# AN EXTENDED WORK INTEGRAL FOR PULSATING STARS

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This paper contains a preliminary report on research aimed at extending the pulsational work integral to next highest order beyond the linear regime.

We begin by writing the integral in fully nonlinear form (Cox and Giuli 1968), normalized by the stellar luminosity L

$$F \equiv \frac{1}{L} < \frac{dW}{dt} > = \frac{1}{\Pi} \int_0^{\Pi} dt \int_M \frac{P}{L} \frac{\partial}{\partial t} (1/\rho)\, dm \; . \qquad (1)$$

To evaluate F in the linear theory, one expands P and $\rho$ in a series of small perturbations, discarding terms higher than first order. For the fundamental mode with the frequency $\omega_0$, one obtains

$$F_0 = a_0 \lambda_0^2 \; ,$$

while for the first overtone, frequency $\omega_1$, the result is

$$F_1 = a_1 \lambda_1^2 .$$

Here $a_0$ and $a_1$ represent integrals over the star. The scale factors $\lambda_0$ and $\lambda_1$ have been explicitly inserted to indicate that the linear results are arbitrary to within a multiplicative constant for each mode. These scale factors enter quadratically since the P and $\rho$ expansions each carry the arbitrary constant.

The work integral (1) may be extended in a straightforward way by keeping higher order terms in P and $\rho$. These terms are in turn evaluated using the iterative theory of Simon (1980). Following considerable algebraic and trigonometric manipulation, to be described elsewhere, one obtains

$$F = a_0\lambda_0^2 + a_1\lambda_1^2 + b_{00}\lambda_0^2 + b_{11}\lambda_1^2 + b_{01}\lambda_0^2\lambda_1^2 \; . \qquad (2)$$

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The first two terms on the RHS of (2) are just the linear expressions given above; the remaining terms represent nonlinear corrections, all of which scale quartically, as indicated. The coefficients $b_{00}$, $b_{11}$ and $b_{01}$ are again integrals over the star. We shall take them to be strictly negative, meaning that they always produce a damping effect. Such a choice is supported by the lack of any observational or theoretical evidence for hard self-excited pulsations. The coefficients $b_{00}$ and $b_{11}$ are self-damping terms for the fundamental and first overtone modes respectively. All of the interaction between the two modes is subsumed in the coefficient $b_{01}$.

Equation (2) may be written in simpler form by setting

$$x = \lambda_0^2 \qquad y = \lambda_1^2$$

we then have

$$F = a_0 x + a_1 y + b_{00} x^2 + b_{11} y^2 + b_{01} xy \ . \tag{3}$$

The work integral (1) is defined such that for $F > 0$ energy is fed into the pulsation with the result that, in a crude sense, the amplitudes x and/or y must grow. Conversely, if $F < 0$, x and/or y ought to decline. If one considers an amplitude-amplitude diagram (y vs. x) it is convenient to think of F as a "force" which, when positive, pushes system points away from the origin and, when negative, tends to restore them.

In such a scheme the curve $F = 0$ represents an equilibrium locus. According to Eq. (3) such curves are conic sections, i.e., ellipses or hyperboli. Figure 1 gives some typical examples of the former while Figures 2 and 3 illustrate the latter. In all of these illustrations we have plotted y against x. Since both quantities are defined so as to be positive, only the first quadrant of our diagrams is physically meaningful.

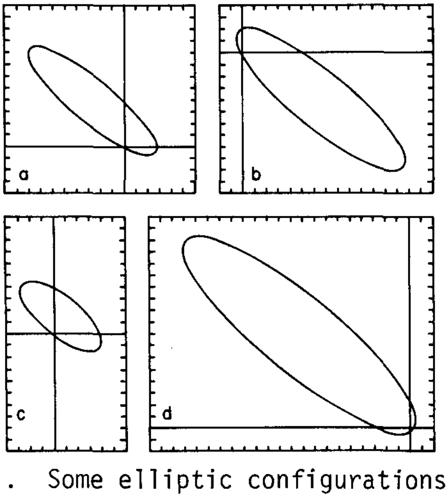


Figure 1. Some elliptic configurations for $F = 0$.

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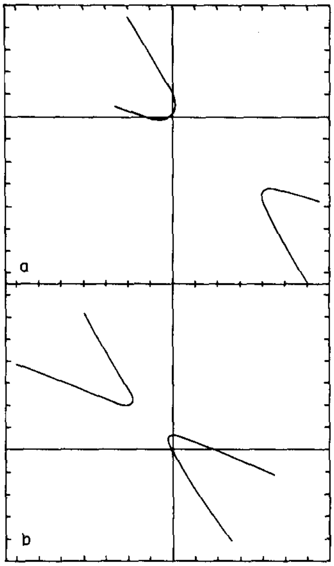


Figure 2. Some hyperbolic configurations for F = 0.

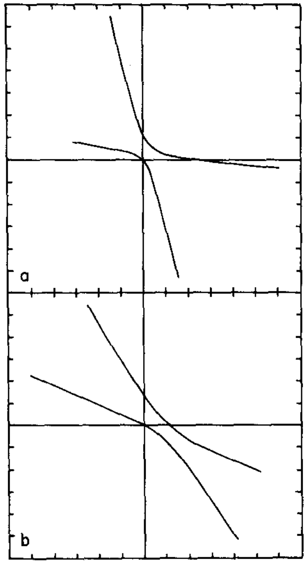


Figure 3. Further hyperbolic configurations for F = 0.

Consider Figure 1. Given the assumption of strictly negative b-coefficients, it is easy to show that F is positive everywhere inside the ellipses and negative everywhere outside. This means that system points must eventually gravitate to the F = 0 curve. The same holds true for the physically real portions of the hyperboli shown in Figs. 2 and 3.

However, because in the language of our scheme, the F = 0 locus is "force-free," a system point may seemingly move along the curve at will. Thus our analysis must be broadened in order to treat the stability of individual points on the F = 0 locus. In this regard it turns out to be instructive to discuss a special case — namely that which arises by setting $b_{01} = 0$ in Eq. (3). The corresponding F = 0 curve is an ellipse, illustrated in Figure 4. Because there is no interaction in this case, physical intuition easily predicts the outcome. Provided there is linear driving at both frequencies (i.e., $a_0$, $a_1 > 0$), each mode must separately attain its limiting amplitude, viz. $y = -a_1/b_{11}$, $x = -a_0/b_{00}$. The final state is thus a superposition of the individual oscillations, its (x,y) location indicated in Figure 4 by the intersection of two dashed lines.

With this result in mind we shall treat the general case by performing on Eq. (3) a rotation of coordinates into a new system, (x',y'). The primed coordinates are defined so that $b'_{01} = 0$. Such a transformation is always possible. In the new frame, we have

$$F' = F_{x'} + F_{y'} \quad , \tag{4}$$

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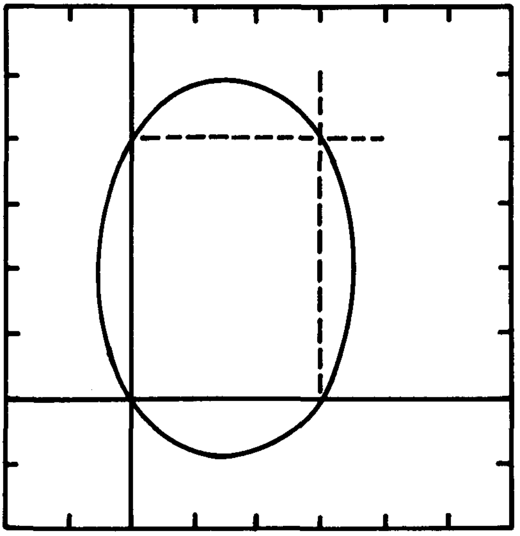

Figure 4. $F=0$ for the special case $b_{01}=0$.

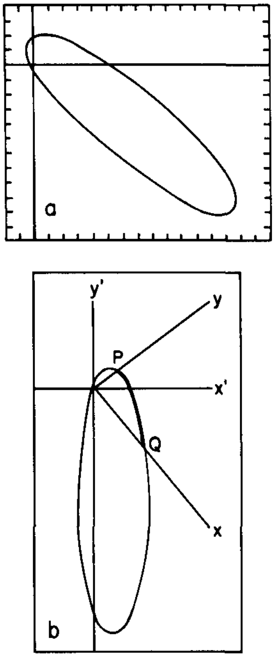


Figure 5. Rotation of an ellipse into the primed system.

where

$$F_{x'} = x'(a_0' + b_{00}'x'),\ F_{y'} = y'(a_1' + b_{11}'y') \quad . \tag{5}$$

From Eqs. (4) and (5) one sees that the terms $F_{x'}$ and $F_{y'}$ are uncoupled, the former depending only on x', the latter on y'.

In the spirit of our previous analysis, we shall now think of $F_{x'}$ and $F_{y'}$ as "orthogonal forces" operating in the primed frame. Since the new coordinates are no longer restricted to positive values, it becomes necessary to modify our description of how $F_{x'}$ and $F_{y'}$ act. This we shall do by analogy with elementary mechanics. Thus a positive $F_{x'}$ ($F_{y'}$) will tend to push the system away from the y'(x') axis, while a negative $F_{x'}(F_{y'})$ will restore the system toward the y'(x') axis.

With these definitions in hand, it becomes clear that, in the primed system, two necessary conditions for equilibrium are

$$F_{x'} = 0 \qquad \text{and} \qquad F_{y'} = 0 \quad . \tag{6}$$

Any point satisfying Eq. (6) must of course lie on the equilibrium locus $F=0$. However, the conditions (6) are not sufficient: one must also guarantee that points satisfying Eq. (6) are stable against small displacements.

Figure 5a repeats the ellipse already pictured in Figure 1b. In Figure 5b this ellipse is shown rotated into the primed coordinate system. The original axes are indicated. An inspection of this figure yields, after some reflection, the following requirements for the stability against small perturbations of points satisfying Eq. (6):

$$x' \frac{\partial F_{x'}}{\partial x'} < 0 \,, \qquad\qquad y' \frac{\partial F_{y'}}{\partial y'} < 0 \quad . \tag{7}$$

Eqs. (6) and (7) together constitute a necessary and sufficient condition for the existence of a point of stable double mode pulsation in the primed frame. It is not difficult to show that for the elliptic configuration there is always one and only one point satisfying (6) and (7), while in the hyperbolic case no point ever satisfies both these requirements. If the configuration is an ellipse a double-mode state will exist provided that the stable point in question lies on the physically real part of the curve (shown in Figure 5b as the heavy arc PQ). After considerable algebraic and trigonometric manipulation one may establish that this will be the case if and only if two simple parameters, U and V, satisfy the following conditions

$$U > 0 \,, \qquad V > 0 \,, \tag{8}$$

where

$$U \equiv b_{01}a_1 - 2b_{11}a_0 \tag{9}$$

$$V \equiv b_{01}a_0 - 2b_{00}a_1 \quad . \tag{10}$$

If Eq. (8) is not satisfied no stable points exist along the physically real portion of the curve $F = 0$. However, either or both of the endpoints P and Q may still be stable. To see this one notes that, in primed space, the x and y axes form impenetrable boundaries through which no system point may pass. Thus if the $F_{x'}$ and $F_{y'}$ forces are such as to jam the system up against these barriers at P and/or Q, either or both of these points may be stable. A further examination of Figure 5b shows that this will happen provided that

$$F_{x'} < 0 \qquad \text{and} \qquad F_{y'} > 0 \tag{11}$$

at the points in question.

Manipulating the conditions (11) one may demonstrate that they can be expressed in terms of the same two parameters, U and V, given by Eqs. (9) and (10). The table which follows summarizes all possible outcomes for the system governed by Eq. (3):

| Final state | U | V | Configuration* |
|---|---|---|---|
| x only | + | - | E or H |
| y only | - | + | E or H |
| x or y | - | - | H |
| x and y | + | + | E |

*E = ellipse; H = hyperbola

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We note that the final states given above are equivalent to those found by Stellingwerf (1975) and Simon, Cox and Hodson (1980). Since the signs of U and V determine limiting amplitude behavior, the curves U = 0 and V = 0, plotted on, say, the H-R diagram, will form modal selection boundaries. Unfortunately, space limitations preclude further treatment of this question in the present contribution. A continuation of the discussion, as well as details of the calculations we have outlined, will be provided shortly in another place.

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## DISCUSSION

J. COX: This is very interesting. Can you predict where the double mode pulsators should be?

SIMON: According to my analysis the two transition lines on the H-R diagram must cross.

J. COX: If I remember correctly, the intersection of the first overtone and fundamental blue edges is kind of bright and you need to go down from that somewhat.

SIMON: You can get some hint from the periods at which double mode pulsators appear, but I don't know, can you get transition lines to cross?

A. COX: Who says those lines have to meet at the intersection of the blue edges?

SIMON: Stellingwerf is the only one who actually said that. According to my theory, they also have to meet at the intersection of the blue edges.

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