\log\tilde{H}_1^*(t,s,T)\big)^{1/2}}\geq 1\right\}$$

$$\leq 2\sum_k P\left\{\sup_{T\in A_kA}\frac{|Z(T)|}{\theta^k\big(y+(2+\epsilon)\log\log\theta^{|k|}\big)^{1/2}}\geq 1\right\}$$

$$\leq 2\sum_k P\left\{\sup_{T_k'\leq T\leq T_k}\frac{|Z(T)|}{H_1^*(t,s,T_k)\big(y+(2+\epsilon)\log\log\theta^{|k|}\big)^{1/2}}\geq\frac{1}{\theta}\right\}$$

$$\leq 4\sum_k P\left\{\frac{|Z(T_k)|}{H_1^*(t,s,T_k)\big(y+(2+\epsilon)\log\log\theta^{|k|}\big)^{1/2}}\geq\frac{1}{\theta}\right\}$$

$$\leq 8\sum_k \exp\Big(-\frac{y+(2+\epsilon)\log\log\theta^{|k|}}{2\theta^2}\Big)$$
$$\leq c(\epsilon)\exp\Big(-\frac{y}{2\theta^2}\Big)$$
$$\leq c(\epsilon)\exp\Big(-\frac{y}{2+\epsilon}\Big)$$

as desired.

We now turn to the proof of (2.13). Let

$$t_{j+k}=\Big(\Big[\frac{t\,2^{2^{j+k}}}{a_0}\Big]+1\Big)a_0/2^{2^{j+k}},\quad j=0,1,2,\ldots.$$

Using (2.15), we obtain

(2.27)
$$\sup_{T\in A}\sup_{b_{1,T}\leq T\leq b_{2,T}}\sup_{0\leq s\leq a_0}|X(t+s,T)-X(t,T)|$$
$$\leq \sup_{T\in A}\sup_{b_{1,T}\leq T\leq b_{2,T}}\sup_{0\leq s\leq a_0}\Big|X\big((t+s)_k,T\big)-X(t_k,T)\Big|$$
$$+\sum_{j=0}^{\infty}\sup_{T\in A}\sup_{b_{1,T}\leq T\leq b_{2,T}}\sup_{0\leq s\leq a_0}\Big|X\big((t+s)_{k+j},T\big)-X\big((t+s)_{k+j+1},T\big)\Big|$$
$$+\sum_{j=0}^{\infty}\sup_{T\in A}\sup_{b_{1,T}\leq T\leq b_{2,T}}\sup_{0\leq s\leq a_0}|X(t_{t+j+1},T)-X(t_{k+j},T)|.$$

Similarly to (2.16)–(2.18), we have

(2.28) $$H_1\big(t_k,(t+s)_k-t_k,T\big)\leq H_1(t,a_0,T)+H_{11}(t,s,T,K,a_0)$$

(2.29) $$H_1\big((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},T\big)\leq 2H_1(t+s,a_0/2^{2^{k+j}},T)$$

(2.30) $$H_1(t_{k+j+1},t_{k+j}-t_{k+j+1},T)\leq 2H_1(t,a_0/2^{2^{k+j}},T)$$

for each $b_{1,T}\leq t\leq b_{2,T}$, $0\leq s\leq a_0$. From (2.28) and (2.25) it follows that

(2.31)
$$P\Big\{\sup_{T\in A}\sup_{b_{1,T}\leq T\leq b_{2,T}}\sup_{0\leq s\leq a_0}\Big|X\big((t+s)_k,T\big)-X(t_k,T)\Big|\Big/ \Big\{\Big(H_1(t,a_0,T)+H_{11}(t,s,T,K,a_0)\Big)\cdot\Big(x^2+(2+\epsilon)\log\log\big(\tilde H_1(t,a_0,T)+H_{11}(t,s,T,K,a_0)\big)\Big)^{1/2}\Big\}\geq 1\Big\}$$
$$\leq c(\epsilon)2^{2^{k+1}}\sup_{T\in A}\Big(\frac{b_{2,T}-b_{1,T}}{a_0}+1\Big)\exp\Big(-\frac{x^2}{2+\epsilon}\Big).$$

Similarly, by (2.29), (2.30) and (2.25) we have for each $x_j>0$

(2.32)
$$P\Big\{\sup_{T\in A}\sup_{b_{1,T}\leq t\leq b_{2,T}}\sup_{0\leq s\leq a_0}\Big|X\big((t+s)_{k+j+1},T\big)-X\big((t+s)_{k+j},T\big)\Big|\Big/ \Big\{4H_1(t+s,a_0/2^{2^{k+j}},T)\big(x_j^2+\log\log\tilde H_1(t+s,a_0/2^{2^{k+j}},T)\big)^{1/2}\Big\}\geq 1\Big\}$$
$$\leq c(1/2)2^{2^{k+j+1}}\sup_{T\in A}\Big(\frac{b_{2,T}-b_{1,T}}{a_0}+1\Big)\exp(-x_j^2/2),$$

$$P\Big\{\sup_{T\in A}\sup_{b_{1,T}\le t\le b_{2,T}}\sup_{0\le s\le a_0}|X(t_{k+j},T)-X(t_{k+j+1},T)| \Big/ \Big\{4H_1(t,a_0/2^{2^{k+j}},T)\cdot\big(x_j^2+\log\log\tilde H_1(t,a_0/2^{2^{k+j}},T)\big)^{1/2}\Big\}\ge 1\Big\} \le c(1/2)2^{2^{k+j+1}}\sup_{T\in A}\Big(\frac{b_{2,T}-b_{1,T}}{a_0}+1\Big)\exp(-x_j^2/2). \tag{2.33}$$

From (2.27), (2.31)–(2.33) we conclude

(2.34)

$$P\Big\{\sup_{T\in A}\sup_{b_{1,T}\le t\le b_{2,T}}\sup_{0\le s\le a_0}\frac{|X(t+s,T)-X(t,T)|}{I_1+I_2+I_3}\ge 1\Big\} \le c(\epsilon)\sup_{T\in A}\Big(\frac{b_{2,T}-b_{1,T}}{a_0}+1\Big)\cdot\Big(2^{2^{k+1}}\exp\Big(-\frac{x^2}{2+\epsilon}\Big)+\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp\big(-x_j^2/2\big)\Big),$$

where

$$I_1=\big(H_1(t,a_0,T)+H_{11}(t,s,T,k,a_0)\big)\cdot\Big(x^2+(2+\epsilon)\log\log\big(\tilde H_1(t,a_0,T)+H_{11}(t,s,T,K,a_0)\big)\Big)^{\frac12},$$

$$I_2=4\sum_{j=0}^{\infty}H_1\Big(t+s,\frac{a_0}{2^{2^{k+j}}},T\Big)\Big(x_j^2+\log\log\tilde H_1\Big(t+s,\frac{a_0}{2^{2^{k+j}}},T\Big)\Big)^{\frac12},$$

$$I_3=4\sum_{j=0}^{\infty}H_1\Big(t,\frac{a_0}{2^{2^{k+j}}},T\Big)\Big(x_j^2+\log\log\tilde H_1\Big(t,\frac{a_0}{2^{2^{k+j}}},T\Big)\Big)^{\frac12}.$$

Let $x_j^2=x^2+2^{k+j+2}$. Then, proceeding as in the proof of (2.23) and (2.24) we can obtain

$$I_2\le 8x\int_{2^{k-2}}^{\infty}\frac{H_1(t+s,a_0e^{-z},T)}{z}\,dz+40\int_{2^{\frac{k}{2}-1}}^{\infty}H_1(t+s,a_0e^{-z^2},T)\,dz +8\int_{2^{k-2}}^{\infty}\frac{H_1(t+s,a_0e^{-z},T)}{z}\big(\log\log\tilde H_1(t+s,a_0e^{-z},T)\big)^{\frac12}\,dz, \tag{2.35}$$

as well as

$$I_3\le 8x\int_{2^{k-2}}^{\infty}\frac{H_1(t,a_0e^{-z},T)}{z}\,dz+40\int_{2^{\frac{k}{2}-1}}^{\infty}H_1(t,a_0e^{-z^2},T)\,dz +8\int_{2^{k-2}}^{\infty}\frac{H_1(t,a_0e^{-z},T)}{z}\big(\log\log\tilde H_1(t,a_0e^{-z},T)\big)^{\frac12}\,dz. \tag{2.36}$$

Now (2.13) follows from (2.34)–(2.36) and (2.22).

From the proof of (2.25), we can see that

$$P\Big\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{H_1^*(t,s,T^*)y^{\frac12}}\ge 1\Big\}\le 8\exp(-y/2) \tag{2.37}$$

holds true for every $y>0$ and $H_1^*(t,s,T^*)\ge H(t,s,T^*)$. Along the lines of the proof of (2.13), with (2.37) instead of (2.25), we obtain also that (2.14) holds true. This completes the proof of Lemma 2.3.

For studying the increments of $X(t,v)$ in t and v, we give below another large deviation result for $X(t,v)$. Put

$$H_{21}(t,s,v,u,K) = 2H_2\Big(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\Big) + 2H_2\Big(t,s,v+u-\frac{c_T}{K},\frac{2c_T}{K}\Big) + H_2\Big(t,\frac{a_T}{K},v,c_T\Big(1+\frac{2}{K}\Big)\Big) + 2H_2\Big(t+s,\frac{a_T}{K},v,c_T\Big(1+\frac{2}{K}\Big)\Big),$$

$$H_2^1(t,s,v,z) = H_2\Big(t,z,v-\frac{c_T}{K},c_T\Big(1+\frac{2}{K}\Big)\Big) + H_2\Big(t+s,z,v-\frac{c_T}{K},c_T\Big(1+\frac{2}{K}\Big)\Big),$$

$$H_{22}(t,s,v,K) = 40\int_{2^{k-2}}^{\infty} H_2^1(t,s,v,a_Te^{-z})/z\,dz,$$

$$H_{23}(t,s,v,K) = 40\int_{2^{\frac{k}{2}-1}}^{\infty} H_2^1(t,s,v,a_Te^{-z^2})\,dz.$$

REMARK 2.1. Here and in the sequel, we write $H_2(t,s,v,u) = H_2(t,s,0,u)$ if $v<0$.

LEMMA 2.4. *Assume that $H_2(t,s,v,u)$ is non-decreasing in s and u and that for each t, s, $a \ge v \ge v' > 0$ we have*

$$EX\big(R(t,s,v',a-v')\big)X\big(R(t,s,v,a-v)\big) \ge EX^2\big(R(t,s,v,a-v)\big). \tag{2.38}$$

Then we have

$$P\Big\{\sup_{0\le v\le D_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,v,u)\big)\big| \Big/ \Big\{x\big(H_2(t,s,v,u)+H_{21}(t,s,v,u,K)+H_{22}(t,s,v,K)\big)+H_{23}(t,s,v,K)\Big\} \ge 1\Big\} \le 56\cdot 2^{2^{k+2}}\Big(\frac{D_T}{c_T}+1\Big)\Big(\frac{b_T}{a_T}+1\Big)e^{-x^2/2}. \tag{2.39}$$

REMARK 2.2. We recall that $X\big(R(t,s,v,u)\big)$ is defined on a rectangle $[t,t+s]\times[v,v+u]$ (*cf.* (1.11)). Intuitively, this makes it easy to understand the subdivision of $X\big(R(t,s,v,u)\big)$ in the proof of Lemma 2.4 that follows.

PROOF. Let

$$t_{k+j} = \Big(\Big[\frac{t2^{2^{k+j}}}{a_T}\Big]+1\Big)a_T \Big/ 2^{2^{k+j}},$$

$$v'_{k+j} = \Big(\Big[\frac{v2^{2^{k+j}}}{c_T}\Big]+1\Big)c_T \Big/ 2^{2^{k+j}}.$$

We have

$$\begin{aligned}\big|X\big(R(t,s,v,u)\big)\big| &\le \big|X\big(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)\big)\big| \\ &\quad + \big|X\big(R(t+s,(t+s)_k-(t+s),v'_k,(v+u)'_k-v'_k)\big)\big| \\ &\quad + \big|X\big(R(t,t_k-t,v'_k,(v+u)'_k-v'_k)\big)\big| \\ &\quad + \big|X\big(R(t,s,v,v'_k-v)\big)\big| + \big|X\big(R(t,s,v+u,(v+u)'_k-v+u)\big)\big| \\ &\le \big|X\big(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)\big)\big|\end{aligned}$$

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$$+\sum_{j=0}^{\infty}\Big|X\Big(R((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},v'_k,(v+u)'_k-v'_k)\Big)\Big|$$

$$(2.40)\qquad +\sum_{j=0}^{\infty}\Big|X\Big(R(t_{k+j+1},t_{k+j}-t_{k+j+1},v'_k,(v+u)'_k-v'_k)\Big)\Big|$$

$$+\Big|X\Big(R(t,s,v,v'_k-v)\Big)\Big|+\Big|X\Big(R(t,s,v+u,(v+u)'_k-(v+u))\Big)\Big|.$$

From (2.38) it follows that we have

$$(2.41)\qquad H_2(t,s,v,u)\geq H_2(t,s,v',u')$$

for each $v'\geq v$, $v+u\geq v'+u'$, and with the help of (2.41), we get

$$\begin{aligned}H_2\big(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k\big)&\\ \leq H_2(t,s,v,u)&+H_2\big(t+s,(t+s)_k-(t+s),v'_k,(v+u)'_k-v'_k\big)\\ &+H_2\big(t,t_k-t,v'_k,(v+u)'_k-v'_k\big)+H_2(t,s,v,v'_k-v)\\ &+H_2\big(t,s,v+u,(v+u)'_k-(v+u)\big)\\ \leq H_2(t,s,v,u)&+H_{21}(t,s,v,u,K).\end{aligned}$$

Corresponding to (2.17) and (2.18), we have

$$H_2\big((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},v'_k,(v+u)'_k-v'_k\big)\leq 2H_2\Big(t+s,\frac{a_T}{2^{2^{k+j}}},v,c_T\Big(1+\frac{2}{K}\Big)\Big),$$

$$H_2\big(t_{k+j+1},t_{k+j}-t_{k+j+1},v'_k,(v+u)'_k-v'_k\big)\leq 2H_2\Big(t,\frac{a_T}{2^{2^{k+j}}},v,c_T\Big(1+\frac{2}{K}\Big)\Big).$$

Then, for every $x_j>0$,

$$(2.42)\qquad P\left\{\sup_{0\leq v\leq D_T}\sup_{0\leq u\leq c_T}\sup_{|t|\leq b_T}\sup_{0\leq s\leq a_T}\frac{\Big|X\Big(R\big(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k\big)\Big)\Big|}{x\Big(H_2(t,s,v,u)+H_{21}(t,s,v,u,K)\Big)}\geq 1\right\}$$
$$\leq 4\cdot 2^{2^{k+2}}\Big(\frac{D_T}{c_T}+1\Big)\Big(\frac{b_T}{a_T}+1\Big)e^{-x^2/2},$$

(2.43)
$$P\left\{\sup_{0\leq v\leq D_T}\sup_{0\leq u\leq c_T}\sup_{|t|\leq b_T}\sup_{0\leq s\leq a_T}\frac{\sum_{j=0}^{\infty}\Big|X\Big(R\big((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},v'_k,(v+u)'_k-v'_k\big)\Big)\Big|}{2\sum_{j=0}^{\infty}x_jH_2\big(t+s,\frac{a_T}{2^{2^{k+j}}},v,c_T(1+\frac{1}{K})\big)}\geq 1\right\}$$
$$\leq 4\Big(\frac{D_T}{c_T}+1\Big)\Big(\frac{b_T}{a_T}+1\Big)\sum_{j=0}^{\infty}2^{2^{k+1}+2^{k+j+1}}\exp(-x_j^2/2),$$

(2.44)
$$P\left\{\sup_{0\leq v\leq D_T}\sup_{0\leq u\leq c_T}\sup_{|t|\leq b_T}\sup_{0\leq s\leq a_T}\frac{\sum_{j=0}^{\infty}\Big|X\Big(R\big(t_{k+j+1},t_{k+j}-t_{k+j+1},v'_k,(v+u)'_k-v'_k\big)\Big)\Big|}{2\sum_{j=0}^{\infty}x_jH_2\big(t,\frac{a_T}{2^{2^{k+j}}},v,c_T(1+\frac{1}{K})\big)}\geq 1\right\}$$
$$\leq 4\Big(\frac{D_T}{c_T}+1\Big)\Big(\frac{b_T}{a_T}+1\Big)\sum_{j=0}^{\infty}2^{2^{k+1}+2^{k+j+1}}\exp(-x_j^2/2).$$

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Let $x_j^2 = x^2 + 2^{k+j+2}$. If follows that

$$\sum_{j=0}^{\infty} 2^{2^{k+1}+2^{k+j+1}} e^{-x_j^2/2} \le 2K^2 e^{-x^2/2}. \tag{2.45}$$

Similarly to (2.23) and (2.24), we have

(2.46)

$$\sum_{j=0}^{\infty} 2x_j H_2\Big(t+s, \frac{a_T}{2^{2^{k+j}}}, v, c_T\Big(1+\frac{2}{K}\Big)\Big) \le 4x \int_{2^{k-2}}^{\infty} \frac{H_2\big(t+s, a_T e^{-z}, v, c_T(1+\frac{2}{K})\big)}{z}\,dz + 20\int_{2^{\frac{k}{2}-1}}^{\infty} H_2\Big(t+s, a_T e^{-z^2}, v, c_T\Big(1+\frac{2}{K}\Big)\Big)\,dz,$$

(2.47)

$$\sum_{j=0}^{\infty} 2x_j H_2\Big(t, \frac{a_T}{2^{2^{k+j}}}, v, c_T\Big(1+\frac{1}{K}\Big)\Big) \le 4x \int_{2^{k-2}}^{\infty} \frac{H_2\big(t, a_T e^{-z}, v, c_T\big(1+\frac{1}{K}\big)\big)}{z}\,dz + 20\int_{2^{\frac{k}{2}-1}}^{\infty} H_2\Big(t, a_T e^{-z^2}, v, c_T\Big(1+\frac{1}{K}\Big)\Big)\,dz.$$

We now deal with $X\big(R(t,s,v,v_k'-v)\big)$ and $X\Big(R\big(t,s,v+u,(v+u)_k'-(v+u)\big)\Big)$. For each $y>0$ we have

(2.48)

$$P\Big\{\sup_{0\le v\le D_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,v,v_k'-v)\big)\big|\ge y\Big\}$$
$$\le P\Big\{\max_{0\le i\le \frac{D_T}{c_T}K}\ \sup_{\frac{ic_T}{K}\le v\le \frac{(i+1)c_T}{K}}\ \sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,v,v_k'-v)\big)\big|\ge y\Big\}$$
$$\le \sum_{i=0}^{[\frac{D_T}{c_T}K]} P\Big\{\sup_{\frac{ic_T}{K}\le v\le\frac{(i+1)c_T}{K}}\ \sup_{|t|\le b_T}\sup_{0\le s\le a_T}\Big|X\Big(R\Big(t,s,v,\frac{(i+1)c_T}{K}-v\Big)\Big)\Big|\ge y\Big\}.$$

Let $d_i = (i+1)c_T/K$. We show below that for each fixed t, s

$$P\Big\{\sup_{\frac{ic_T}{K}\le v\le\frac{(i+1)c_T}{K}} \frac{\big|X\big(R(t,s,v,d_i-v)\big)\big|}{H_2(t,s,v-\frac{c_T}{K},\frac{2c_T}{K})}\ge y\Big\}\le 4\exp(-y^2/2) \tag{2.49}$$

for each $y>0$.

Let $Z(v)$ be an independent increment process with $Z(d_i - v) \overset{\mathcal{D}}{=} X\big(R(t,s,v,d_i-v)\big)$ for $\frac{ic_T}{K}\le v\le \frac{(i+1)c_T}{K}$. Then for each $v>v'$

$$EZ(d_i-v)Z(d_i-v') = EZ^2(d_i-v) = EX^2\big(R(t,s,v,d_i-v)\big) \le EX\big(R(t,s,v,d_i-v)\big)X\big(R(t,s,v',d_i-v')\big),$$

where the last inequality is from (2.38). By (2.41) we find that

$$H_2\Big(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\Big)\ge H_2\Big(t,s,\frac{ic_T}{K},\frac{c_T}{K}\Big) \tag{2.50}$$

for each $\frac{ic_T}{K} \le v \le \frac{(i+1)c_T}{K}$. Using the Slepian inequality and (2.50), we obtain

$$
\begin{aligned}
P\Big\{ \sup_{\frac{ic_T}{K} \le v \le \frac{(i+1)c_T}{K}} \frac{\big|X\big(R(t,s,v,d_i-v)\big)\big|}{H_2(t,s,v-\frac{c_T}{K},\frac{2c_T}{K})} \ge y \Big\}
&\le P\Big\{ \sup_{\frac{ic_T}{K} \le v \le \frac{(i+1)c_T}{K}} \frac{X\big(R(t,s,v,d_i-v)\big)}{H_2(t,s,v-\frac{c_T}{K},\frac{2c_T}{K})} \ge y \Big\} \\
&\quad + P\Big\{ \sup_{\frac{ic_T}{K} \le v \le \frac{(i+1)c_T}{K}} \frac{-X\big(R(t,s,v,d_i-v)\big)}{H_2(t,s,v-\frac{c_T}{K},\frac{2c_T}{K})} \ge y \Big\} \\
&\le P\Big\{ \sup_{\frac{ic_T}{K} \le v \le \frac{(i+1)c_T}{K}} \frac{Z(d_i-v)}{H_2(t,s,v-\frac{c_T}{K},\frac{2c_T}{K})} \ge y \Big\} \\
&\quad + P\Big\{ \sup_{\frac{ic_T}{K} \le v \le \frac{(i+1)c_T}{K}} \frac{-Z(d_i-v)}{H_2(t,s,v-\frac{c_T}{K},\frac{2c_T}{K})} \ge y \Big\} \\
&\le 2P\Big\{ \frac{|Z(d_i-\frac{ic_T}{K})|}{H_2(t,s,\frac{ic_T}{K},\frac{c_T}{K})} \ge y \Big\} \\
&= 2P\Big\{ \frac{\big|X\big(R(t,s,\frac{ic_T}{K},d_i-\frac{ic_T}{K})\big)\big|}{H_2(t,s,\frac{ic_T}{K},\frac{c_T}{K})} \ge y \Big\} \\
&\le 4\exp(-y^2/2).
\end{aligned}
$$

This proves (2.49).

Proceeding along the lines of the proof of (2.14), by (2.49) we can get

$$
\begin{aligned}
P\Big\{ \sup_{\frac{ic_T}{K} \le v \le \frac{(i+1)c_T}{K}} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{\big|X\big(R(t,s,v,d_i-v)\big)\big|}{xI_1(t,s,v,K)+I_2(t,s,v,K)} \ge 1 \Big\} \\
\le 8\cdot 2^{2^{k+1}}\Big(\frac{b_T}{a_T}+1\Big)\exp(-x^2/2),
\end{aligned}
\tag{2.51}
$$

where

$$
\begin{aligned}
I_1(t,s,v,K) = H_2\Big(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\Big) + 16\int_{2^{k-2}}^{\infty}\Big(\frac{1}{z}\Big)\Bigg(H_2\Big(t,a_Te^{-z},v-\frac{c_T}{K},c_T\Big(1+\frac{1}{K}\Big)\Big) \\
+ H_2\Big(t+s,a_Te^{-z},v-\frac{c_T}{K},c_T\Big(1+\frac{1}{K}\Big)\Big)\Bigg)\,dz,
\end{aligned}
$$

$$
\begin{aligned}
I_2(t,s,v,K) = 40\int_{2^{\frac{k}{2}-1}}^{\infty}\Bigg(H_2\Big(t,a_Te^{-z^2},v-\frac{c_T}{K},c_T\Big(1+\frac{1}{K}\Big)\Big) \\
+ H_2\Big(t+s,a_Te^{-z^2},v-\frac{c_T}{K},c_T\Big(1+\frac{1}{K}\Big)\Big)\Bigg)\,dz.
\end{aligned}
$$

Combining (2.48) and (2.51) yields

$$P\Big\{\sup_{0\le v\le D_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{\big|X\big(R(t,s,v,v_k'-v)\big)\big|}{xI_1(t,s,v,K)+I_2(t,s,v,K)}\ge 1\Big\}\le 16\cdot 2^{2^{k+2}}\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{D_T}{c_T}+1\Big)\exp(-x^2/2). \tag{2.52}$$

Similarly, we have

$$P\left\{\sup_{0\le v\le D_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{\Big|X\Big(R\big(t,s,v+u,(v+u)_k'-(v+u)\big)\Big)\Big|}{xI_1(t,s,v+u,K)+I_2(t,s,v+u,K)}\ge 1\right\}\le 16\cdot 2^{2^{k+2}}\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{D_T}{c_T}+1\Big)\exp(-x^2/2) \tag{2.53}$$

and (2.39) now follows from (2.40)–(2.47), (2.52) and (2.53). This completes the proof of Lemma 2.4.

LEMMA 2.5. *Assume that the conditions of Lemma 2.4 are satisfied. Then, for any* $A\subset\mathbb{R}^+$, *we have*

(2.54)

$$P\Big\{\sup_{T\in A}\sup_{0\le v\le D_T}\sup_{0\le u\le c_0}\sup_{|t|\le b_T}\sup_{0\le s\le a_0}\big|X\big(R(t,s,v,u)\big)\big|\Big/$$
$$\Big\{H_{23}^*(t,s,v,u,K)+x\Big(H_2(t,s,v,u)+H_{21}^*(t,s,v,u,K)+H_{22}^*(t,s,v,u,K)\Big)\Big\}\ge 1\Big\}$$
$$\le 56\cdot 2^{2^{k+2}}\sup_{T\in A}\Big(\frac{D_T}{c_0}+1\Big)\Big(\frac{b_T}{a_0}+1\Big)\exp(-x^2/2),$$

where H_{21}^*, H_{22}^*, H_{23}^* *are defined as* H_{21}, H_{22}, H_{23} *are but with* a_0 *and* c_0 *instead of* a_T *and* c_T *respectively.*

PROOF. Let

$$t_{k+j}=\Big(\Big[\frac{t2^{2^{k+j}}}{a_0}\Big]+1\Big)a_0\Big/2^{2^{k+j}},\qquad v_{k+j}'=\Big(\Big[\frac{v2^{2^{k+j}}}{c_0}\Big]+1\Big)c_0\Big/2^{2^{k+j}}.$$

Then (2.40) remains true. The rest of the proof is exactly the same as that of Lemma 2.4. The details are omitted here.

In the rest of this section we consider some particular cases of Lemmas 2.3 and 2.5.

LEMMA 2.6. *Let* $A\subset\mathbb{R}^+$, $s_0>0$, $b_{1,T}\le b_{2,T}$. *Assume that (2.12) is satisfied and that there exist positive numbers* c *and* α *such that*

$$\frac{H_1(t,s,T)}{s^\alpha}\le c\frac{H_1(t,s_1,T)}{s_1^\alpha} \tag{2.55}$$

for each $T \in A$, $b_{1,T} \le t \le b_{2,T}$, $0 \le s \le s_1 \le s_0$. *Then, for every* $0 < \varepsilon < 1/(1 + c^{1+\alpha})$, *there exists a positive constant* $c(\varepsilon)$, *depending only on* c, α, ε, *such that*

(2.56)

$$P\Big\{ \sup_{T\in A} \sup_{b_{1,T}\le t\le b_{2,T}} \sup_{0\le s\le s_0} |X(t+s,T) - X(t,T)| \;/\; \Big\{\varepsilon\big(H_1(t+s,s_0,T) + H_1(t,s_0,T)\big) \cdot \Big(\log\log\big(\tilde{H}_1(t+s,s_0,T) + \tilde{H}_1(t,s_0,T)\big)\Big)^{1/2} + \big(H_1(t,s_0,T) + H_1(t+s,\varepsilon s_0,T)\big) \cdot \Big(x^2 + 2\log\log\Big(\tilde{H}_1(t,s_0,T) + H_1(t+s,\varepsilon s_0,T)\Big)\Big)^{1/2}\Big\} \ge 1+\varepsilon\Big\} \le c(\varepsilon) \sup_{T\in A}\Big(\frac{b_{2,T} - b_{1,T}}{s_0} + 1\Big)\exp\Big(-\frac{x^2}{2+\varepsilon}\Big)$$

and

$$P\Big\{ \sup_{T\in A} \sup_{b_{1,T}\le t\le b_{2,T}} \sup_{0\le s\le s_0} \frac{|X(t+s,T) - X(t,T)|}{x\big(H_1(t,s_0,T^*) + H_1(t,\varepsilon s_0,T^*)\big)} \ge 1+\varepsilon\Big\} \le c(\varepsilon)\sup_{T\in A}\Big(\frac{b_{2,T} - b_{1,T}}{s_0} + 1\Big)\exp\Big(-\frac{x^2}{2+\varepsilon}\Big) \tag{2.57}$$

for each $x \ge 1$, *where* $T^* = \sup\{T : T \in A\}$.

PROOF. The basic idea of this proof is that on assuming the condition (2.55) and taking k large enough, the terms H_{11}, H_{12}, H_{13} and H_{14} of (2.13) become negligible when compared to the main term H_1 of the same denominator. It follows from (2.55) that

$$H_1(t, s_0 y, T) \le \frac{cy^\alpha}{\varepsilon^\alpha} H_1(t, s_0\varepsilon, T) \tag{2.58}$$

for each $0 < y \le \varepsilon$. Then we have

$$\begin{aligned} H_{12}(t,s,T,K,s_0) &\le \frac{4c}{\varepsilon^\alpha}\int_{2^{k-2}}^\infty \frac{e^{-z\alpha}}{z}\big(H_1(t,\varepsilon s_0,T) + H_1(t+s,s_0\varepsilon,T)\big)\,dz \\ &\le \frac{4c}{\alpha\varepsilon^\alpha} e^{-\alpha 2^{k-2}}\big(H_1(t,\varepsilon s_0,T) + H_1(t+s,s_0\varepsilon,T)\big) \\ &\le \frac{\varepsilon}{8}\big(H_1(t,\varepsilon s_0,T) + H_1(t+s,s_0\varepsilon,T)\big), \end{aligned}$$

provided $k \ge 2 + \frac{32c}{\alpha^2\varepsilon^{\alpha+1}}$.

Similarly, noting that $x\log\log(x + \frac{1}{x})$ is increasing on $(0,\infty)$, we have

$$\max\big(H_{11}(t,s,T,K,s_0), H_{13}(t,s,T,K,s_0)\big) \le \frac{\varepsilon}{8}\big(H_1(t,s_0\varepsilon,T) + H_1(t+s,s_0\varepsilon,T)\big)$$

and

$$\begin{aligned} H_{14}(t,s,T,K,s_0) &\le \frac{\varepsilon}{8} H_1(t,s_0,T)\big(\log\log\tilde{H}_1(t,s_0,T)\big)^{1/2} \\ &\quad + \frac{\varepsilon}{8} H_1(t+s,s_0,T)\big(\log\log\tilde{H}_1(t+s,s_0,T)\big)^{1/2}, \end{aligned}$$

provided that $k \ge 2 + \frac{64c}{\alpha^2\varepsilon^{\alpha+1}}$.

Combining the above inequalities with (2.13) yields (2.56). The proof of (2.57) is similar and is omitted here.

LEMMA 2.7. *Let $A \subset \mathbb{R}^+$, c_0, $a_0 > 0$. Assume that (2.38) is satisfied and that there exist positive numbers c and α such that*

$$\frac{H_2(t,s,v,u)}{s^\alpha} \le c\frac{H_2(t,s_1,v,u)}{s_1^\alpha} \tag{2.59}$$

for each $T \in A$, $|t| \le b_T$, $0 \le s \le s_1 \le a_0$, $0 \le v \le D_T + c_0$, $0 \le u \le 2c_0$. Then, for every $0 < \varepsilon < 1$, there exists $c(\varepsilon)$ depending only on ε, c, α such that

(2.60)

$$\begin{aligned} P\Big\{&\sup_{T\in A}\sup_{0\le v\le D_T}\sup_{0\le u\le c_0}\sup_{|t|\le b_T}\sup_{0\le s\le a_0}\big|X\big(R(t,s,v,u)\big)\big| \Big/ \\ &\qquad \Big\{x\Big(H_2(t,a_0,v,c_0(1+\varepsilon)\big) + H_2\big(t,\varepsilon a_0, v-\varepsilon c_0, c_0(1+\varepsilon)\big) \\ &\qquad + 2H_2(t,a_0,v-\varepsilon c_0, 2\varepsilon c_0) + 2H_2(t,a_0,v+u-\varepsilon c_0,2\varepsilon c_0) \\ &\qquad + 3H_2\big(t+s,\varepsilon a_0, c_0(1+\varepsilon)\big)\Big)\Big\} \ge 1\Big\} \\ &\le c(\varepsilon)\sup_{T\in A}\Big(\frac{D_T}{c_0}+1\Big)\Big(\frac{b_T}{a_T}+1\Big)\exp\Big(-\frac{x^2}{2+\varepsilon}\Big) \end{aligned}$$

for each $x \ge 1$.

PROOF. By (2.59) we have

$$H_2(t,a_0y,v,u) \le \frac{cy^\alpha}{\varepsilon^\alpha}H_2(t,a_0\varepsilon,v,u) \tag{2.61}$$

for each $0 < y \le \varepsilon$. Along the lines of the proof of Lemma 2.6, we obtain by (2.61) and (2.41) that

$$\begin{aligned} H_{21}^*(t,s,v,u,K) &\le 2H_2(t,s,v-\varepsilon c_0,2\varepsilon c_0) + 2H_2(t,s,v+u-\varepsilon c_0,2\varepsilon c_0) \\ &\qquad + \frac{\varepsilon}{8}H_2\big(t,a_0,v,c_0(1+\varepsilon)\big) + 2H_2\big(t+s,\varepsilon a_0,v,c_0(1+\varepsilon)\big), \\ H_{22}^*(t,s,v,K) &\le \frac{\varepsilon}{8}\Big(H_2\big(t,\varepsilon a_0,v-\varepsilon c_0,c_0(1+\varepsilon)\big) + H_2\big(t+s,\varepsilon a_0,c_0(1+\varepsilon)\big)\Big), \\ H_{23}^*(t,s,v,K) &\le \frac{\varepsilon}{8}\Big(H_2\big(t,\varepsilon a_0,v-\varepsilon c_0,c_0(1+\varepsilon)\big) + H_2\big(t+s,\varepsilon a_0,c_0(1+\varepsilon)\big)\Big), \end{aligned}$$

provided $k \ge \frac{2}{\varepsilon} + \frac{128c}{\alpha^2 e^{\alpha+1}}$. Now (2.60) follows from the above inequalities and by (2.54).

3. **Proofs of theorems.**

PROOF OF THEOREM 1.1. We first prove (1.19). For every $0 < \varepsilon < 1/2$, by (1.17) there exists a constant N such that

$$\sup_{|t|\le b_T+a_T}\sup_{0\le s\le a_T}\frac{H_1(t+s,a_T/N,T)}{H_1(t,a_T,T)} \le \varepsilon \tag{3.1}$$

for each $T \ge N$. Let $1 < \theta < \min\big(1+\frac{1}{N}, 1+\frac{\varepsilon^2}{38}\big)$. Put

$$A_i = \{T : \theta^i < a_T \le \theta^{i+1}\}, \quad -\infty < i < \infty,$$

$$B_j = \{T : \theta^j < \frac{b_T}{a_T} + 1 \le \theta^{j+1}\}, \quad j = 0,1,2,\dots.$$

Clearly, (1.18) implies $\frac{b_T}{a_T} \longrightarrow \infty$ as $T \longrightarrow \infty$ and $A_i B_j = \emptyset$ if $|i| \geq \theta^{\varepsilon j}$, where j is sufficiently large. Hence we have

(3.2)

$$\limsup_{T\to\infty} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,a_T,T)\Big(2\big(\log\frac{b_T}{a_T}+\log\log\tilde{H}_1(t,a_T,T)\big)\Big)^{1/2}}$$

$$\leq \limsup_{j\to\infty} \sup_{T\in B_j} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,a_T,T)\Big(2\big(\log\frac{b_T}{a_T}+\log\log\tilde{H}_1(t,a_T,T)\big)\Big)^{1/2}}$$

$$\leq \limsup_{j\to\infty} \max_{|i|\leq\theta^{\varepsilon j}} \sup_{T\in B_jA_i} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,a_T,T)\Big(2\big(\log\frac{b_T}{a_T}+\log\log\tilde{H}_1(t,a_T,T)\big)\Big)^{1/2}}$$

$$\leq \limsup_{j\to\infty} \max_{|i|\leq\theta^{\varepsilon j}} \sup_{T\in B_jA_i} \sup_{|t|\leq b_T} \sup_{0\leq s\leq \theta^{i+1}} \frac{(1+\varepsilon)|X(t+s,T)-X(t,T)|}{H_1(t,\theta^{i+1},T)\Big(2\big(\log\theta^j+\log\log\tilde{H}_1(t,\theta^{i+1},T)\big)\Big)^{1/2}}.$$

Here we have used, and in the sequel we will frequently use, (3.1) and the fact that $x\log\log\big(x+\frac{1}{x}\big)$is non-decreasing on $(0,\infty)$.

Using Lemma 2.6 and (3.1) again, we get

(3.3)

$$P\left\{\sup_{T\in B_jA_i} \sup_{|t|\leq b_T} \sup_{0\leq s\leq\theta^{i+1}} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,\theta^{i+1},T)\Big(2\big(\log\theta^j+\log\log\tilde{H}_1(t,\theta^{i+1},T)\big)\Big)^{1/2}} \geq (1+\varepsilon)^3\right\}$$

$$\leq c(\varepsilon) \sup_{T\in B_jA_i}\left(\frac{b_T}{\theta^{i+1}}+1\right)\exp\big(-(1+\varepsilon)^2\log\theta^j\big)$$

$$\leq c(\varepsilon)\theta^{-2\varepsilon j}$$

and hence

(3.4)

$$P\left\{\max_{|i|\leq\theta^{\varepsilon j}} \sup_{T\in B_jA_i} \sup_{|t|\leq b_T} \sup_{0\leq s\leq\theta^{i+1}} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,\theta^{i+1},T)\Big(2\big(\log\theta^j+\log\log\tilde{H}_1(t,\theta^{i+1},T)\big)\Big)^{1/2}} \geq (1+\varepsilon)^3\right\} \leq c(\varepsilon)\theta^{-\varepsilon j}.$$

It follows from (3.2), (3.4) and the Borel-Cantelli lemma that

(3.5)

$$\limsup_{T\to\infty} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} |X(t+s,T)-X(t,T)| \Big/ \left\{H_1(t,a_T,T)\cdot\left(2\left(\log\frac{b_T}{a_T}+\log\log\tilde{H}_1(t,a_T,T)\right)\right)^{1/2}\right\} \leq (1+\varepsilon)^4 \quad \text{a.s.}$$

This proves (1.19) by the arbitrariness of ε.

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We deal next with (3.8) which, together with (1.19), will imply also (1.23). Note that

(3.6)
$$\begin{aligned}
&\liminf_{T\to\infty}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_1(t,a_T,T)(2\log\frac{b_T}{a_T})^{1/2}}\\
&\quad\ge\liminf_{j\to\infty}\inf_{|i|\le\theta^{\varepsilon j}}\inf_{T\in B_jA_i}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_1(t,a_T,T)(2\log\frac{b_T}{a_T})^{1/2}}\\
&\quad\ge\liminf_{j\to\infty}\inf_{|i|\le\theta^{\varepsilon j}}\inf_{T\in B_jA_i}\sup_{0\le t\le b_T}\frac{|X(t+\theta^{i+1},T)-X(t,T)|}{(1+\varepsilon)H_1(t,\theta^{i+1},T)(2\log\theta^j)^{1/2}}\\
&\qquad-\limsup_{j\to\infty}\sup_{|i|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_i}\sup_{0\le t\le b_T}\frac{|X(t+\theta^{i+1},T)-X(t+a_T,T)|}{H_1(t,\theta^{i+1},T)(2\log\theta^j)^{1/2}}\\
&\quad\ge\liminf_{j\to\infty}\inf_{|i|\le\theta^{\varepsilon j}}\inf_{T\in B_jA_i}\sup_{0\le l\le\theta^{j-2}}\frac{\left|X\big((l+1)\theta^{i+1},T\big)-X(l\theta^{i+1},T)\right|}{(1+\varepsilon)H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}}\\
&\qquad-\limsup_{j\to\infty}\sup_{|i|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_i}\sup_{a_T\le t\le b_T+a_T}\sup_{0\le s\le(\theta-1)\theta^i}\frac{|X(t+s,T)-X(t,T)|}{H_1(t-a_T,\theta^{i+1},T)(2\log\theta^j)^{1/2}}.
\end{aligned}$$

Along the lines of the proof of (1.19), by (1.20), (1.17), we have

(3.7)
$$\begin{aligned}
&\limsup_{j\to\infty}\sup_{|i|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_i}\sup_{a_T\le t\le b_T+a_T}\sup_{0\le s\le(\theta-1)\theta^i}\frac{|X(t+s,T)-X(t,T)|}{H_1(t-a_T,\theta^{i+1},T)(2\log\theta^j)^{1/2}}\\
&\quad\le\limsup_{j\to\infty}\sup_{|i|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_i}\sup_{a_T\le t\le b_T+a_T}\sup_{0\le s\le(\theta-1)\theta^i}\frac{\varepsilon|X(t+s,T)-X(t,T)|}{H_1\big(t,(\theta-1)\theta^i,T\big)(2\log\theta^j)^{1/2}}\\
&\quad\le\varepsilon\quad\text{a.s.}
\end{aligned}$$

Let

(3.8) $$C_k:=C(k,l,i)=\{T:\theta^k<H_1(l\theta^{i+1},\theta^{i+1},T)\le\theta^{k+1}\},\quad -\infty<k<\infty,$$

(3.9) $$T_{k,l}:=T(k,l,i,j)=\sup\{T:T\in C_kA_iB_j\},$$

(3.10) $$T^*_{k,l}:=T^*(k,l,i,j)=\inf\{T:T\in C_kA_iB_j\}.$$

Put $Y(l,T)=X((l+1)\theta^{i+1},T)-X(l\theta^{i+1},T)$. Let $Z(l,T)$ be a two-parameter Gaussian process such that for each fixed l, $Z(l,T)$ is an independent increment process with $Z(l,T)\overset{\mathcal{D}}{=}Y(l,T)$, and $EZ(l,T)Z(n,T')=EY(l,T)Y(n,T')$ for $n\ne l$. Then, by (2.12) we have

$$\begin{gathered}
EY^2(l,T)=EZ^2(l,T),\\
EY(l,T)Y(n,T)=EZ(l,T)Z(n,T),\\
EY(l,T)Y(n,T')=EZ(l,T)Z(n,T'),\quad\text{for } l\ne n\\
EY(l,T)Y(l,T')\ge EY^2(l,T\wedge T')=EZ(l,T)Z(l,T').
\end{gathered}$$

Thus, we can use Lemma 2.2 and obtain

$$P\Big\{\inf_{T\in B_jA_i}\max_{0\le l\le\theta^{j-2}}\frac{Y(l,T)}{H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\}$$

(3.11)
$$\begin{aligned}&=1-P\Big\{\bigcap_{T\in B_jA_i}\bigcup_{0\le l\le\theta^{j-2}}\Big\{\frac{Y(l,T)}{H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}}\ge\frac{1}{(1+\varepsilon)^2}\Big\}\Big\}\\&\le1-P\Big\{\bigcap_{T\in B_jA_i}\bigcup_{0\le l\le\theta^{j-2}}\Big\{\frac{Z(l,T)}{H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}}\ge\frac{1}{(1+\varepsilon)^2}\Big\}\Big\}\\&=P\Big\{\bigcup_{T\in B_jA_i}\bigcap_{0\le l\le\theta^{j-2}}\Big\{\frac{Z(l,T)}{H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\}\Big\}.\end{aligned}$$

From (1.20) it is easy to see that $C_kB_j=\emptyset$ if $|k|\ge\theta^j$, when j is sufficiently large. Hence

(3.12)
$$\begin{aligned}&P\Big\{\bigcup_{T\in B_jA_i}\bigcap_{0\le l\le\theta^{j-2}}\Big\{\frac{Z(l,T)}{H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\}\Big\}\\&\quad\le\sum_{|k|\le\theta^j}P\Big\{\bigcup_{T\in B_jA_iC_k}\bigcap_{0\le l\le\theta^{j-2}}\Big\{\frac{Z(l,T)}{H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\}\Big\}\\&\quad\le\sum_{|k|\le\theta^j}P\Big\{\bigcap_{0\le l\le\theta^{j-2}}\bigcup_{T\in B_jA_iC_k}\Big\{\frac{Z(l,T)}{H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\}\Big\}\\&\quad\le\sum_{|k|\le\theta^j}P\Big\{\max_{0\le l\le\theta^{j-2}}\frac{Z(l,T_{k,l})}{H_1(l\theta^{i+1},\theta^{i+1},T_{k,l})(2\log\theta^j)^{1/2}}\le\frac{\theta}{1+\varepsilon}\Big\}\\&\qquad+\sum_{|k|\le\theta^j}P\Big\{\max_{0\le l\le\theta^{j-2}}\sup_{T\in B_jA_iC_k}\frac{|Z(l,T_{k,l})-Z(l,T)|}{H_1(l\theta^{i+1},\theta^{i+1},T_{k,l})(2\log\theta^j)^{1/2}}\ge\frac{\theta\varepsilon}{(1+\varepsilon)^2}\Big\}.\end{aligned}$$

Noting that $Z(l,T)$ is an independent increment process for l fixed, we have

$$\begin{aligned}E\big(Z(l,T_{k,l})-Z(l,T^*_{k,l})\big)^2&=EZ^2(l,T_{k,l})-EZ^2(l,T^*_{k,l})\\&=EY^2(l,T_{k,l})-EY^2(l,T^*_{k,l})\\&\le\theta^{2(k+1)}-\theta^{2k}\le(\theta^2-1)EY^2(l,T_{k,l})\\&=(\theta^2-1)H_1^2(l\theta^{i+1},\theta^{i+1},T_{k,l})\end{aligned}$$

and hence

(3.13)
$$\begin{aligned}&\sum_{|k|\le\theta^j}P\Big\{\max_{0\le l\le\theta^{j-2}}\sup_{T\in B_jA_iC_k}\frac{|Z(l,T_{k,l})-Z(l,T)|}{H_1(l\theta^{i+1},\theta^{i+1},T_{k,l})(2\log\theta^j)^{1/2}}\ge\frac{\theta\varepsilon}{(1+\varepsilon)^2}\Big\}\\&\quad\le\sum_{|k|\le\theta^j}\sum_{l=0}^{\theta^{j-2}}P\Big\{\sup_{T\in B_jA_iC_k}\frac{|Z(l,T_{k,l})-Z(l,T)|}{H_1(l\theta^{i+1},\theta^{i+1},T_{k,l})(2\log\theta^j)^{1/2}}\ge\frac{\varepsilon}{2}\Big\}\\&\quad\le\sum_{|k|\le\theta^j}\sum_{l=0}^{\theta^{j-2}}2P\Big\{\frac{|Z(l,T_{k,l})-Z(l,T^*_{k,l})|}{H_1(l\theta^{i+1},\theta^{i+1},T_{k,l})(2\log\theta^j)^{1/2}}\ge\frac{\varepsilon}{2}\Big\}\\&\quad\le4\sum_{|k|\le\theta^j}\sum_{l=0}^{\theta^{j-2}}\exp\Big(-\frac{\varepsilon^2\log\theta^j}{4(\theta^2-1)}\Big)\\&\quad\le4\theta^{-2j}.\end{aligned}$$

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Here $1 < \theta < 1 + \frac{\varepsilon}{32}$ was used.

When (1.21a) is satisfied, using the Slepian lemma we have

$$
\begin{aligned}
P\Big\{ \max_{0\le l\le \theta^{j-2}} \frac{Z(l,T_{k,l})}{H_1(l\theta^{i+1},\theta^{i+1},T_{k,l})(2\log\theta^j)^{1/2}} \le \frac{\theta}{1+\varepsilon}\Big\} \\
&\le \prod_{l=0}^{\theta^{j-2}} P\Big\{ \frac{Z(l,T_{k,l})}{H_1(l\theta^{i+1},\theta^{i+1},T_{k,l})(2\log\theta^j)^{1/2}} \le \frac{\theta}{1+\varepsilon}\Big\} \\
&= \prod_{l=0}^{\theta^{j-2}} P\Big\{ \frac{Y(l,T_{k,l})}{H_1(l\theta^{i+1},\theta^{i+1},T_{k,l})(2\log\theta^j)^{1/2}} \le \frac{\theta}{1+\varepsilon}\Big\} \\
&\le \prod_{l=0}^{\theta^{j-2}} \Big(1-\exp\Big(-\frac{\theta^2}{1+\varepsilon}\log\theta^j\Big)\Big) \\
&\le \exp\Big(-\theta^{\varepsilon j/4}\Big) \\
&\le \theta^{-4j}.
\end{aligned}
\tag{3.14}
$$

If, on the other hand, (1.21b) is satisfied, then $T_{k,l}$ does not depend on l by (3.8). Hence, using (1.21b) and the Slepian lemma again, we again have (3.14). Therefore, we conclude from (3.11)–(3.14) that

$$
P\Big\{ \inf_{T\in B_jA_i} \max_{0\le l\le \theta^{j-2}} \frac{Y(l,T)}{H_1(l\theta^{i+1},\theta^{i+1},T)(2\log\theta^j)^{1/2}} \le \frac{1}{(1+\varepsilon)^2}\Big\} \le 5\theta^{-2j}
\tag{3.15}
$$

for every sufficiently large j.

Combining (3.6), (3.7), (3.15) with the Borel-Cantelli lemma yields

$$
\liminf_{T\to\infty} \sup_{0\le t\le b_T} \frac{|X(t+a_T,T)-X(t,T)|}{H_1(t,a_T,T)(2\log\frac{b_T}{a_T})^{1/2}} \ge \frac{1}{(1+\varepsilon)^2}-\varepsilon \quad \text{a.s.}
\tag{3.16}
$$

This proves (1.22), by (1.19), (1.20) and by the arbitrariness of ε, and also completes the proof of Theorem 1.1.

PROOF OF THEOREM 1.2. For every $0 < \varepsilon < 1/2$, let N be a positive number such that (3.1) holds true. Put $\theta = \min(1+\frac{1}{N}, 1+\frac{\varepsilon^2}{32})$. Define

$$
\begin{aligned}
A_{k,i} &= \Big\{T : \theta^k < H_1\big((\theta-1)ia_T, a_T, T\big) \le \theta^{k+1}\Big\}, \quad -\infty < k < \infty \\
A_{k,i,j} &= \Big\{T : \theta^j < 1 + \frac{b_T}{a_T} \le \theta^{j+1}, T \in A_{k,i}\Big\}, \quad j = 0,1,2,\dots \\
T_{k,i,j} &= \sup\{T : T \in A_{k,i,j}\}, \\
G(i,T) &= H_1\big((i+1)(\theta-1)a_T, a_T, T\big).
\end{aligned}
$$

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Then, by (3.1)

$$\limsup_{T\to\infty} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,a_T,T)\Big(2\big(\log\frac{b_T}{a_T}+\log\log\tilde{H}_1(t,a_T,T)\big)\Big)^{1/2}}$$

$$\le \limsup_{T\to\infty} \max_{|i|\le\frac{b_T}{(\theta-1)a_T}} \sup_{i(\theta-1)a_T<t\le(i+1)(\theta-1)a_T} \sup_{0\le s\le a_T} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,a_T,T)\Big(2\big(\log\frac{b_T}{a_T}+\log\log\tilde{H}_1(t,a_T,T)\big)\Big)^{1/2}}$$

(3.17)
$$\le \limsup_{T\to\infty} \max_{|i|\le\frac{b_T}{(\theta-1)a_T}} \sup_{i(\theta-1)a_T<t\le(i+1)(\theta-1)a_T} \sup_{0\le s\le a_T} \frac{(1+\varepsilon)|X(t+s,T)-X(t,T)|}{G(i,T)\Big(2\big(\log\frac{b_T}{a_T}+\log\log\tilde{G}(i,T)\big)\Big)^{1/2}}$$

$$\le \limsup_{j+|k|\to\infty} \max_{|i|\le\frac{\theta^{j+1}}{\theta-1}} \sup_{T\in A_{k,i,j}} \sup_{i(\theta-1)a_T<t\le(i+1)(\theta-1)a_T} \sup_{0\le s\le a_{T_{k,i,j}}} \frac{(1+\varepsilon)\theta|X(t+s,T)-X(t,T)|}{G(i,T_{k,i,j})\Big(2\big(\log\theta^j+\log\log\tilde{G}(i,T_{k,i,j})\big)\Big)^{1/2}}$$

$$\le \limsup_{j+|k|\to\infty} \max_{|i|\le\frac{\theta^{j+1}}{\theta-1}} \sup_{T\in A_{k,i,j}} \sup_{i(\theta-1)a_T<t\le(i+1)(\theta-1)a_T} \sup_{0\le s\le a_{T_{k,i,j}}} \frac{(1+\varepsilon)\theta|X(t+s,T)-X(t,T)|}{H_1(t,a_{T_{k,i,j}},T_{k,i,j})\big(2(\log\theta^j+\log\log\theta^{|k|})\big)^{1/2}},$$

where in the last inequality the assumption that $H_1(t,s,T)$ is non-decreasing in t on $(0,\infty)$ and non-increasing in t on $(-\infty,0)$ is used.

In terms of (3.1) and (2.57), we obtain

(3.18)
$$P\Bigg\{\max_{|i|\le\frac{\theta^{j+1}}{\theta-1}} \sup_{T\in A_{k,i,j}} \sup_{i(\theta-1)a_T<t\le(i+1)(\theta-1)a_T} \sup_{0\le s\le a_{T_{k,i,j}}} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,a_{T_{k,i,j}},T_{k,i,j})\big(2(\log\theta^j+\log\log\theta^{|k|})\big)^{1/2}} \ge (1+\varepsilon)^2\Bigg\}$$
$$\le c(\varepsilon)\sum_{|i|\le\frac{\theta^{j+1}}{\theta-1}} \exp\big(-(1+\varepsilon)(\log\theta^j+\log\log\theta^{|k|})\big)$$
$$\le c(\varepsilon)\theta^{-\varepsilon j}(|k|+1)^{-(1+\varepsilon)}.$$

From (3.17), (3.18) and the Borel-Cantelli lemma we conclude

$$\lim_{T\to\infty} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{|X(t+s,T)-X(t,T)|}{H_1(t,a_T,T)\Big(2\big(\log\frac{b_T}{a_T}+\log\log\tilde{H}_1(t,a_T,T)\big)\Big)^{1/2}} \le (1+\varepsilon)^4 \quad \text{a.s.}$$

This proves (1.19) by the arbitrariness of ε.

We show next that (1.22) and (1.23) hold true. It suffices to prove that (1.20), (1.16) and $b_T \geq c_0 > 0$ imply (1.18). It is clear that (1.20) implies $\frac{b_T}{a_T} \longrightarrow \infty$ as $T \longrightarrow \infty$. So, when $0 < a_T \leq 1$, (1.18) is obvious. When $a_T \geq 1$, then by (1.16)

$$a_T^\alpha \leq \frac{cH_1(0, a_T, T)}{H_1(0, 1, T)} \leq \frac{cH_1(0, a_T, T)}{H_1(0, 1, 1)}$$

for $T \geq 1$, which together with (1.20) also yields (1.18). This proves that (1.18) is satisfied. From Theorem 1.1 it follows that (1.22) and (1.23) are true.

PROOF OF THEOREM 1.3. For every $0 < \varepsilon < 1/2$, by (1.26) there exists a positive N such that

(3.19)
$$\sup_{|t| \leq b_T + a_T} \sup_{0 \leq s \leq a_T} \sup_{0 \leq v \leq D_T + c_T} \sup_{-\frac{c_T}{N} \leq u \leq c_T} \frac{H_2(t+s, a_T, v+u, \frac{c_T}{N}) + H_2(t+s, \frac{a_T}{N}, v+u, c_T)}{H_2(t, a_T, v, c_T)} \leq \varepsilon$$

for $T \geq N$. Let $1 < \theta < 1 + \frac{1}{N}$. Put

$$A_k = \{T : \theta^k < a_T \leq \theta^{k+1}\}, \quad -\infty < k < \infty$$
$$B_i = \{T : \theta^i < c_T \leq \theta^{i+1}\}, \quad -\infty < i < \infty$$
$$G_j = \Big\{T : \theta^j < \Big(\frac{b_T}{a_T} + 1\Big)\Big(\frac{D_T}{c_T} + 1\Big) \leq \theta^{j+1}\Big\}, \quad j = 0, 1, 2, \ldots$$

Clearly, (1.27) implies that

$$\Big(\frac{b_T}{a_T} + 1\Big)\Big(\frac{D_T}{c_T} + 1\Big) \longrightarrow \infty \quad \text{as } T \longrightarrow \infty,$$

$A_k G_j = \emptyset$ and $B_k G_j = \emptyset$ if $|k| \geq \theta^{\varepsilon j}$, when j is sufficiently large. Hence we have

(3.20)
$$\begin{aligned}
&\limsup_{T \to \infty} \sup_{0 \leq v \leq D_T} \sup_{0 \leq u \leq c_T} \sup_{|t| \leq b_T} \sup_{0 \leq s \leq a_T} \\
&\qquad \frac{\Big|X\Big(R(t,s,v,u)\Big)\Big|}{H_2(t, a_T, v, c_T)\Big(2\Big(\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1) + \log\log \tilde{H}_2(t, a_T, v, c_T)\Big)\Big)^{1/2}} \\
&\leq \limsup_{j \to \infty} \max_{|k|, |i| \leq \theta^{\varepsilon j}} \sup_{T \in A_k B_i G_j} \sup_{0 \leq v \leq D_T} \sup_{0 \leq u \leq c_T} \sup_{|t| \leq b_T} \sup_{0 \leq s \leq a_T} \\
&\qquad \frac{\Big|X\Big(R(t,s,v,u)\Big)\Big|}{H_2(t, a_T, v, c_T)\Big(2\Big(\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1) + \log\log \tilde{H}_2(t, a_T, v, c_T)\Big)\Big)^{1/2}} \\
&\leq \limsup_{j \to \infty} \max_{|k|, |i| \leq \theta^{\varepsilon j}} \sup_{T \in A_k B_i G_j} \sup_{0 \leq v \leq D_T} \sup_{0 \leq u \leq \theta^{i+1}} \sup_{|t| \leq b_T} \sup_{0 \leq s \leq \theta^{k+1}} \\
&\qquad \frac{(1+\varepsilon)\Big|X\Big(R(t,s,v,u)\Big)\Big|}{H_2(t, \theta^{k+1}, v, \theta^{i+1})\Big(2\Big(\log \theta^j + \log\log \tilde{H}_2(t, \theta^{k+1}, v, \theta^{i+1})\Big)\Big)^{1/2}}.
\end{aligned}$$

Using (3.19) and (2.60), we derive

(3.21)

$$P\Bigg\{ \max_{|k|,|i|\le\theta^{\varepsilon j}} \sup_{T\in A_kB_iG_j} \sup_{0\le v\le D_T} \sup_{0\le u\le\theta^{i+1}} \sup_{|t|\le b_T} \sup_{0\le s\le\theta^{k+1}} \frac{\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,\theta^{k+1},v,\theta^{i+1})\Big(2\big(\log\theta^j+\log\log\tilde{H}_2(t,\theta^{k+1},v,\theta^{i+1})\big)\Big)^{1/2}} \ge (1+\varepsilon)^4\Bigg\}$$

$$\le c(\varepsilon)\sum_{|k|,|i|\le\theta^{\varepsilon j}} \sup_{T\in A_kB_iG_j}\Big(\frac{D_T}{\theta^{i+1}}+1\Big)\Big(\frac{b_T}{\theta^{k+1}}+1\Big)\exp\big(-(1+\varepsilon)^2\log\theta^j\big)$$

$$\le c(\varepsilon)\sum_{|k|,|i|\le\theta^{\varepsilon j}} \theta^j\exp\big(-(1+\varepsilon)^2\log\theta^j\big)$$

$$\le c(\varepsilon)\theta^{-\varepsilon^2 j}.$$

It follows from (3.20), (3.21) and the Borel- Cantelli lemma that

$$\limsup_{T\to\infty} \sup_{0\le v\le D_T} \sup_{0\le u\le c_T} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,a_T,v,c_T)\Big(2\big(\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)+\log\log\tilde{H}_2(t,a_T,v,c_T)\big)\Big)^{1/2}}$$

$$\le (1+\varepsilon)^5 \quad \text{a.s.}$$

This proves (1.28) by the arbitrariness of ε.

We now consider (1.31). Note that

$$\liminf_{T\to\infty} \sup_{0\le v\le D_T} \sup_{0\le t\le b_T} \frac{\big|X\big(R(t,a_T,v,c_t)\big)\big|}{H_2(t,a_T,v,c_T)\big(2\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)\big)^{1/2}}$$

$$\ge \liminf_{j\to\infty} \inf_{|k|,|i|\le\theta^{\varepsilon j}} \inf_{T\in A_kB_iG_j} \sup_{0\le v\le D_T} \sup_{0\le t\le b_T} \frac{\big|X\big(R(t,a_T,v,c_T)\big)\big|}{H_2(t,a_T,v,c_T)\big(2\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)\big)^{1/2}}$$

$$\ge \liminf_{j\to\infty} \inf_{|k|,|i|\le\theta^{\varepsilon j}} \inf_{T\in A_kB_iG_j} \sup_{0\le v\le D_T} \sup_{0\le t\le b_T} \frac{\big|X\big(R(t,\theta^{k+1},v,\theta^{i+1})\big)\big|}{(1+\varepsilon)H_2(t,\theta^{k+1},v,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$-\limsup_{j\to\infty} \sup_{|k|,|i|\le\theta^{\varepsilon j}} \sup_{T\in A_kB_iG_j} \sup_{\theta^i\le v\le\theta^i+D_T} \sup_{0\le u\le(\theta-1)\theta^i} \sup_{0\le t\le b_T} \sup_{0\le s\le\theta^{k+1}} \frac{\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,\theta^{k+1},v-\theta^i,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$-\limsup_{j\to\infty} \sup_{|k|,|i|\le\theta^{\varepsilon j}} \sup_{T\in A_kB_iG_j} \sup_{0\le v\le D_T} \sup_{0\le u\le\theta^{i+1}} \sup_{\theta^k\le t\le\theta^k+b_T} \sup_{0\le s\le(\theta-1)\theta^k} \frac{\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t-\theta^k,\theta^{k+1},v,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$+\sum_{j=0}^{\infty}\Big|X\big(R((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)$$

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(3.22)

$$\geq \liminf_{j\to\infty} \inf_{|k|,|i|\leq\theta^{\varepsilon j}} \min_{m\cdot n\geq\theta^j} \max_{0\leq l\leq m} \max_{0\leq p\leq n} \frac{\left|X\left(R(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})\right)\right|}{(1+\varepsilon)H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$-\varepsilon \limsup_{j\to\infty} \sup_{|k|,|i|\leq\theta^{\varepsilon j}} \sup_{T\in A_kB_iG_j} \sup_{0\leq v\leq\theta^i+D_T} \sup_{0\leq u\leq(\theta-1)\theta^i} \sup_{0\leq t\leq b_T} \sup_{0\leq s\leq\theta^{k+1}} \frac{\left|X\left(R(t,s,v,u)\right)\right|}{H_2(t,\theta^{k+1},v,(\theta-1)\theta^i)(2\log\theta^j)^{1/2}}$$

$$-\varepsilon \limsup_{j\to\infty} \sup_{|k|,|i|\leq\theta^{\varepsilon j}} \sup_{T\in A_kB_iG_j} \sup_{0\leq v\leq D_T} \sup_{0\leq u\leq\theta^{i+1}} \sup_{0\leq t\leq\theta^k+b_T} \sup_{0\leq s\leq(\theta-1)\theta^k} \frac{\left|X\left(R(t,s,v,u)\right)\right|}{H_2(t,(\theta-1)\theta^k,v,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$:= I_1 - I_2 - I_3.$$

Along the lines of the proof of (1.28) and by (1.29) we can obtain

$$I_2 + I_3 \leq 2\varepsilon \quad \text{a.s.} \tag{3.23}$$

For I_1, in terms of (1.30), we can apply the Slepian lemma and get

$$P\left\{\min_{mn\geq\theta^j} \max_{0\leq l\leq m} \max_{0\leq p\leq n} \frac{X(R((l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1}))}{H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}} \leq \frac{1}{(1+\varepsilon)^2}\right\}$$

$$\leq \sum_{mn\geq\theta^j} \prod_{0\leq l\leq m} \prod_{0\leq p\leq n} P\left\{\frac{X(R((l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1}))}{H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}} \leq \frac{1}{(1+\varepsilon)^2}\right\}$$

$$\leq \sum_{m,n:mn\geq\theta^j} \left(1 - \exp\left(-\frac{\log\theta^j}{(1+\varepsilon)^2}\right)\right)^{(m+1)(n+1)}$$

$$\leq \sum_{m,n:mn\geq\theta^j} \exp\left(-(m+1)(n+1)\theta^{-\frac{j}{(1+\varepsilon)^2}}\right)$$

$$\leq \theta^{2j}\exp(-\theta^{j\varepsilon})$$

$$\leq \theta^{-j}$$

for every sufficiently large j. This proves

$$I_1 \geq \frac{1}{(1+\varepsilon)^3} \quad \text{a.s.} \tag{3.24}$$

by the Borel-Cantelli lemma. From the above inequalities we finally conclude

(3.25)

$$\liminf_{T\to\infty} \sup_{0\leq v\leq D_T} \sup_{0\leq t\leq b_T} \frac{\left|X\left(R(t,a_T,v,c_T)\right)\right|}{H_2(t,a_T,v,c_T)\left(2\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)\right)^{1/2}} \geq \frac{1}{(1+\varepsilon)^3} - 2\varepsilon \quad \text{a.s.}$$

Now (1.31) follows from (3.25), (1.28) and (1.29), and so does also (1.32). This completes the proof of Theorem 1.3.

PROOF OF THEOREM 1.4. For every $0 < \varepsilon < 1/2$, let N satisfy (3.19) and $1 < \theta < 1 + \frac{1}{N}$. Put

$$A_{k,l,p} = \Big\{T : \theta^k < H_2\big((\theta-1)la_T, a_T, p(\theta-1)c_T, c_T\big) \le \theta^{k+1}\Big\}, \quad -\infty < k < \infty,$$
$$G_j = \Big\{T : \theta^j < \Big(1 + \frac{b_T}{a_T}\Big)\Big(1 + \frac{D_T}{c_T}\Big) \le \theta^{j+1}\Big\}, \quad j = 0, 1, 2, \ldots,$$
$$T^* := T(k, l, p, j) = \sup\{T : T \in A_{k,l,p} G_j\}.$$

Then

$$\limsup_{T\to\infty} \sup_{0\le v\le D_T} \sup_{0\le u\le c_T} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,a_T,v,c_T)\Big(2\Big(\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1) + \log\log \tilde H_2(t,a_T,v,c_T)\Big)\Big)^{1/2}}$$
$$\le \limsup_{j+i\to\infty} \max_{(|l|+1)(p+1)\le\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\ge i} \sup_{T\in A_{k,l,p}G_j} \sup_{p(\theta-1)c_T\le v\le(p+1)(\theta-1)c_T} \sup_{0\le u\le c_T} \sup_{l(\theta-1)a_T\le t\le(l+1)(\theta-1)a_T}$$
$$\sup_{0\le s\le a_T} \frac{\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,a_T,v,c_T)\Big(2\Big(\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1) + \log\log \tilde H_2(t,a_T,v,c_T)\Big)\Big)^{1/2}}$$
$$\le \limsup_{j+i\to\infty} \max_{(|l|+1)(p+1)\le\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\ge i} \sup_{T\in A_{k,l,p}G_j} \sup_{p(\theta-1)c_T\le v\le(p+1)(\theta-1)c_T} \sup_{0\le u\le c_{T^*}}$$
$$\sup_{l(\theta-1)a_T l+1)(\theta-1)a_T} \sup_{0\le s\le a_{T^*}} \frac{(1+\varepsilon)\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,a_{T^*},v,c_{T^*})\big(2(\log\theta^j + \log\log\theta^{|k|})\big)^{1/2}}$$
$$:= \limsup_{j+i\to\infty} \max_{(|l|+1)(p+1)\le\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\ge i} I_{k,l,p,j}.$$

According to (3.19) and (2.60), we arrive at

$$P\Big\{\max_{(|l|+1)(p+1)\le\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\ge i} I_{k,l,p,j} \ge (1+\varepsilon)^3\Big\}$$
$$\le c(\varepsilon) \sum_{l,p:(|l|+1)(p+1)\le\frac{\theta^{j+2}}{\theta-1}} \sum_{|k|\ge i} \sup_{T\in A_{k,l,p}G_j} \Big(\frac{c_T}{c_{T^*}}+1\Big)$$
$$\cdot\Big(\frac{a_T}{a_{T^*}}+1\Big)\exp\big(-(1+\varepsilon)(\log\theta^j + \log\log\theta^{|k|})\big)$$
$$\le c(\varepsilon) \sum_{l,p:(|l|+1)(p+1)\le\frac{\theta^{j+2}}{\theta-1}} \sum_{|k|\ge i} \theta^{-(1+\varepsilon)j}(|k|+1)^{-1-\varepsilon}$$
$$\le c(\varepsilon)\theta^j \cdot j \cdot \theta^{-(1+\varepsilon)j}(i+1)^{-\varepsilon}$$
$$\le c(\varepsilon)\theta^{-\frac{\varepsilon}{2}j}(i+1)^{-\varepsilon}$$

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and hence

$$\begin{aligned}
&\lim_{N\to\infty} P\Big\{ \bigcup_{j+i\geq 2N} \Big\{ \max_{(|l|+1)(p+1)\leq\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\geq i} I_{k,l,p,j} \geq (1+\varepsilon)^3 \Big\}\Big\} \\
&\quad \leq \lim_{N\to\infty} P\Big\{ \bigcup_{j=0}^{\infty} \bigcup_{i=N}^{\infty} \Big\{ \max_{(|l|+1)(p+1)\leq\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\geq i} I_{k,l,p,j} \geq (1+\varepsilon)^3 \Big\}\Big\} \\
&\qquad + \lim_{N\to\infty} P\Big\{ \bigcup_{j=N}^{\infty} \bigcup_{i=0}^{\infty} \Big\{ \max_{(|l|+1)(p+1)\leq\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\geq i} I_{k,l,p,j} \geq (1+\varepsilon)^3 \Big\}\Big\} \\
&\quad \leq \lim_{N\to\infty} \sum_{j=0}^{\infty} P\Big\{ \max_{(|l|+1)(p+1)\leq\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\geq N} I_{k,l,p,j} \geq (1+\varepsilon)^3 \Big\} \\
&\qquad + \lim_{N\to\infty} \sum_{j=N}^{\infty} P\Big\{ \max_{(|l|+1)(p+1)\leq\frac{\theta^{j+2}}{\theta-1}} \sup_{|k|\geq 0} I_{k,l,p,j} \geq (1+\varepsilon)^3 \Big\} \\
&\quad \leq \lim_{N\to\infty} \sum_{j=0}^{\infty} c(\varepsilon)\cdot\theta^{-\frac{\varepsilon}{2}j}(N+1)^{-\varepsilon} + \lim_{N\to\infty} \sum_{j=N}^{\infty} c(\varepsilon)\theta^{-\frac{\varepsilon}{2}j} \\
&\quad = 0.
\end{aligned}$$

This proves

$$\limsup_{T\to\infty} \sup_{0\leq v\leq D_T} \sup_{0\leq u\leq c_T} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \frac{\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,a_T,v,c_T)\Big(2\big(\log(\frac{b_T}{a_T}+1)(\frac{D_T}{c_T}+1)+\log\log\tilde{H}_2(t,a_T,v,c_T)\big)\Big)^{1/2}} \leq (1+\varepsilon)^3 \quad \text{a.s.},$$

which implies that (1.28) holds true by the arbitrariness of ε.

REMARK 3.1. From the proof of Theorem 1.3 one can see that if in addition to the conditions of Theorem 1.4, also (1.29), (1.30) and

$$\log\log\Big(a_T + c_T + \frac{1}{b_T} + \frac{1}{D_T}\Big) = O\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(1+\frac{D_T}{c_T}\Big)\Big) \quad \text{as } T\to\infty$$

are satisfied, then (1.31) and (1.32) are true.

PROOF OF COROLLARY 1.1. Since $H_2^2(t,s,v,u) = su$, $H_1^2(t,s,v) = sv$ and $\{W(x,y), -\infty < x < \infty, 0 \leq y < \infty\}$ is an independent increment process, it is easy to verify that the conditions in Theorem 1.1 are satisfied. This implies that the conclusion of Corollary 1.1 holds true.

The proof of Corollary 1.2 is trivial and is omitted.

PROOF OF COROLLARIES 1.3 AND 1.4. Noting that $H_1^2(t,s,v) = s(1-s)v$, $H_2^2(t,s,v,u) = s(1-s)u$ and that (1.36) and $b_T \leq 1$ imply $a_T \to 0$ as $T \to 0$, we have that (1.16), (1.17), (1.25), (1.26) are satisfied. For each $v \geq u$, $0 \leq t+s \leq 1$, $t \geq 0$, $s \geq 0$ we have

$$\begin{aligned} &E\Big(K(t+s,v) - K(t,v)\Big)\Big(K(t+s,u) - K(t,u)\Big) \\ &\qquad = \Big((t+s) - (t+s)^2\Big)u - \Big(t - t(t+s)\Big)u - \Big(t - t(t+s)\Big)u + (t-t^2)u \\ &\qquad = s(1-s)u = E\Big(K(t+s,u) - K(t,u)\Big)^2 \end{aligned}$$

and for $j > l$

$$\begin{aligned} &E\Big(K\big((j+1)s, v\big) - K(js,v)\Big)\Big(K\big((l+1)s,v\big) - K(ls,v)\Big) \\ &\quad = \Big((l+1)s\big(1-(j+1)s\big) - (l+1)s(1-js) - ls\big(1-(j+1)s\big) + ls(1-js)\Big)v \\ &\quad = -sv \leq 0. \end{aligned}$$

This shows that (2.12) and (1.21b) are satisfied. Similarly, we can verify that (2.38) and (1.30) are satisfied. Now Corollaries 1.3 and 1.4 follow from Theorems 1.1 and 1.3.

PROOF OF COROLLARY 1.5. Noting that

$$EX(t,v)X(s,u) = \int_0^{v\wedge u} \exp\big(-\lambda(y)|t-s|\big)\frac{\gamma(y)}{\lambda(y)}\,dy$$

for $v, u > 0$, we can verify that (2.12), (1.21b), (2.38) are satisfied. We show below that $H^2(s,T)/s^\alpha$ is increasing in s on $(0, a_T)$, where $\alpha = \frac{1}{6(c+1)}$. Let

$$f(s) = H^2(s,T)/s^\alpha = 2\int_0^T \frac{\gamma(x)}{\lambda(x)}\big(1 - \exp(-\lambda(x)s)\big)\,dx/s^\alpha.$$

Then, by (1.38)

$$\begin{aligned} f'(s) &= 2s^{-\alpha-1}\Big(-\alpha\int_0^T \frac{\gamma(x)}{\lambda(x)}\Big(1 - \exp\big(-\lambda(x)s\big)\Big)\,dx + \int_0^T s\gamma(x)\exp\big(-\lambda(x)s\big)\,dx\Big) \\ &\geq 2s^{-\alpha-1}\Big(-\alpha\int_{0\leq x\leq T:\lambda(x)\geq\frac{1}{s}} \frac{\gamma(x)}{\lambda(x)}\,dx - \alpha s\int_{0\leq x\leq T:\lambda(x)\leq\frac{1}{s}} \gamma(x)\,dx \\ &\qquad + \frac{s}{3}\int_{0\leq x\leq T:\lambda(x)\leq\frac{1}{s}} \gamma(x)\,dx\Big) \\ &\geq 2s^{-\alpha-1}\Big(-\alpha(c+1)s\int_{0\leq x\leq T:\lambda(x)\leq\frac{1}{s}} \gamma(x)\,dx + \frac{s}{3}\int_{0\leq x\leq T:\lambda(x)\leq\frac{1}{s}} \gamma(x)\,dx\Big) \\ &> 0, \end{aligned}$$

provided $0 < \alpha < \frac{1}{3(c+1)}$, as desired. Therefore, (1.16) is satisfied. Corollary 1.5 now follows from Theorem 1.1.

PROOF OF COROLLARY 1.6. Put $M = \sup_{0<x\leq d+b}\lambda(x)$. Then $M < \infty$ by (1.40). It follows from (1.41) that

$$\int_0^{d+b} \gamma(x)\,dx < \infty. \tag{3.26}$$

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Clearly, for $0 < s \le 1/M, 0 < v + c \le d + b$,

$$\frac{s}{3}\int_v^{v+c}\gamma(x)\,dx \le \int_v^{v+c}\frac{\gamma(x)}{\lambda(x)}\Big(1-\exp\big(-\lambda(x)s\big)\Big)\,dx = \frac{1}{2}H^2(s,v,c) \le s\int_v^{v+c}\gamma(x)\,dx,$$

which implies that (1.25) is satisfied. We show next that (1.26) is also satisfied. It suffices to prove that we have

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\sup_{0\le v\le d+\frac{b}{2}}\sup_{-\delta c_T\le u\le c_T}\frac{H^2(a_T, v+u, \delta c_T)}{H^2(a_T, v, c_T)} = 0. \tag{3.27}$$

By (1.40) again, we see that

$$\frac{H^2(a_T, v, c)}{2a_T\int_v^{v+c}\gamma(x)dx} \longrightarrow 1$$

as $T \longrightarrow \infty$, uniformly in $0 < v \le d + b, 0 < c \le 1$. Hence, equivalently, it is enough to show that

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\sup_{0\le v\le d+\frac{b}{2}}\sup_{-\delta c_T\le u\le c_T}\frac{\int_{v+u}^{v+u+\delta c_T}\gamma(x)\,dx}{\int_c^{v+c_T}\gamma(x)\,dx} = 0. \tag{3.28}$$

Noting that $\gamma(x)$ is a positive continuous function on $(0,\infty)$, we have

$$0 < \inf_{\frac{1}{3}\le x\le d+b}\gamma(x) \le \sup_{\frac{1}{3}\le x\le d+b}\gamma(x) < \infty.$$

Hence

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\sup_{\frac{1}{2}\le v\le d+\frac{b}{2}}\sup_{-\delta c_T\le u\le c_T}\frac{\int_{v+u}^{v+u+\delta c_T}\gamma(x)\,dx}{\int_c^{v+c_T}\gamma(x)\,dx} \le \lim_{\delta\to 0}\delta\cdot\frac{\sup_{\frac{1}{3}\le x\le d+b}\gamma(x)}{\inf_{\frac{1}{3}\le x\le d+b}\gamma(x)} = 0. \tag{3.29}$$

If $0 < v \le \frac{1}{2}, -\delta c_T \le u \le \frac{c_T}{2}$, then

$$\int_v^{v+c_T}\gamma(x)\,dx \ge \int_{v+\frac{2}{3}c_T}^{v+c_T}\gamma(x)\,dx \ge \frac{c_T}{3}\gamma(x_0), \tag{3.30}$$

where $v + \frac{c_T}{3} \le x_0 \le v + c_T$. By (1.41) we find

$$\begin{aligned}\int_{v+u}^{v+u+\delta c_T}\gamma(x)\,dx &\le c x_0^{1-\alpha}\gamma(x_0)\int_{v+u}^{v+u+\delta c_T}\frac{1}{x^{1-\alpha}}\,dx\\ &= \frac{c}{\alpha}x_0^{1-\alpha}\gamma(x_0)\big((v+u+\delta c_T)^\alpha-(v+u)^\alpha\big)\\ &\le \frac{c}{\alpha}(v+c_T)^{1-\alpha}\gamma(x_0)\big((v+u+\delta c_T)^\alpha-(v+u)^\alpha\big)\\ &\le \frac{6c(\delta^\alpha+\delta)c_T\gamma(x_0)}{\alpha}.\end{aligned} \tag{3.31}$$

If $0 < v \le \frac{1}{2}, \frac{c_T}{2} \le u \le c_T$, then

$$\int_{v+u}^{v+u+\delta c_T}\gamma(x)\,dx = \gamma(y_0)\delta c_T, \tag{3.32}$$

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where $u+v \le y_0 \le v+u+\delta c_T$. From (1.41) again, we obtain

$$
\begin{aligned}
\int_v^{v+c_T} \gamma(x)\,dx &\ge \int_v^{v+\frac{c_T}{2}} \gamma(x)\,dx \\
&\ge \frac{\gamma(y_0)}{c y_0^{1/\alpha}} \int_v^{v+\frac{c_T}{2}} x^{\frac{1}{\alpha}}\,dx \\
&\ge \frac{\alpha\gamma(y_0)}{2c y_0^{1/\alpha}} \left(\left(v+\frac{c_T}{2}\right)^{1+\frac{1}{\alpha}} - v^{1+\frac{1}{\alpha}} \right) \qquad (3.33)\\
&\ge \frac{\alpha\gamma(y_0)}{2c\cdot 2^{\frac{1}{\alpha}}(v+c_T)^{\frac{1}{\alpha}}} \left(\left(v+\frac{c_T}{2}\right)^{1+\frac{1}{\alpha}} - v^{1+\frac{1}{\alpha}} \right) \\
&\ge \frac{\alpha\gamma(y_0)c_T}{c\cdot(12)^{\frac{1}{\alpha}+1}}.
\end{aligned}
$$

Combining (3.29)–(3.33) yields (3.28). This proves that (1.26) is satisfied. Corollary 1.6 follows from Theorem 1.3.

The proofs of Corollaries 1.7–1.9 are trivial. The details are omitted here.

ACKNOWLEDGEMENTS. We are most grateful to a referee for carefully reading our manuscript, for pointing out a number of misprints and for many insightful comments and suggestions.

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