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ON DEGENERATE FULLY NONLINEAR ELLIPTIC EQUATIONS IN BALLS

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We establish derivative estimates and existence theorems for the Dirichlet and Neumann problems for nonlinear, degenerate elliptic equations of the form $F(D^2u) = g$ in balls. The degeneracy arises through the possible vanishing of the function g and the degenerate Monge-Ampère equation is covered as a special case.

This note concerns boundary value problems of Dirichlet and Neumann type for degenerate nonlinear elliptic equations of the form.

$$F(D^2u) = g \;, \tag{1}$$

where the function F is similar to those treated by Caffarelli, Nirenberg and Spruck [2]. The degeneracy arises through the vanishing of the non-negative function g. Here we shall restrict attention to balls in Euclidean n space, $\mathbb{R}^n$, making use of an argument which originated in [10], but hopefully this special case will shed some light on the general situation. The results here also improve previous work on the elliptic Monge-Ampère equation,

$$\det D^2u = g \;, \tag{2}$$

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but again, we stress, only in balls. The Dirichlet problem for (2) has received considerable attention in recent years ([1], [4], [5], [6], [8], [11]). The Neumann problem was recently treated by Lions, Trudinger and Urbas [10].

To formulate appropriate conditions on F, we let $\mathcal{S}^n$ denote the linear space of $n \times n$ real symmetric matrices and suppose that F is defined on an open convex cone $\Gamma \subset \mathcal{S}^n$, with vertex at the origin and containing the cone of positive matrices. We then assume $F \in C^0(\overline{\Gamma}) \cap C^2(\Gamma)$ satisfies:

(i) F is positive on Γ and vanishes on $\partial\Gamma$,

(ii) F is elliptic on Γ which means F' is positive there,

(iii) F is concave on Γ,

(iv) $F(r) \to \infty$ as $r \to \infty$, (that is, as the minimum eigenvalue $\to \infty$).

Further hypotheses will be invoked as needed. For the Monge-Ampere equation, we note that these hypotheses are fulfilled when Γ is the cone of positive matrices and

$$F(r) = (\det r)^{1/n} . \tag{3}$$

In conjunction with equation (1) in the unit ball B in Euclidean n space $\mathbb{R}^n$, we consider Dirichlet boundary conditions,

$$u = \phi \quad \text{on} \quad \partial B , \tag{4}$$

and Neumann boundary conditions of the form

$$\frac{\partial u}{\partial r} + \gamma u = \psi \quad \text{on} \quad \partial B , \tag{5}$$

where ϕ, γ, ψ are functions defined on ∂B. Through the method of continuity, the solvability of these problems for sufficiently smooth g, ϕ, γ, ψ depends upon *a priori* estimates of solutions and their first and second derivatives which we now establish. Throughout we shall assume that a solution u of equation (1) is *admissible* in the sense that $D^2u \subset \overline{\Gamma}$.

Solution estimates

For these we may consider any sufficiently smooth domain Ω in $\mathbb{R}^n$ and a boundary condition of the form,

$$\beta \cdot Du + \gamma u = \psi \quad \text{on} \quad \partial\Omega \tag{6}$$

where $\beta \cdot \nu \geq 0$, $\gamma \geq \gamma_0$, ν denotes the unit outer normal to $\partial\Omega$ and γ_0 is a positive constant. Taking $0 \in \Omega$ and setting $w(x) = A|x|^2$ we can by virtue of (iv) make $F(D^2 w)$ arbitrarily large by taking the constant A sufficiently large. Fixing A so that

$$F(D^2 w) \geq \sup_\Omega g ,$$

it then follows from the comparison principle that

$$\frac{1}{\gamma_0}(\inf_{\partial\Omega} \psi - 2A \sup_{\partial\Omega}|\beta| \operatorname{diam} \Omega) \leq u \leq \frac{1}{\gamma_0} \sup_{\partial\Omega} \phi . \tag{7}$$

First derivative estimates

For the Dirichlet problem, these are elementary and essentially covered by [2]. Differentiating (1), we obtain for $k = 1, \ldots, n$,

$$F^{ij}(D^2 u) D_{ijk} u = D_k g \tag{8}$$

where $[F^{ij}] = F'$. Now, using the concavity of F, we have

$$F(D^2 w) - F(D^2 u) \leq F^{ij}(D^2 u) D_{ij}(w-u)$$

so that

$$\begin{aligned} F^{ij}(D^2 u) D_{ij} w &\geq F(D^2 w) - F(D^2 u) + F^{ij}(D^2 u) D_{ij} u \\ &\geq F(D^2 w) - g \end{aligned} \tag{9}$$

by virtue again of the concavity of F and equation (1). With the constant A again chosen sufficiently large, we obtain for any domain Ω,

$$\sup_\Omega |Du| \leq \sup_{\partial\Omega} |Du| + C|g|_{1;\Omega} . \tag{10}$$

But then $\sup_{\partial B} |Du|$ is readily bounded with the aid of barrier functions of the form

(11) $$w^{\pm} = \pm A(1 - |x|^2) + \phi \ .$$

As a result we obtain,

(12) $$\sup_B |Du| \le C(|g|_1 + |\phi|_2)$$

where C depends on n and trace $F'(I)$. A first derivative estimate for the Neumann problem (5) will be deduced by interpolation of the second derivative estimate.

Second derivative estimates

We set

(13) $$v = \Delta u - x_i x_j D_{ij} u$$

noting that on ∂B,

(14) $$v = \Delta_{\partial B} u + (n-1) \frac{\partial u}{\partial r} ,$$

where $\Delta_{\partial B}$ is the Laplace Beltrami operator in S^{n-1}. By computation, as in [10], we have

$$F^{ij} D_{ij} v = \Delta g - x_i x_j D_{ij} g - F^{ij,pq} (\delta_{k\ell} - x_k x_\ell) D_{ijk} u D_{pq\ell} u$$
$$- 2F^{ij} D_i (x_k x_\ell) D_{jk\ell} u - F^{ij} D_{ij} (x_k x_\ell) D_{k\ell} u$$

where

$$F^{ij,pq}(a) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{pq}}$$

for any $a = [a_{ij}] \in \mathcal{S}^n$. Consequently using the concavity of F and (8),

(15) $$F^{ij} D_{ij} v \ge \Delta g - x_i x_j D_{ij} g - 4x_i D_i g - 2F^{ij} D_{ij} u$$
$$\ge \Delta g - x_i x_j D_{ij} g - 4r \frac{\partial g}{\partial r} - 2g \ .$$

Hence if the Dirichlet boundary condition (4) is satisfied, we get using (9), (11)

$$\sup_B v \le \sup_{\partial B} v + C|g|_{2;B} \tag{16}$$
$$\le C(|\phi|_{2;\partial B} + |g|_{2;B})$$

where C depends only on n and trace $F'(I)$. Now if we impose a further condition on Γ, namely

$$a \le (\text{trace } a)I \tag{17}$$

for any $a \in \Gamma$, we deduce from (16) the *interior* estimate

$$|D^2u(x)| \le \frac{C}{1-|x|}(|\phi|_{2;\partial B} + |g|_{2;B}) \tag{18}$$

where C depends only on n. Note that the dependence on trace $F'(I)$ from (12) disappears since (17) implies trace $a \ge 0$, whence u is subharmonic. For the Monge Ampère equation (2), the estimate (18) extends the interior second derivative estimates of Pogorelov [11] and Trudinger and Urbas [13] to non-negative inhomogeneous terms g (but only in balls!).

Next suppose that the Neumann boundary condition (5) is satisfied. By computation we then obtain on ∂B,

$$\frac{\partial v}{\partial r} = (\delta_{ij} - x_i x_j)D_{ij}\left(\frac{\partial u}{\partial r}\right) - 2\Delta u \tag{19}$$
$$= \Delta_{\partial B}\left(\frac{\partial u}{\partial r}\right) - 2\Delta u + (n-1)\frac{\partial^2 u}{\partial r^2},$$
$$= \Delta_{\partial B}(\psi - \gamma u) - 2\Delta u + (n-1)\frac{\partial^2 u}{\partial r^2},$$

so that combining (15) and (19), we get an estimate

$$v \le C\left[|g|_{2;B} + |\psi|_{2;B} + |u|_{1;B} + \sup_{\partial B}\left|\frac{\partial^2 u}{\partial r^2}\right|\right] \tag{20}$$

where C depends on n and $|\gamma|_{2;B}$. To estimate $\frac{\partial^2 u}{\partial r^2}$, we now compute

$$F^{ij}D_{ij}(x_kD_ku - \gamma u) = 2F^{ij}D_{ij}u + x_kD_kg - F^{ij}D_{ij}(\gamma u) \tag{21}$$

so that with barrier considerations similar to (11), we obtain

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(22) $$\sup_{\partial B} \left|\frac{\partial^2 u}{\partial r^2}\right| \le C\left[|g|_{1;B} + |\psi|_{2;B} + |u|_{1;B}\right]$$

and hence conbining (20) and (22), we have

(23) $$\sup_{\partial B} \Delta u \le C\left[|g|_{1;B} + |\psi|_{2;B} + |u|_{1;B}\right] .$$

But since

(24) $$F^{ij} D_{ij} \Delta u \ge \Delta g \quad \text{in} \quad B ,$$

we can infer a global bound

(25) $$\sup_{B} \Delta u \le C\left[|g|_{1;B} + |\psi|_{2;B} + |u|_{1;B}\right] .$$

Using (25) and the concavity of F, we then obtain the full second derivative bound,

(26) $$\sup_{B} |D^2 u| \le C\left[|g|_{2;B} + |\psi|_{2;\partial B} + |u|_{1;B}\right]$$

where C depends on n, $|\gamma|_{2;\partial B}$ and $F'(I)$, and finally by interpolation,

(27) $$|u|_{2;B} \le C\left[|g|_{2;B} + |\psi|_{2;\partial B}\right] .$$

Existence theorems.

The second derivative estimates (18) and (27) can now be used to solve the corresponding boundary value problems through approximation by nondegenerate problems. In the case of the Dirichlet problem we replace the function g by $g+\epsilon$ for positive ϵ and approximate the boundary value ϕ by $C^4(\partial B)$ functions. If the functions F are further restricted to fulfill all the conditions of Caffarelli, Nirenberg and Spruck [2], the approximating Dirichlet problems can be solved classically. In particular for their prime examples, the elementary symmetric functions of the eigenvalues of $D^2 u$,

(28) $$F_k(D^2 u) = \left\{\sum_{i_1<\dots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}\right\}^{1/k} , \quad k=1,\dots,n ,$$

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the cone Γ is the component of the set where F_k is positive which contains the positive cone. We thus conclude from the estimates (7), (12), (18) and [2, Theorem 3]:

THEOREM 1. *Let* $g \geq 0$, $\in C^{1,1}(\bar{B})$, $\phi \in C^{1,1}(\partial B)$. *Then there exists a unique solution* $u \in C^{0,1}(\bar{B}) \cap C^{1,1}(B)$ *with* $D^2u \subset \bar{\Gamma}$ *of the Dirichlet problems,*

$$F_k(D^2u) = g \quad in \quad B\,, \quad u = \phi \quad on \quad \partial B\,. \tag{29}$$

Observe that we can in particular allow g to vanish everywhere in B. In the Monge-Ampère case, this was proved in [13] while the case when $\phi = 0$, $g \not\equiv 0$ was established by Chen [3].

For Neumann boundary conditions (5) the results are somewhat stronger as we need only assume F satisfies conditions (i) to (iv). Here we again replace g by $g + \epsilon$ for positive ϵ so the approximating equations become uniformly elliptic with respect to solutions with bounded second derivatives. Invoking the second derivative Hölder estimates of Lions and Trudinger [9], (see also [7], [12]) we can then establish the classical solvability of the approximating boundary value problems. Thus we obtain the following.

THEOREM 2. *Let* F *satisfy (i) to (iv) with respect to some cone* Γ, $g \geq 0$, $\in C^{1,1}(\bar{B})$, $\gamma, \psi \in C^{1,1}(\partial B)$ *with* $\gamma \geq \gamma_0$ *for some positive constant* γ_0. *Then there exists a unique solution* $u \in C^{1,1}(\bar{B})$ *with* $D^2u \subset \bar{\Gamma}$ *of the boundary value problem,*

$$F(D^2u) = g \quad in \quad B\,, \quad \frac{\partial u}{\partial r} + \gamma u = \psi \quad on \quad \partial B\,. \tag{30}$$

It is plausible to conjecture that Theorem 1 and 2 extend at least from balls to sufficiently smooth uniformly convex domains but even in the Monge Ampère case, this is an open question.

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