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QUANTITATIVE OSCILLATION ESTIMATES FOR ALMOST-UMBILICAL CLOSED HYPERSURFACES IN EUCLIDEAN SPACE

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Abstract

We prove ϵ-closeness of hypersurfaces to a sphere in Euclidean space under the assumption that the traceless second fundamental form is δ-small compared to the mean curvature. We give the explicit dependence of δ on ϵ within the class of uniformly convex hypersurfaces with bounded volume.

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1. Introduction

In this paper we investigate the potential of the traceless second fundamental form, also called the *umbilicity tensor*,

$$\mathring{A} = A - \frac{\operatorname{tr}(A)}{n} g,$$

of a hypersurface embedded in the Euclidean space to pinch other geometric quantities of the hypersurface. Questions like this arise from the well-known fact that $\mathring{A} = 0$ implies that the hypersurface must be a sphere. It is natural to ask if this behaviour is continuous, in the sense that a small traceless second fundamental form implies closeness to a sphere. During the last decade, substantial progress has been made towards a better understanding of this question. In 2005, Camillo De Lellis and Stefan Müller [7] proved the estimate

$$\inf_{\lambda \in \mathbb{R}} \|A - \lambda g\|_{L^2(M)} \le C \|\mathring{A}\|_{L^2(M)}$$

for hypersurfaces $M \subset \mathbb{R}^3$. From this, the authors deduced $W^{2,2}$-closeness to a sphere. One year later, in [8], the authors made a step towards uniform closeness and showed that in addition the metric is C^0-close to the standard sphere metric. In 2011, one

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of De Lellis's PhD students, Daniel Perez, proved in the class of hypersurfaces with volume 1 and bounded second fundamental form, that for given $\epsilon > 0$ there exists $\delta > 0$, such that a δ-small traceless second fundamental form yields ϵ-closeness to a sphere (compare [11, Corollary 1.2]). He used an argument via contradiction and it does not seem possible to extract the ϵ-dependence of δ from his proof. In [11, page xvi] the author posed the derivation of a quantitative dependence as an open problem. In this paper we tackle this problem and prove the following theorem.

THEOREM 1.1. *Let $n \geq 2$ and $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be the smooth, isometric embedding of a closed, connected, orientable and strictly mean-convex hypersurface. Let $0 < \alpha < 1$. Then there exists $c > 0$, such that whenever we have $\epsilon < c|M|^{1/n}$ and the pointwise estimate*

$$\|\mathring{A}\| \leq H|M|^{-(2+\alpha)/n}\epsilon^{2+\alpha} \tag{1.1}$$

holds, then M is strictly convex and

$$M \subset B_{\sqrt{(n/\lambda_1(M))}+\epsilon}(x_0)\backslash B_{\sqrt{(n/\lambda_1(M))}-\epsilon}(x_0).$$

The constant c depends on n, α, $\|\tilde{A}\|_\infty$ and $\|\tilde{A}^{-1}\|_\infty$, where $|M| = \mathrm{vol}(M)$, $\tilde{A} = |M|^{1/n}A$, $\lambda_1(M)$ is the first nonzero eigenvalue of the Laplace–Beltrami operator on M and x_0 is the centre of mass of M.

Thus in the class of uniformly convex hypersurfaces of unit volume we obtain ϵ-closeness to a sphere, if $\mathring{A}$ is of order $\epsilon^{2+\alpha}$ and ϵ is sufficiently small. A more detailed description of the notation involved here is presented in Section 2.

Note that a similar result by Roth has recently appeared in [13]. In more general ambient spaces, he proves quasi-isometry of hypersurfaces to the sphere under certain assumptions, including smallness of the gradient of the second fundamental form.

The author's motivation to find a quantitative dependence like this arose from his work on inverse curvature flows in the Euclidean space. In [15, Appendix A] Oliver Schnürer derived a pinching estimate of the traceless second fundamental form for hypersurfaces evolving by the inverse Gauss curvature flow in $\mathbb{R}^3$. Ben Andrews applied estimates like this to bound the difference between circumradius r_+ and inradius r_- of the surface in [1, Section 4]. However, we are not aware whether those methods may be transferred to higher dimensions. Clearly, Theorem 1.1 provides an estimate of $r_+ - r_-$ in terms of $\mathring{A}$. Indeed, we will apply this estimate to prove asymptotical roundness of hypersurfaces solving an inverse curvature flow equation in $\mathbb{R}^{n+1}$ (cf. [14]).

Let us give an overview of the main ingredients in the proof. We need a result which somehow yields the transition from *qualitative* to *quantitative*. We found the following result due to Julien Roth. We formulate a special case and only the statements which are of interest to our proof.

THEOREM 1.2 [12, Theorem 1]. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed in $\mathbb{R}^{n+1}$. Assume that*

$|M| = 1$ and $H_2 > 0$. Then for any $p \geq 2$ and $\epsilon > 0$ there exists a constant $C_\epsilon = C_\epsilon(n, \|H\|_\infty, \|H_2\|_{2p})$, such that if

$$\lambda_1(M)\Big(\int_M H\Big)^2 - n\|H_2\|_{2p}^2 > -C_\epsilon \tag{1.2}$$

is satisfied, then

$$M \subset B_{\sqrt{(n/\lambda_1)}+\epsilon}(x_0)\backslash B_{\sqrt{(n/\lambda_1)}-\epsilon}(x_0),$$

where x_0 is the centre of mass of M and H_2 is the second normalised elementary symmetric polynomial.

This theorem is a generalisation of [6] to higher kth mean curvatures. There are also generalisations to ambient spaces of bounded sectional curvature (cf. [10]). At first glance, it does not seem to be a quantitative result, but a rather tedious scanning of the proof shows that C_ϵ can be chosen to be of order ϵ^2: compare Section 3 below.

Certainly, this ϵ^2 gives insight into the question of where the order $\epsilon^{2+\alpha}$ comes from in Theorem 1.1. It is an interesting question whether, and if so how, this could be improved.

Thus we have to derive (1.2) from (1.1). Firstly, we need to relate the first eigenvalue of the Laplacian to the traceless second fundamental form. This transition has another stop at the Ricci tensor. The following result, due to Erwann Aubry, relates the Ricci tensor to λ_1. It was proven in [3], but is accessible more easily in [4, Theorem 1.6]. Again, we only cite the aspects which are relevant to our work.

THEOREM 1.3 [4, Theorem 1.6]. *For any $p > n/2$ there exists $C(n, p)$ such that, if M^n is a complete manifold with*

$$\int_M (\underline{\mathrm{Ric}} - (n-1))_-^p < \frac{|M|}{C(n, p)}, \tag{1.3}$$

then M is compact and satisfies

$$\lambda_1(M) \geq n\Big(1 - C\Big(\frac{1}{|M|}\int_M (\underline{\mathrm{Ric}} - (n-1))_-^p\Big)^{1/p}\Big).$$

Here, $\underline{\mathrm{Ric}} = \underline{\mathrm{Ric}}(x)$ denotes the smallest eigenvalue of the Ricci tensor at $x \in M$ and for $y \in \mathbb{R}$ we set $y_- = \max(0, -y)$.

The other quantities in (1.2) can be controlled with the help of (1.1) quite easily. Thus the only ingredient left is to control the Ricci tensor in (1.3). The following result, due to Perez [11], and also to De Lellis and Müller [7] for $n = 2$, is helpful.

THEOREM 1.4 [11, Theorem 1.1]. *Let $n \geq 2$, $p \in (n, \infty)$ and $c_0 > 0$. Then there exists $C(n, p, c_0) > 0$ such that, for any smooth, closed and connected hypersurface $M \subset \mathbb{R}^{n+1}$ with*

$$|M| = 1$$

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and

$$\|A\|_p \le c_0,$$

we have

$$\min_{\mu\in\mathbb{R}} \|A - \mu g\|_p \le C\|\mathring{A}\|_p. \tag{1.4}$$

This result will enable us to move, via the Ricci tensor, to an estimate of λ_1 and to finally provide the estimate (1.2). Then the result follows. The largest technical difficulty is that we need L^∞ bounds, where Theorems 1.3 and 1.4 only make statements on L^p norms. We show how to handle this in Section 4.

Note that we will not need to know the explicit value of μ_0 in (1.4), where the minimum is attained. However, this is another interesting question with some history. According to [11, page 50], Gerhard Huisken suggested an inverse mean curvature flow approach to prove that the minimum is attained at

$$\mu = \frac{1}{|M|}\int_M H.$$

In [11, page 52, Ch. 3.4], this is proven for $n \ge 2$, $p = 2$ and for closed convex hypersurfaces. Unfortunately, the case $p = 2$ is not enough in our case. Hence, we have to deal with the small technical difficulty that μ_0 is not explicitly known.

We wish to mention as well that there is literature on spherical closeness in terms of lower bounds on the principal curvatures (cf. [5]). This is a somewhat different issue since we want to provide arbitrary closeness.

Our detailed analysis of the problem at hand begins with an explanation of our notation.

2. Notation and preliminaries

In this paper we consider closed embedded hypersurfaces $M^n \subset \mathbb{R}^{n+1}$. We follow the notation as it appears in the references as closely as possible.

$g = (g_{ij})$ denotes the induced metric of M^n, $A = (h_{ij})$ the second fundamental form and κ_i, $i = 1, \ldots, n$, the principal curvatures ordered pointwise,

$$\kappa_1 \le \cdots \le \kappa_n.$$

The volume of M is

$$|M| = \int_M 1\, d\mu,$$

where μ is the canonical surface measure associated to g.

$\lambda_1(M)$ denotes the first nonzero eigenvalue of $-\Delta$, where Δ is the Laplace–Beltrami operator on (M, g).

For $k = 1, \ldots, n$ we define

$$H_k = \binom{n}{k}^{-1} \sum_{1\le i_1<\cdots<i_k\le n} \kappa_{i_1}\cdots\kappa_{i_k}.$$

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This includes the definition of the mean curvature,

$$H = \frac{1}{n}\sum_{i=1}^{n}\kappa_i,$$

which deviates from some of the references. It corresponds to the notation in [12]. Thus the traceless second fundamental form is

$$\mathring{A} = A - Hg.$$

For smooth tensor fields on M, $T = (t^{i_1\ldots i_k}_{j_1\ldots j_l})$, we define the pointwise norms to be

$$\|T\| = \sqrt{t^{i_1\ldots i_k}_{j_1\ldots j_l}\, t^{j_1\ldots j_l}_{i_1\ldots i_k}},$$

where indices are lowered or lifted with respect to the induced metric of the hypersurface the tensor field is defined on. With the help of this definition we may define L^p norms on a subset $\Omega \subset M$ to be

$$\|T\|_{p,\Omega} = \left(\int_\Omega \|T\|^p\right)^{1/p},$$

where the surface measure to be used is implicitly included in the set of integration Ω. Analogously we set

$$\|T\|_{\infty,\Omega} = \sup_\Omega \|T\|.$$

The tensor $\mathrm{Ric} = (R_{ij})$ is the Ricci tensor and $R = \mathrm{tr}(\mathrm{Ric}) = R^i_i$ the scalar curvature. $\underline{\mathrm{Ric}}(x)$ denotes the smallest eigenvalue of the Ricci tensor at $x \in M$.

For M^n the symbol $\tilde{M}^n$ always denotes the normalised manifold

$$\tilde{M} = |M|^{-1/n} M \hookrightarrow \mathbb{R}^{n+1}$$

with $|\tilde{M}| = 1$. The corresponding rescaled geometric quantities are denoted with a tilde as well, for example,

$$\tilde{g} = (\tilde{g}_{ij}), \quad \tilde{A} = (\tilde{h}_{ij}).$$

Finally,

$$B_r(x_0) \subset \mathbb{R}^{n+1}$$

denotes an $(n+1)$-dimensional ball in $\mathbb{R}^{n+1}$ with radius r and centre x_0.

3. Qualitative closeness revisited

In this section we turn our attention to Theorem 1.2 which connects λ_1 with closeness to a sphere. We state how the constant C_ϵ involved here depends on ϵ, whereafter we indicate how this can be deduced from the corresponding sequence of lemmas in [12]. We prove the following proposition.

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Proposition 3.1. *In the situation of Theorem 1.2, let* $0 < \epsilon < 2/(3\|H\|_\infty)$. *If (1.2) holds for*

$$C_\epsilon = \frac{1}{2}\min\Big(L\sqrt{\frac{n}{\lambda_1(M)}}\epsilon^2, L\Big),$$

where L is bounded and uniformly positive whenever $\|H\|_\infty$ *and* $\|H_2\|_{2p}$ *range in compact subsets of* $(0,\infty)$, *then we have*

$$M \subset B_{\sqrt{(n/\lambda_1(M))}+\epsilon}(x_0)\backslash B_{\sqrt{n/\lambda_1(M))}-\epsilon}(x_0).$$

Proof. We will spot and note the relevant formulae in [12], always showing how they depend on the geometric quantities and on ϵ. There is a sequence of constants from which we arrive at C_ϵ. We start with [12, page 297, Lemma 2.1]. First of all, it is required that

$$C_\epsilon < \frac{n}{2}\|H_2\|_{2p}^2.$$

Equation (5) in [12] yields

$$A_1 = \frac{2\|H\|_\infty^2}{\|H_2\|_{2p}^2}.$$

Then [12, page 298, Lemma 2.2] yields

$$A_2 = \frac{A_1}{n\|H_2\|_{2p}^2}.$$

The proofs of [12, Lemmas 2.4 and 2.5] imply that A_3 and A_4 are of a similar form. Finally, the author cites a lemma implying an L^∞-estimate on the function

$$\varphi = |X|\Big(|X| - \sqrt{\frac{n}{\lambda_1(M)}}\Big)^2,$$

where X is the position vector field with respect to the centre of mass, x_0, of M. The lemma is (cf. [12, Lemma 3.1]):

For $p \geq 2$ *and any* $\eta > 0$, *there exists* $K_\eta(n, \|H\|_\infty, \|H_2\|_{2p})$ *such that, if (1.2) holds with* $C_\epsilon = K_\eta$, *then* $\|\varphi\|_\infty \leq \eta$.

Essentially, the proof of this lemma is given in [6, page 188, proof of Lemma 3.1] (also compare [12, Section 6]). Here one sees that this K_η can be chosen to be

$$K_\eta = \min\Big(\frac{\eta}{(L'A_4)^4}, c_n\Big) > 0,$$

where L' is just of the same form as A_4. Now, in [12, page 301] the author defines

$$\begin{aligned}\eta(\epsilon) &= \min\Big(\Big(\sqrt{\frac{n}{\lambda_1(M)}} - \epsilon\Big)\epsilon^2, \frac{1}{27\|H\|_\infty^3}\Big)\\ &\geq \min\Big(\frac{1}{3}\sqrt{\frac{n}{\lambda_1(M)}}\epsilon^2, \frac{1}{27\|H\|_\infty^3}\Big),\end{aligned}$$

since $\epsilon < 2/(3\|H\|_\infty)$ and

$$\lambda_1(M) \le \frac{1}{n-1}\|R\|_\infty \le n\|H\|_\infty^2$$

(compare [9, Theorem 3.1]). He concludes that

$$M \subset B_{\sqrt{(n/\lambda_1(M))}+\epsilon}(x_0)\backslash B_{\sqrt{(n/\lambda_1(M))}-\epsilon}(x_0)$$

under assumption (1.2) with

$$C_\epsilon = \frac{1}{2}\min\Big(\frac{n}{2}\|H_2\|_{2p}^2, c_n, \frac{1}{3(L'A_4)^4}\sqrt{\frac{n}{\lambda_1(M)}}\epsilon^2, \frac{1}{27(L'A_4)^4\|H\|_\infty^3}\Big),$$

which has the form claimed in the proposition. □

4. Quantitative spherical closeness

Now we come to the proof of the main result. We state it again for convenience.

THEOREM 4.1. *Let $n \ge 2$ and $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be the smooth, isometric embedding of a closed, connected, orientable and mean-convex hypersurface. Let $0 < \alpha < 1$. Then there exists $c > 0$, such that whenever we have $\epsilon < c|M|^{1/n}$ and the pointwise estimate*

$$\|\mathring{A}\| \le H|M|^{-(2+\alpha)/n}\epsilon^{2+\alpha} \tag{4.1}$$

holds, then M is strictly convex and

$$X(M) \subset B_{\sqrt{(n/\lambda_1(M))}+\epsilon}(x_0)\backslash B_{\sqrt{(n/\lambda_1(M))}-\epsilon}(x_0).$$

The constant c depends on n, α, $\|\tilde{A}\|_\infty$ and $\|\tilde{A}^{-1}\|_\infty$, where $|M| = \mathrm{vol}(M)$, $\tilde{A} = |M|^{1/n}A$, $\lambda_1(M)$ is the first nonzero eigenvalue of the Laplace–Beltrami operator on M and x_0 is the centre of mass of M.

PROOF. In this proof, $\tilde{C}_i$, $i \in \mathbb{N}$, always denote generic constants which depend at most on n, α, $\|\tilde{A}\|_\infty$ and $\|\tilde{A}^{-1}\|_\infty$. Set

$$p = n + 1$$

and let

$$k = \frac{6}{\alpha}.$$

For the rescaled surfaces

$$\tilde{M} = |M|^{-1/n}M,$$

we find from Theorem 1.4 that

$$\|\tilde{A} - \mu_0\tilde{g}\|_{kp} \le \tilde{C}_1\|\mathring{\tilde{A}}\|_{kp}, \tag{4.2}$$

where $\mu_0 = \mu_0(n, \alpha, \|\tilde{A}\|_\infty, \|\tilde{A}^{-1}\|_\infty)$.

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The first condition we put on the constant c is to satisfy

$$c < \left(\frac{1}{\sqrt{n(n-1)}}\right)^{1/(2+\alpha)}.$$

Then (4.1) yields the strict convexity of $\tilde{M}$, due to [2, Lemma 2.2]. μ_0 is strictly positive, since obviously we have

$$\inf_{\tilde{M}} \tilde{\kappa}_1 \leq \mu_0 \leq \sup_{\tilde{M}} \tilde{\kappa}_n.$$

Define

$$\hat{M} = \mu_0 \tilde{M}.$$

Then

$$\|\hat{A} - \hat{g}\|_{kp} = \left(\int_{\hat{M}} \mu_0^{-kp} \|\tilde{A} - \mu_0 \tilde{g}\|^{kp}\right)^{1/kp} = \mu_0^{(n/kp)-1} \|\tilde{A} - \mu_0 \tilde{g}\|_{kp}.$$

Define the set

$$\hat{P} = \{\hat{x} \in \hat{M} : \|\hat{A}(\hat{x}) - \hat{g}(\hat{x})\| < 1\}.$$

Its complement has volume

$$|\hat{P}^c| \leq \int_{\hat{P}^c} \|\hat{A} - \hat{g}\|^{kp} \leq \mu_0^{n-kp} \|\tilde{A} - \mu_0 \tilde{g}\|_{kp}^{kp}.$$

In order to apply Theorem 1.3, we need an estimate on the Ricci tensor $\widehat{\mathrm{Ric}} = (\hat{R}_{ij})$. By the Gaussian formula,

$$\hat{R}_{ij} = n\hat{H}\hat{h}_{ij} - \hat{h}_{ik}\hat{h}_j^k.$$

Let $\hat{x} \in \hat{P}$ and $\xi \in T_{\hat{x}}\hat{M}$. Then

$$\begin{aligned} \hat{R}_{ij}\xi^i\xi^j &= n\hat{H}\hat{h}_{ij}\xi^i\xi^j - \hat{h}_{ik}\hat{h}_j^k\xi^i\xi^j \\ &= n(\hat{H} - 1)(\hat{h}_{ij} - \hat{g}_{ij})\xi^i\xi^j + n(\hat{h}_{ij} - \hat{g}_{ij})\xi^i\xi^j \\ &\quad + n(\hat{H} - 1)\|\xi\|^2 + (n-1)\|\xi\|^2 - 2(\hat{h}_{ij} - \hat{g}_{ij})\xi^i\xi^j \\ &\quad - (\hat{h}_{ik} - \hat{g}_{ik})(\hat{h}_j^k - \delta_j^k)\xi^i\xi^j, \end{aligned} \tag{4.3}$$

from which we obtain at $\hat{x}$,

$$\|\widehat{\mathrm{Ric}} - (n-1)\hat{g}\| \leq \tilde{C}_2 \|\hat{A} - \hat{g}\|,$$

since $\|\hat{A} - \hat{g}\| < 1$. In the notation of Theorem 1.3 we obtain

$$\begin{aligned} \int_{\hat{M}} (\underline{\widehat{\mathrm{Ric}}} - (n-1))_-^{kp} &\leq \int_{\hat{P}} \tilde{C}_2^{kp} \|\hat{A} - \hat{g}\|^{kp} + \int_{\hat{P}^c} (\underline{\widehat{\mathrm{Ric}}} - (n-1))_-^{kp} \\ &\leq (\tilde{C}_2^{kp}\mu_0^{n-kp} + (n-1)^{kp}\mu_0^{n-kp}) \|\tilde{A} - \mu_0\tilde{g}\|_{kp}^{kp} \\ &= \tilde{C}_3 \|\tilde{A} - \mu_0 \tilde{g}\|_{kp}^{kp}. \end{aligned}$$

Thus Theorem 1.3 will be applicable under condition (4.1) if we choose c small enough to ensure the last of the following inequalities (note that in the first inequality

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we use (4.2)):

$$\begin{aligned}
\tilde{C}_3\|\tilde{A}-\mu_0\tilde{g}\|_{kp}^{kp} &\le \tilde{C}_3\tilde{C}_1^{kp}\|\mathring{\tilde{A}}\|_{kp}^{kp} = \tilde{C}_3\tilde{C}_1^{kp}|M|^{kp/n-1}\|\mathring{A}\|_{kp}^{kp}\\
&\le \tilde{C}_3\tilde{C}_1^{kp}|M|^{-((1+\alpha)kp+n)/n}\epsilon^{(2+\alpha)kp}\|H\|_{kp}^{kp}\\
&= \tilde{C}_3\tilde{C}_1^{kp}|M|^{-((2+\alpha)kp)/n}\epsilon^{(2+\alpha)kp}\|\tilde{H}\|_{kp}^{kp}\\
&< \tilde{C}_3\tilde{C}_1^{kp}c^{(2+\alpha)kp}\|\tilde{H}\|_{kp}^{kp}\\
&\overset{!}{<} \frac{|\hat{M}|}{C(n,kp)} = \frac{\mu_0^n}{C(n,kp)},
\end{aligned}$$

where $C(n,kp)$ is the constant from Theorem 1.3. Thus $c = c(n,\alpha,\|\tilde{A}\|_\infty,\|\tilde{A}^{-1}\|_\infty)$ can also be chosen, such that this chain of inequalities is true. We may apply Theorem 1.3 to conclude that

$$\begin{aligned}
(\lambda_1(\hat{M}) &\ge n\Big(1 - C(n,kp)\Big(\frac{1}{|\hat{M}|}\int_{\hat{M}}(\widehat{\mathrm{Ric}}-(n-1))_-^{kp}\Big)^{1/kp}\Big)\\
&\ge n(1 - C(n,kp)\mu_0^{-n/kp}\tilde{C}_1\tilde{C}_3^{1/kp}\|\tilde{H}\|_{kp}\tilde{\epsilon}^{2+\alpha}),
\end{aligned}$$

where $\tilde{\epsilon} = |M|^{-1/n}\epsilon$. We obtain

$$\lambda_1(\tilde{M}) \ge \mu_0^2 n(1-\tilde{C}_4\tilde{\epsilon}^{2+\alpha}), \tag{4.4}$$

with a new constant $\tilde{C}_4$.

We now want to apply Theorem 1.2. Therefore we need estimates of the curvature integrals. First, note that

$$\tilde{H}_2 = \frac{1}{n(n-1)}\tilde{R}.$$

A similar calculation to (4.3) shows that at any point

$$\tilde{x} \in \tilde{P}_\gamma = \{\tilde{x}\in\tilde{M} : \|\tilde{A}-\mu_0\tilde{g}\| < \gamma\}, \quad 0<\gamma<1,$$

we have

$$\|\tilde{R}_{ij} - \mu_0^2(n-1)\tilde{g}_{ij}\| \le \tilde{C}_5(n,\mu_0)\|\tilde{A}-\mu_0\tilde{g}\|. \tag{4.5}$$

Furthermore,

$$|\tilde{P}_\gamma^c|\gamma^{kp} \le \int_{\tilde{P}_\gamma^c}\|\tilde{A}-\mu_0\tilde{g}\|^{kp} \le \tilde{C}_1^{kp}\|\mathring{\tilde{A}}\|_{kp}^{kp} \le \tilde{C}_6\tilde{\epsilon}^{(2+\alpha)kp}$$

and thus

$$|\tilde{P}_\gamma^c| \le \tilde{C}_6\Big(\frac{\tilde{\epsilon}^{2+\alpha}}{\gamma}\Big)^{kp}.$$

We estimate

$$\begin{aligned}
\Big(\int_{\tilde{M}}\tilde{H}_2^{2p}\Big)^{1/p} &= \Big(\int_{\tilde{P}_\gamma}\Big(\frac{\tilde{R}}{n(n-1)}\Big)^{2p} + \int_{\tilde{P}_\gamma^c}\Big(\frac{\tilde{R}}{n(n-1)}\Big)^{2p}\Big)^{1/p}\\
&\le \Big\|\frac{\tilde{R}}{n(n-1)}\Big\|_{2p,\tilde{P}_\gamma}^2 + \Big\|\frac{\tilde{R}}{n(n-1)}\Big\|_{2p,\tilde{P}_\gamma^c}^2\\
&\le (\mu_0^2 + \tilde{C}_5\|\tilde{A}-\mu_0\tilde{g}\|_{2p,\tilde{P}_\gamma})^2 + |\tilde{P}_\gamma^c|^{1/p}\|\tilde{H}\|_\infty^4,
\end{aligned} \tag{4.6}$$

where we use $\tilde{H}_2^{1/2} \le \tilde{H}$ and (4.5).

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Furthermore, from (4.5),

$$\Big(\int_{\tilde M}\tilde H\Big)^2 \ge \Big(\int_{\tilde P_\gamma}\Big(\frac{\tilde R}{n(n-1)}\Big)^{1/2}\Big)^2 \ge \Big(|\tilde P_\gamma|\sqrt{\mu_0^2-\tilde C_5\gamma}\Big)^2$$
$$= |\tilde P_\gamma|^2\mu_0^2 - |\tilde P_\gamma|^2\tilde C_5\gamma \tag{4.7}$$

for all

$$0<\gamma<\frac{\mu_0^2}{\tilde C_5}.$$

From (4.4), (4.6) and (4.7) we obtain

$$\begin{aligned}\lambda_1(\tilde M)\Big(\int_{\tilde M}\tilde H\Big)^2 - n\|\tilde H_2\|_{2p}^2 &\ge (\mu_0^2 n - \mu_0^2 n\tilde C_4\tilde\epsilon^{2+\alpha})(|\tilde P_\gamma|^2\mu_0^2 - |\tilde P_\gamma|^2\tilde C_5\gamma)\\ &\quad - n\mu_0^4 - n\tilde C_5^2\gamma^2 - 2n\mu_0^2\tilde C_5\gamma - n|\tilde P_\gamma^c|^{1/p}\|\tilde H\|_\infty^4\\ &\ge -\tilde C_7|\tilde P_\gamma^c| - \tilde C_7\gamma - \tilde C_7\tilde\epsilon^{2+\alpha} - \tilde C_7\Big(\frac{\tilde\epsilon^{2+\alpha}}{\gamma}\Big)^k,\end{aligned}$$

where $\tilde C_7$ is a new constant. According to Theorem 1.2 and Proposition 3.1 there exists $C_{\tilde\epsilon}$, which can be chosen as

$$C_{\tilde\epsilon} = \frac{1}{2}\min\Big(L\sqrt{\frac{n}{\lambda_1(\tilde M)}}\tilde\epsilon^2, L\Big),$$

such that whenever $\tilde\epsilon < 2/(3\|\tilde H\|_\infty)$ and

$$\lambda_1(\tilde M)\Big(\int_{\tilde M}\tilde H\Big)^2 - n\|\tilde H_2\|_{2p}^2 > -C_{\tilde\epsilon},$$

we can conclude that

$$\tilde M \subset B_{\sqrt{(n/\lambda_1(\tilde M))}+\tilde\epsilon}(\tilde x_0)\backslash B_{\sqrt{(n/\lambda_1(\tilde M))}-\tilde\epsilon}(\tilde x_0).$$

Now define

$$\gamma = \tilde\epsilon^{2+(\alpha/2)}.$$

Then

$$\begin{aligned}\tilde C_7\Big(\Big(\frac{\tilde\epsilon^{2+\alpha}}{\gamma}\Big)^{kp} + \Big(\frac{\tilde\epsilon^{2+\alpha}}{\gamma}\Big)^k + \gamma + \tilde\epsilon^{2+\alpha}\Big) &\le \tilde C_7(\tilde\epsilon^{(\alpha kp)/2} + \tilde\epsilon^{(\alpha k)/2} + \tilde\epsilon^{2+(\alpha/2)} + \tilde\epsilon^{2+\alpha})\\ &= \tilde C_7(\tilde\epsilon^{3p} + \tilde\epsilon^3 + \tilde\epsilon^{2+(\alpha/2)} + \tilde\epsilon^{2+\alpha})\\ &< \frac{1}{2}\min\Big(L\sqrt{\frac{n}{\lambda_1(\tilde M)}}\tilde\epsilon^2, L\Big),\end{aligned}$$

for all $0<\tilde\epsilon<c$, if c is small enough in its dependence on n, α, $\|\tilde A\|_\infty$ and $\|\tilde A^{-1}\|_\infty$, such that the requirements for γ, namely

$$\gamma < \min\Big(1, \frac{\mu_0^2}{\tilde C_5}\Big),$$

are fulfilled as well.

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We conclude, rescaling again,

$$M \subset B_{\sqrt{(n/\lambda_1(M))}+\epsilon}(x_0) \backslash B_{\sqrt{(n/\lambda_1(M))}-\epsilon}(x_0),$$

the desired result. □

REMARK 4.2. The previous result is easier to comprehend if one restricts to the class of hypersurfaces of bounded volume and modulus of convexity, namely

$$0 < c \leq |M| \leq C$$

and

$$0 < cg \leq A \leq Cg.$$

Then, in order to prove ϵ-closeness, one has to find constants $c > 0$ and $\beta > 0$, such that

$$\|A - Hg\| \leq cH\epsilon^{2+\beta},$$

where c must not depend on ϵ. Then applying Theorem 4.1 with $\alpha = \beta/2$, one concludes ϵ-closeness for small $0 < \epsilon < \epsilon_0$.

5. Concluding remarks and open questions

We must remark that this result is only a first step towards a better understanding of the stability problem. It helps to control the order of δ with respect to ϵ which is sufficient for first applications in geometric flows (compare [14]).

However, two things will be desirable in this context. Firstly, there would be direct applications to geometric flows if one could improve the order $\epsilon^{2+\alpha}$. We are not aware of the existence of such a result. Secondly, pinching results for the first eigenvalue of the Laplacian are known in other ambient spaces (cf. [10]). It would be interesting, with immediate applications to curvature flows in those spaces, if results like ours could be deduced in those settings as well.

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