UNCOUNTABLE EXISTENTIALLY CLOSED GROUPS IN LOCALLY FINITE GROUP CLASSES

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1. Introduction. In this paper, $\mathfrak{X}$ will always denote a local class of locally finite groups, which is closed with respect to subgroups, homomorphic images, extensions, and with respect to cartesian powers of finite $\mathfrak{X}$-groups. Examples for $\mathfrak{X}$ are the classes $L\mathfrak{F}_\pi$ of all locally finite π-groups and $L(\mathfrak{F}_\pi \cap \mathfrak{S})$ of all locally soluble π-groups (where π is a fixed set of primes). In [**4**], a wreath product construction was used in the study of existentially closed $\mathfrak{X}$-groups (= e.c. $\mathfrak{X}$-groups); the restrictive type of construction available in [**4**] permitted results for only countable groups. This drawback was then removed partially in [**5**] with the help of permutational products. Nevertheless, the techniques essentially only permitted amalgamation of $\mathfrak{X}$-groups with locally nilpotent π-groups. Thus, satisfactory results could be obtained for $L\mathfrak{F}_p$-groups (resp. locally nilpotent π-groups) [**6**], while the theory remained incomplete in all other cases.

It is the purpose of the present note to close this gap. We can do so by using a new construction, which is related to both Krasner–Kaloujnine embeddings and permutational products. It is derived from the observation that, whenever $N \trianglelefteq G$, then the right regular representation $G \to \mathrm{Sym}(G)$ coincides with a Krasner–Kaloujnine embedding, if we regard $N\,Wr\,G/N$ as a permutation group on $N \times T$ (where T is a transversal of N in G) and identify $N \times T$ canonically with G. Thus, if $G \in \mathfrak{X}$, then the image of the right regular representation lies in the intersection of $N\,Wr\,G/N$ with the constricted symmetric group on G [**3**, p. 180]. The latter is locally finite, and so our assumptions about $\mathfrak{X}$ ensure that the intersection is an $\mathfrak{X}$-group. Therefore, in the construction, we basically just try to find enough elements in this intersection in order to obtain appropriate $\mathfrak{X}$-supergroups of given $G \in \mathfrak{X}$. This is accomplished by a certain choice of T and some further modifications.

The basic construction is given in Section 2. It turns out to be much easier than the previous ones, and it allows us to reprove all previous theorems in full generality for $\mathfrak{X}$-groups. In this paper, we will just fill the remaining gaps. In Section 3, we remove the countability assumption from the theorems of [**4**, §4]. The results of [**5**, §4] about complements in countable e.c. $\mathfrak{X}$-groups are generalized in Section 4 to results about partial complements in e.c. $\mathfrak{X}$-groups. We also supplement our theorems about algebraically closed (a.c.) $L(\mathcal{F}_\pi \cap \mathfrak{G})$-groups [**8**]. Finally, Section 5 contains a treatment of amalgamation in $L(\mathcal{F}_\pi \cap \mathfrak{G})$ which is in line with [**7**].

Note that it remains open whether the restriction to splitting groups in H. Ensel's results [**1**] about e.c. Sylow tower groups is redundant.

2. The construction. Let $\bar{\ }: G \to H$ be a homomorphism of $\mathfrak{X}$-groups with kernel N. Fix $U \leq V \leq G$ such that V is finite. Choose left transversals R of $U \cap N$ in U (or equivalently of N in UN with $R \subseteq U$), and S of UN in G, and T of $\bar{G}$ in H. Then $H = T\bar{S}\bar{R}$. Put $\Omega = G \times H$, and regard the unrestricted regular wreath product $W =$

$G \mathit{Wr}\, H$ as a permutation group on Ω in the usual way, i.e.,

$$(g', h')^{f\cdot h} = (g' \cdot (h')f, h'h) \qquad \text{for all } f: H \to G,\ h \in H.$$

Note that $W = \{f \cdot h \mid f: H \to G$ and $h \in H\}$ where $f_1h_1f_2h_2 = f_1f_2^{(h_1^{-1})}h_1h_2$ and $(h)(f_1f_2^{(h_1^{-1})}) = (h)f_1 \cdot (hh_1)f_2$ for all $h \in H$. Define embeddings

$$\sigma: H \to W \quad \text{and} \quad \tau: G \to W$$

via

$$(g_0, t_0\bar{s}_0\bar{r}_0)^{h\sigma} = (g_1, t_1\bar{s}_1\bar{r}_1) \quad \text{where} \quad t_1\bar{s}_1\bar{r}_1 = t_0\bar{s}_0\bar{r}_0h \quad \text{and} \quad g_1 = g_0s_0s_1^{-1};$$

and

$$(g_0, t_0\bar{s}_0\bar{r}_0)^{g\tau} = (g_1, t_1\bar{s}_1\bar{r}_1) \quad \text{where} \quad t_1\bar{s}_1\bar{r}_1 = t_0\bar{s}_0\bar{r}_0\bar{g} \quad \text{and} \quad g_1 = g_0s_0r_0g \cdot r_1^{-1}s_1^{-1}.$$

Observe that τ is a standard embedding with respect to $\bar{\cdot}$ in the sense of [**2**] (the countermap $\cdot * : H \to G$ has to be defined via $(\overline{tsr})^* = sr$ here). Finally, define an embedding

$$\mu: V \mathit{Wr}\, \bar{U} \to W$$

via

$$(f \cdot \bar{u})\mu = \hat{f} \cdot \bar{u} \quad \text{where} \quad (\overline{tsr})\hat{f} = s \cdot (\bar{r})f \cdot s^{-1} \qquad \text{for all } t \in T,\ s \in S,\ r \in R.$$

Obviously, $\operatorname{Im}\mu \leq \Delta \cdot \bar{U}$ where $\Delta = \{f: H \to G \mid (\overline{tsr})f \in sVs^{-1}$ for all $t \in T,\ s \in S,\ r \in R\}$.

THEOREM 2.1. (a) $W_0 = \langle H\sigma, G\tau, \Delta\bar{U}\rangle \in \mathfrak{X}$.

(b) *$H\sigma$ is a complement to the base group of W, and $\sigma \mid \bar{U} = \mu \mid \bar{U} = id_{\bar{U}}$.*

(c) *If $\kappa: U \to (U \cap N) \mathit{Wr}\, \bar{U} \leq V \mathit{Wr}\, \bar{U}$ denotes the Krasner–Kaloujnine embedding with respect to the transversal R (i.e., $u\kappa = f_u \cdot \bar{u}$ for all $u \in U$, where $(\bar{u}')f_u = (\bar{u}')^* \cdot u \cdot (\bar{u}'\bar{u})^{*-1}$ for all $u' \in U$, and where $*: \bar{U} \to U$ is given by $\bar{r}^* = r$ for all $r \in R$), then the diagram*

$$\begin{tikzcd} U \arrow[r, "\kappa"] \arrow[dr, "\tau"'] & V \mathit{Wr}\, \bar{U} \arrow[d, "\mu"] \\ & W \end{tikzcd}$$

commutes.

Proof. (a) It suffices to show that W_0 is locally finite, since the argument of [**5**, p. 1999] will then ensure that $W_0 \in \mathfrak{X}$. Let $G_0 \leq G$ and $H_0 \leq H$ be finite with $V \leq G_0$ and $\bar{G}_0 \leq H_0$. By [**9**, Lemma 5.3] it suffices to show that the transitivity systems of $Q = \langle H_0\sigma, G_0\tau, \Delta\bar{U}\rangle$ are boundedly finite. Fix $\omega_0 = (g_0, t_0\bar{s}_0\bar{r}_0) \in \Omega$. We will show that

$$\omega_0^Q \subseteq \Omega_0 = \{(g, \overline{tsr}) \mid \overline{tsr} \in t_0\bar{s}_0\bar{r}_0H_0 \text{ and } g \in g_0s_0G_0s^{-1}\},$$

whence $|\omega_0^Q| \leq |\Omega_0| \leq |H_0| \cdot |G_0|$.

To this end, let $\omega_1 = (g_1, t_1\bar{s}_1\bar{r}_1) \in \Omega_0$. Then $t_1\bar{s}_1\bar{r}_1 = t_0\bar{s}_0\bar{r}_0x$ and $g_1 = g_0s_0y \cdot s_1^{-1}$ for suitable $x \in H_0$, $y \in G_0$. Suppose that $\omega_1^q = \omega_2 = (g_2, t_2\bar{s}_2\bar{r}_2)$. If $q = h\sigma$ for some $h \in H_0$, then $t_2\bar{s}_2\bar{r}_2 = t_1\bar{s}_1\bar{r}_1h = t_0\bar{s}_0\bar{r}_0xh \in t_0\bar{s}_0\bar{r}_0H_0$ and $g_2 = g_1s_1s_2^{-1} = g_0s_0y \,.\, s_1^{-1}s_1s_2^{-1} = g_0s_0y \cdot s_2^{-1} \in g_0s_0G_0s_2^{-1}$. If $q = g\tau$ for some $g \in G_0$, then $t_2\bar{s}_2\bar{r}_2 = t_1\bar{s}_1\bar{r}_1\bar{g} = t_0\bar{s}_0\bar{r}_0x\bar{g} \in t_0\bar{s}_0\bar{r}_0H_0$ (observe that $\bar{G}_0 \leq H_0$) and $g_2 = g_1s_1r_1g \cdot r_2^{-1}s_2^{-1} = g_0s_0y \cdot s_1^{-1}s_1r_1g \cdot r_2^{-1}s_2^{-1} = g_0s_0y \cdot r_1g \cdot r_2^{-1}s_2^{-1} \in g_0s_0G_0s_2^{-1}$ (observe that $R \subseteq U \leq V \leq G_0$). If $q = f \cdot \bar{u} \in \Delta \,.\, \bar{U}$, then

$t_2\bar{s}_2\bar{r}_2 = t_1\bar{s}_1\bar{r}_1\bar{u} = t_0\bar{s}_0\bar{r}_0x\bar{u} \in t_0\bar{s}_0\bar{r}_0H_0$ (observe that $\bar{U} \le \bar{V} \le \bar{G}_0 \le H$); in particular $s_2 = s_1$, and thus $g_2 = g_1 \cdot (t_1\bar{s}_1\bar{r}_1)f \in g_0s_0y \cdot s_1^{-1}s_1V \cdot s_1^{-1} \subseteq g_0s_0G_0s_2^{-1}$ (observe that $V \le G_0$). This shows that $\omega^q \in \Omega_0$ for all $q \in Q$.

(b) is obvious.

(c) Fix $u \in U$. Let $\omega_0^{u\kappa\mu} = \omega_1$ with $\omega_i = (g_i, t_i\bar{s}_i\bar{r}_i) \in \Omega$. Then $t_1\bar{s}_1\bar{r}_1 = t_0\bar{s}_0\bar{r}_0\bar{u}$. In particular $s_1 = s_0$, and we obtain $g_1 = g_0 \cdot (t_0\bar{s}_0\bar{r}_0)\hat{f}_u = g_0s_0 \cdot (\bar{r}_0)f_u \cdot s_0^{-1} = g_0s_0r_0u \cdot (\bar{r}_0\bar{u})^{*-1}s_0^{-1} = g_0s_0r_0u \cdot r_1^{-1}s_1^{-1}$. This shows that $u\kappa\mu = u\tau$.

The following technical lemma, which can be verified by straightforward calculations, will be essential in our proofs.

LEMMA 2.2. *Let $f \cdot b \in W = A \,Wr\, B$ where $f: B \to A$ and $b \in B - 1$. If $f_1, f_2: B \to A$ satisfy* $\mathrm{supp}(f_i) \subseteq T$ *for a fixed left transversal T of $\langle b \rangle$ in B, then* $[f_1, f_2] = [[f \cdot b, f_1], f_2] \in \langle (f \cdot b)^W \rangle$.

3. Filling some gaps. In this section we will complete the generalization of the results of [**4**, §4] to uncountable e.c. $\mathfrak{X}$-groups (a project begun in [**5**, §2]). The reader should be familiar with the results in [**5**, §2]. In particular, note that every e.c. $\mathfrak{X}$-group G has a unique chief series by [**5**, Theorem 2.3], and that there are three kinds of normal subgroups in G: the groups M and N occurring in the chief factors M/N of G (here, the M's are precisely the normal closures $\langle g^G \rangle$, $g \in G - 1$), and the remaining normal subgroups (which can be obtained as intersections of N's or as unions of M's).

THEOREM 3.1. ([**4**, Theorem 4.8], [**5**, Theorem 2.4]). *If M/N is a chief factor of an e.c. $\mathfrak{X}$-group G, then every finite system of equations and inequalities with coefficients from N, which is solvable in some $\mathfrak{X}$-supergroup of G, has already a solution in every verbal subgroup of M. Moreover, for every $K \trianglelefteq G$ such that K has no maximal normal subgroup, the following statements hold.*

(a) *Every finite system of equations and inequalities with coefficients from K, which is solvable in some $\mathfrak{X}$-supergroup of G, already has a solution in K.*

(b) *Every normal subgroup of K is a normal subgroup in G. In particular, K has a unique chief series, and the normal subgroups of K form a chain.*

(c) *Each automorphism of K, which is induced by conjugation with some element from G, is locally inner.*

Proof. Let $\mathcal{S}$ be a finite system of equations and inequalities with coefficients $n_1, \dots, n_r \in N$, which is solvable in some $\mathfrak{X}$-supergroup of G. Since G is e.c. in $\mathfrak{X}$, there exists a solution $g_1, \dots, g_s$ in G. Choose $g \in M - N$ and a word $w(x_1, \dots, x_v) \ne 1$. Put $U = \langle n_1, \dots, n_r, g \rangle$. Since G is verbally complete [**4**, Theorem 2.1], there exists a finite subgroup $V \le G$ such that $U \le V$ and

$$g_1, \dots, g_s \in \Omega(V'), \tag{3.1}$$

where $\Omega(X) = \langle w(x_1, \dots, x_v) \mid x_i \in X \rangle$ for any group X.

Apply the construction of Section 2 to the canonical epimorphism $\bar{}: G \to G/N$. This yields embeddings $\tau: G \to W_0$ and $\mu: V \,Wr\, \bar{U} \to W_0$ (where $W_0 \in \mathfrak{X}$ is as in Theorem 2.1) such that $\kappa\mu = \tau \mid U$ for some Krasner–Kaloujnine embedding $\kappa: U \to V \,Wr\, \bar{U}$ with respect to a transversal R. Now $n_i\kappa = f_{n_i}$ where $(\bar{r})f_{n_i} = r \cdot n_ir^{-1}$ for all $r \in R$. Hence a solution to $\mathcal{S}\kappa$ in $Z = V \,Wr\, \bar{U}$ is given by $f_{g_1}, \dots, f_{g_s}$ where $(\bar{r})f_{g_i} = r \cdot g_ir^{-1}$ for all $r \in R$.

Because of (3.1) and Lemma 2.2 we have $f_{g_j} \in \Omega(\langle g\kappa^Z \rangle)$, whence $f_{g_1}\mu, \dots, f_{g_s}\mu$ is a solution to $\mathcal{S}\tau$ in $\Omega(\langle g\tau^{W_0} \rangle)$. Since G is e.c. in $\mathfrak{X}$ and $G\tau \le W_0$, there already exists a solution to $\mathcal{S}$ in $\Omega(\langle g^G \rangle) = \Omega(M)$.

The assertions (a), (b) and (c) now follow as in the proof of [**4**, Theorem 4.8].

A counterpart to Theorem 3.1 for factor groups is given by

Theorem 3.2. *Let G be an e.c. $\mathfrak{X}$-group, and let $K \trianglelefteq G$ such that G/K has no minimal normal subgroup. If $\mathcal{S}$ is a finite system of equations and inequalities with coefficients $c_1, \dots, c_r \in G$ and a solution in some $\mathfrak{X}$-supergroup of G, and if $K \cap \langle c_1, \dots, c_r \rangle = 1$, then the system $\mathcal{S}K/K$ has a solution in G/K.*

Proof. Put $C = \langle c_1, \dots, c_r \rangle$. Since C is finite, and since G/K has no minimal normal subgroup, there exists a chief factor M/N in G with $K \le N$ and $M \cap C = 1$. Fix $g \in M - N$, and let $U = \langle C, g \rangle$. Choose $R = C$ and a finite $V \le G$ such that $U \le V$ and $g \in V'$. Apply the construction of Section 2 to the canonical epimorphism $\theta: G \to G/M$. This yields embeddings $\sigma: G/M \to W_0$ and $\tau: G \to W_0$ and $\mu: V\,Wr\,U\theta \to W_0$, where W_0 is an $\mathfrak{X}$-subgroup of $G\,Wr\,G/M$ as in Theorem 2.1. Apply the construction of Section 2 to the composition $\bar{\ }: G \to W_0$ of $\theta: G \to G/M$ and $\sigma: G/M \to W_0$. This yields embeddings $\tilde{\sigma}: W_0 \to \tilde{W}_0$ and $\tilde{\tau}: G \to \tilde{W}_0$ and $\tilde{\mu}: V\,Wr\,\bar{U} \to \tilde{W}_0$, where $\tilde{W}_0$ is an $\mathfrak{X}$-subgroup of $G\,Wr\,W_0$ as in Theorem 2.1.

From Theorem 2.1 we have $\sigma \mid U\theta = \mu \mid U\theta$ and $\kappa\mu = \tau \mid U$ for the Krasner–Kaloujnine embedding $\kappa: U \to V\,Wr\,U\theta$ with respect to R. Observe also, that $\kappa \mid C = \theta \mid C$ by choice of R. Correspondingly, $\tilde{\sigma} \mid \bar{U} = \tilde{\mu} \mid \bar{U}$ and $\tilde{\kappa}\tilde{\mu} = \tilde{\tau} \mid U$ for the Krasner–Kaloujnine embedding $\tilde{\kappa}: U \to V\,Wr\,\bar{U}$ with respect to R, and $\tilde{\kappa} \mid C = \bar{\ } \mid C$. Therefore, $c\tilde{\tau} = c\tilde{\kappa}\tilde{\mu} = \bar{c}\tilde{\mu} = \bar{c}\tilde{\sigma} = c\theta\sigma\tilde{\sigma} = c\theta\mu\tilde{\sigma} = c\kappa\mu\tilde{\sigma} = c\tau\tilde{\sigma}$ for all $c \in C$.

Since G is e.c. in $\mathfrak{X}$, there exists a solution $g_1, \dots, g_s$ to $\mathcal{S}$ in G. Put $D = \langle C, g_1, \dots, g_s \rangle$. Then $g_1\tau\tilde{\sigma}, \dots, g_s\tau\tilde{\sigma}$ is a solution to $\mathcal{S}\tilde{\tau}$ in $\tilde{W}_0$, and $g\tilde{\tau} \in \langle d\tau\tilde{\sigma}^{\tilde{W}_0} \rangle$ for all $d \in D - 1$ (Lemma 2.2). Since G is e.c. in $\mathfrak{X}$, there does already exist a solution $h_1, \dots, h_s$ to $\mathcal{S}$ in G such that $g \in \langle h^G \rangle$ for all $h \in H - 1$, where $H = \langle C, h_1, \dots, h_s \rangle$. The latter implies that $H \cap K \le H \cap N = 1$, whence $h_1K, \dots, h_sK$ is a solution to $\mathcal{S}K/K$ in G/K.

Let M/N be a chief factor of the e.c. $\mathfrak{X}$-group G. If G satisfies the additional assumption

$$\text{for every } g \in G - 1 \text{ there exists a verbal subgroup of } \langle g^G \rangle \text{ different from } \langle g^G \rangle, \tag{3.2}$$

then it follows from [**5**, Theorem 2.6(b)] that Theorem 3.1 holds with N in place of K, while Theorem 3.2 holds with M in place of K.

Theorem 3.3. ([**4**, Theorem 4.9]). *Let M/N be a chief factor of an e.c. $\mathfrak{X}$-group G.*

(a) *If M/N is not central, then $C_{G/N}(M/N) = Z(M/N)$, and M/N is infinite.*

(b) *Denote by $\gamma: G/C_G(M/N) \to \mathrm{Aut}(M/N)$ the canonical embedding and assume the existence of $x_1, x_2 \in Nm$ $(m \in M - N)$ with $o(x_i) = o(Nm)$. If there exists $\alpha \in \mathrm{Aut}(M/N)$ with $Nx_1\alpha = Nx_2$ such that the subgroup $\langle \alpha, \mathrm{Im}\,\gamma \rangle \le \mathrm{Aut}(M/N)$ is an $\mathfrak{X}$-group, then x_1 and x_2 are conjugate in G.*

(c) *Any two elements from Nm $(m \in M - N)$ of order $o(Nm)$ are conjugate in G.*

Proof. (a) follows from Theorem 3.1 as in the proof of [**4**, Theorem 4.9(a)].

(b) Denote epimorphic images modulo N by bars and put $C = C_G(\bar{M})$. At first we will embed $\bar{G}$ into an $\mathfrak{X}$-group H, in which the images of $\bar{x}_1$ and $\bar{x}_2$ are conjugate. In the case when $C = N$, we choose $H = \langle \alpha, \operatorname{Im} \gamma \rangle$ and the embedding $\gamma : \bar{G} \to H$; then $(\bar{x}_1\gamma)^\alpha = \bar{x}_2\gamma$.

Now, suppose that $C \geq M$. The group $\operatorname{Aut}(\bar{M})$ acts on $\bar{M}\, Wr\, G/M$ via

$$(f \cdot Mg)^\beta = f^\beta \cdot Mg \qquad \text{for all } \beta \in \operatorname{Aut}(\bar{M}),\ f : G/M \to \bar{M},\ g \in G, \text{ where}$$
$$(Mh)f^\beta = ((Mh)f)^\beta \qquad \text{for all } h \in G.$$

Since $\bar{M}$ is elementary abelian, the split extension H of $\bar{M}\, Wr\, G/M$ by $\langle \alpha, \operatorname{Im} \gamma \rangle$ is an $\mathfrak{X}$-group. Choose $* : G/M \to G$ such that $(Mg)* \in Mg$ for all $g \in G$. Then an embedding $\delta : \bar{G} \to H$ is given by

$$\bar{g}\delta = (f_{\bar{g}} \cdot Mg) \cdot (Cg)\gamma \qquad \text{for all } g \in G, \text{ where}$$
$$(Mh)f_{\bar{g}} = N(g \cdot (Mhg)^{*-1} \cdot Mh^*) \qquad \text{for all } h \in G.$$

(Note that $(Cg)\gamma = C(Mh^{*-1} \cdot (Mhg)^*)\gamma$.) Moreover, $(\bar{x}_1\delta)^\alpha = \bar{x}_2\delta$, since $f_{\bar{x}_i} \equiv \bar{x}_i$.

Now choose $U = \langle x_1, x_2 \rangle$, and apply the construction of Section 2 to the composition $\bar{\ } : G \to H$ of the canonical epimorphism $G \to G/N$ and the above embedding $G/N \to H$. This yields embeddings $\sigma : H \to W_0$ and $\tau : G \to W_0$ and $\mu : (U \cap N)\, Wr\, \bar{U} \to W_0$ (where $W_0 \in \mathfrak{X}$ is as in Theorem 2.1) such that $\sigma \mid \bar{U} = \mu \mid \bar{U}$ and $\kappa\mu = \tau \mid U$ for some Krasner–Kaloujnine embedding $\kappa : U \to (U \cap N)\, Wr\, \bar{U}$. Because of $o(x_i) = o(\bar{m})$ and [**4**, Lemma 4.2], the element $x_i\kappa$ is conjugate in $(U \cap N)\, Wr\, \bar{U}$ to $\bar{x}_i \in \bar{U}$. Moreover, we have $(\bar{x}_1\mu)^{\alpha\sigma} = (\bar{x}_1^\alpha)\sigma = \bar{x}_2\mu$. Hence $x_1\tau$ and $x_2\tau$ are conjugate in $W_0 \in \mathfrak{X}$. Since G is e.c. in $\mathfrak{X}$, we conclude that x_1 and x_2 are already conjugate in G.

(c) See proof of [**4**, Theorem 4.9(c)].

In the case when $\mathfrak{X} = L(\mathfrak{F}_\pi \cap \mathfrak{S})$, we can even describe the automorphisms between finite subgroups of an e.c. $\mathfrak{X}$-group G, which are induced by conjugation in G. This generalizes [**6**, Theorem 6.1] (see also [**4**, Theorem 5.3]).

THEOREM 3.4. *Let G be an e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-group.*

(a) *An isomorphism $\psi : A \to B$ between finite subgroups of G is induced by conjugation in G, if and only if, for each chief factor M/N in G,*

(1) *$\psi(M \cap A) = M \cap B$ and $\psi(N \cap A) = N \cap B$, and*

(2) *there exists an elementary-abelian group $E \geq M/N$ such that the isomorphism $(M \cap A)N/N \to (M \cap B)N/N$ induced by ψ can be extended to some $\alpha \in \operatorname{Aut}(E)$, such that—for every $g \in G$—conjugation on M/N with Ng can be extended to some $Ng^* \in \operatorname{Aut}(E)$, and such that $\langle \alpha, Ng^* \mid g \in G \rangle \in L(\mathfrak{F}_\pi \cap \mathfrak{S})$.*

(b) *The group of all $\alpha \in \operatorname{Aut}(G)$, which leave every chief factor M/N of G invariant and induce a power π-automorphism on M/N (i.e., an automorphism, which raises each element of M/N to a fixed power, and whose order is a π-number), is contained in the group of all locally inner automorphisms of G.*

Proof. (a) The necessity of the conditions (1) and (2) is obvious. Now suppose that (1) and (2) hold. Let M/N be the unique chief factor in G with $X = A \cap N < A \cap M = A$. As in the proof of Theorem 3.3(b) there exists an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-group $H \geq G/N$ such that the isomorphism $\bar{\psi} : (M \cap A)N/N \to (M \cap B)N/N$ induced by ψ is induced by conjugation

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in H. Since G/N is a.c. in $L(\mathfrak{F}_\pi \cap \mathfrak{G})$ (Theorem 4.3(b)), we conclude that $\bar{\psi}$ is already induced by conjugation in G/N. Thus, we may as well assume that $\bar{\psi} = id$. By induction over $|A|$ we may assume that $\psi \mid X$ is induced by conjugation in G. From Theorem 3.1(c) we obtain that $\psi \mid X$ is even induced by conjugation in N.

Let $\bar{\ }: G \to G/N$ be the canonical epimorphism. Fix $y_1, \ldots, y_r \in A - X$ such that $\overline{M \cap A}$ is the direct product of the $\langle \bar{y}_i \rangle$. Inductively, we will now find elements $h_i \in N$ such that $\psi \mid \langle X, y_1, \ldots, y_i \rangle$ is induced by conjugation with h_i for $1 \le i \le r$. Suppose that h_{i-1} has been found for some i. Let $A_0 = \langle X, y_1, \ldots, y_{i-1} \rangle$. Then we may as well assume that $\bar{\psi} = \text{id}$ and $\psi \mid A_0 = \text{id}$. Let $U = \langle A_0, y_i, y_i\psi \rangle$ and $p = o(\bar{y}_i)$. Choose a finite $V \le G$ such that $U \le V''$. Put $R = R_2R_1$ where $R_1 = \{1, y_i, \ldots, y_i^{p-1}\}$ and where R_2 is a left transversal of X in A_0. Apply the construction of Section 2 to $\bar{\ }$. This yields embeddings $\tau: G \to W_0$ and $\mu: V\,Wr\,\bar{U} \to W_0$ where W_0 is an $L(\mathfrak{F}_\pi \cap \mathfrak{G})$-subgroup of $G\,Wr\,\bar{G}$ as in Theorem 2.1.

Since G is e.c. in $L(\mathfrak{F}_\pi \cap \mathfrak{G})$, and since $N = \langle y_i^G \rangle'$, it suffices to find some $f \in \langle y_i\tau^{W_0} \rangle'$ satisfying $y_i\tau^f = y_i\psi\tau$ and $[A_0\tau, f] = 1$. From Theorem 2.1 we have that $\kappa\mu = \tau \mid U$ for the Krasner–Kaloujnine embedding $\kappa: U \to Z = V\,Wr\,\bar{U}$ with respect to R. It remains to show the existence of $f \in \langle y_i\kappa^Z \rangle'$ satisfying $y_i\kappa^f = y_i\psi\kappa$ and $[A_0\kappa, f] = 1$. Define $f: \bar{U} \to U \le V''$ via $(\overline{ry_i^\nu})f = r \cdot (y_i\psi)^\nu \cdot y_i^{-\nu} \cdot r^{-1}$ for all $r \in R_2$ and $0 \le \nu \le p-1$. Then $f \in \langle y_i\kappa^Z \rangle'$ by Lemma 2.2, and straightforward calculations yield that $y_i\kappa^f = y_i\psi\kappa$. Now, regard some $a \in A_0$. Clearly, $a\kappa = f_a \,.\, \bar{a}$ where $f_a: \bar{U} \to X$. Because of $\psi \mid A_0 = \text{id}$, conjugation with y_i^ν induces the same automorphism on A_0 as conjugation with $(y_i\psi)^\nu$. This implies that $[f_a, f] = 1$, and that $(\overline{ry_i^\nu})f = (\bar{y}_i^\nu)f$ for all $r \in R_2$ and $0 \le \nu \le p-1$. But $\bar{U}$ is abelian, and thus our choice of R ensures that $[\bar{a}, f] = 1$ too.

(b) Observe that the power automorphisms of M/N are contained in the centre of $\text{Aut}(M/N)$.

In the case when $|\pi| \ge 2$ it remains open, whether every e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{G})$-group G acts via conjugation transitively on $M/N - 1$ for each chief factor M/N in G. (Chief factors of locally finite p-groups are central [**3**, 1.B.8].)

THEOREM 3.5. ([**4**, Theorem 4.11(f)]) *If the e.c. $\mathfrak{X}$-group G satisfies* (3.2), *then there exists for every proper subnormal subgroup S of G a chief factor M/N in G such that $N \le S \le M$.*

Proof. Choose m minimal with respect to $S = S_m \lhd S_{m-1} \lhd \ldots \lhd S_1 \lhd G$. Then $S_1 = M$ for some chief factor M/N in G by Theorem 3.1. Assume by induction that $N \le S_k$ for some $k \le m-1$. By Theorem 3.1 there exists $x \in S_{k+1} - N$. Fix $g \in N$. Choose $U = \langle x, g \rangle$ and a finite subgroup $V \le G$ such that $U \le V$ and $g \in (\Omega(V'))'$, where $\Omega(M)$ is a verbal subgroup different from M, which is given from (3.2). Apply the construction of Section 2 to the canonical epimorphism $\bar{\ }: G \to G/N$. This yields embeddings $\tau: G \to W_0$ and $\mu: V\,Wr\,\bar{U} \to W_0$ (where $W_0 \in \mathfrak{X}$ is as in Theorem 2.1) such that $\kappa\mu = \tau \mid U$ for some Krasner–Kaloujnine embedding $\kappa: U \to Z = V\,Wr\,\bar{U}$. From Lemma 2.2 we obtain that $g\kappa \in [[x\kappa, \Omega(\langle x\kappa^Z \rangle)], \Omega(\langle x\kappa^Z \rangle)]$, whence $g\tau \in [[x\tau, \Omega(\langle x\tau^{W_0} \rangle)], \Omega(\langle x\tau^{W_0} \rangle)]$. Since G is e.c. in $\mathfrak{X}$, we already have $g \in [[x, \Omega(\langle x^G \rangle)], \Omega(\langle x^G \rangle)] \le [[x, N], N] \le [[S_{k+1}, S_k], S_k] \le S_{k+1}$. This shows that $N \le S_{k+1}$.

4. Partial complements and algebraically closed groups. Let G be a countable e.c. $\mathfrak{X}$-group satisfying (3.2). In [**5**] we have shown that, if $K \trianglelefteq G$ with $K \ne \langle g^G \rangle$ for all

$g \in G - 1$, then every finite $F \leq G$ with $F \cap K = 1$ is contained in a complement to K in G. This can be generalized as follows.

THEOREM 4.1. ([5, Theorem 4.2]). *Let M/N be a chief factor of the e.c. $\mathfrak{X}$-group G such that $\Omega(M) = N$ for some verbal subgroup $\Omega(M)$ of M. If F is a finite subgroup of G with $N \cap F = 1$, and if $N \cdot F \leq G_0 \leq G$ where $|G_0 : N|$ is countable, then F is contained in a complement to N in G_0.*

Proof. Since G_0 is the union of an ascending chain of groups C satisfying $N \cdot F \leq C \leq G_0$ and $|C : N| < \infty$, we may as well assume that G_0 is finite. Let D be a finite subgroup of G_0 with $F \leq D$ and $G_0 = N \cdot D$. Fix $x \in M - N$. Put $U = \langle D, x \rangle$, and choose a finite $V \leq G$ such that $U \leq \Omega(V')$. Furthermore, let $R = \tilde{R} \cdot F$ where $\tilde{R}$ is a left transversal of $U \cap NF$ in U. Apply the construction of Section 2 to the canonical epimorphism $\bar{\ }: G \to G/N$. This yields embeddings $\sigma : \bar{G} \to W_0$ and $\tau : G \to W_0$ and $\mu : V\, Wr\, \bar{U} \to W_0$ (where $W_0 \in \mathfrak{X}$ is as in Theorem 2.1) such that $\sigma \mid \bar{U} = \mu \mid \bar{U}$ and $\kappa\mu = \tau \mid U$ for the Krasner–Kaloujnine embedding $\kappa : U \to V\, Wr\, \bar{U}$ with respect to R. By choice of R,

$$y\tau = y\kappa\mu = \bar{y}\mu = \bar{y}\sigma \qquad \text{for every } y \in F.$$

For every $u \in U$, we obtain

$$u\tau = u\kappa\mu = f_u\mu \cdot \bar{u}\mu = f_u\mu \cdot \bar{u}\sigma \qquad \text{for suitable } f_u : \bar{U} \to U \cap N, \text{ where}$$

$$f_u\mu \in \Omega(\Delta') \leq \Omega(\langle x\tau^{W_0} \rangle) \qquad \text{by Lemma 2.2.}$$

Moreover,

$$x\tau = f_x\mu \cdot \bar{x}\sigma \in \Delta' \cdot \langle \bar{d}^{\bar{G}} \rangle \sigma \leq \langle \bar{d}\sigma^{W_0} \rangle \qquad \text{whenever } d \in D - N.$$

Since G is e.c. in $\mathfrak{X}$, there does already exist an embedding $\tilde{\sigma} : \bar{D} \to G$ such that $\bar{y}\tilde{\sigma} = y$ for all $y \in F$, and such that $d \in \Omega(\langle x^G \rangle) \cdot \bar{d}\tilde{\sigma}$ for all $d \in D$ and $x \in \langle \bar{d}\tilde{\sigma}^G \rangle$ for all $d \in D - N$. Now $\operatorname{Im} \tilde{\sigma}$ is the desired complement, since the above properties ensure that $F \leq \operatorname{Im} \tilde{\sigma}$, that $d \in N \cdot \bar{d}\tilde{\sigma}$ for all $d \in D$, and that $N \cap \operatorname{Im} \tilde{\sigma} = 1$.

THEOREM 4.2. ([5, Theorem 4.1]). *Let K be a normal subgroup of the e.c. $\mathfrak{X}$-group G which does not occur in any chief factor of G. If F is a finite subgroup of G with $K \cap F = 1$, and if $K \cdot F \leq G_0 \leq G$ where $|G_0 : K|$ is countable, then F is contained in a complement to K in G_0.*

Proof. Again we may assume that $|G_0 : K|$ is finite. Let D be a finite subgroup of G_0 with $F \leq D$ and $G_0 = K \cdot D$. Then there exist chief factors M_1/N_1 and M_2/N_2 in G such that $M_2 \leq K \leq N_1$ and $D \cap N_2 = D \cap K = D \cap N_1$. Denote by $\theta : G \to G/N_2$ and $\bar{\ } : G/N_2 \to G/N_1$ the canonical epimorphisms. Fix $x_i \in M_i - N_i$. Put $U = \langle D, x_1, x_2 \rangle$, and choose a finite $V \leq G$ such that $U \leq V'$. Let $\tilde{R}_1$ be a left transversal of $(U \cap N_1 D)\theta$ in $U\theta$. Then $R_1 = \tilde{R}_1 \cdot D\theta$ is a left transversal of $(U \cap N_1)\theta$ in $U\theta$ (because of $D \cap N_1 = D \cap N_2$). Apply the construction of Section 2 to $\bar{\ }$, with R_1 in place of R. This yields embeddings $\sigma_1 : G/N_1 \to W_0$ and $\tau_1 : G/N_2 \to W_0$, where W_0 is an $\mathfrak{X}$-subgroup of $G/N_2\, Wr\, G/N_1$ as in Theorem 2.1. Denote by $\psi : G \to W_0$ the composition of θ and τ_1. As in the proof of Theorem 4.1 we have

$$x_1\psi = x_1\theta\tau_1 \in \langle \overline{d\theta}\sigma_1^{W_0} \rangle \qquad \text{for every } d \in D - N_1, \text{ and}$$

$$d\psi = d\theta\tau_1 = \overline{d\theta}\sigma_1 \qquad \text{for every } d \in D.$$

In particular,

$$x_1\psi \in \langle d\psi^{W_0}\rangle \qquad \text{for every } d \in D - N_1.$$

Now let $R_2 = \tilde{R}_2 \cdot F$, where $\tilde{R}_2$ is a left transversal of $U \cap N_2F$ in U. Apply the construction of Section 2 to ψ, with R_2 in place of R. This yields embeddings $\sigma_2 : W_0 \to \tilde{W}_0$ and $\tau_2 : G \to \tilde{W}_0$ and $\mu_2 : V\,Wr\,U\psi \to \tilde{W}_0$, where $\tilde{W}_0$ is an $\mathfrak{X}$-subgroup of $G\,Wr\,W_0$ as in Theorem 2.1. As in the proof of Theorem 4.1 we have

$$y\tau_2 = y\psi\sigma_2 \qquad \text{for every } y \in F.$$

For every $u \in U$, we obtain

$$u\tau_2 = f_u\mu_2 \cdot u\psi\sigma_2 \qquad \text{for suitable } f_u : U\psi \to U \cap N_2, \text{ where}$$
$$f_u\mu_2 \in \langle x_2\tau_2^{\tilde{W}_0}\rangle \qquad \text{by Lemma 2.2.}$$

Moreover,

$$x_1\tau_2 = f_{x_1}\mu_2 \cdot x_1\psi\sigma_2 \in \langle d\psi\sigma_2^{\tilde{W}_0}\rangle \qquad \text{for every } d \in D - N_1 = D - N_2.$$

Since G is e.c. in $\mathfrak{X}$, there does already exist an embedding $\bar{\sigma} : D\psi \to G$ such that $\psi\bar{\sigma} \mid F = \mathrm{id}_F$, and such that $d \in \langle x_2^G\rangle \cdot d\psi\bar{\sigma}$ for all $d \in D$ and $x_1 \in \langle d\psi\bar{\sigma}^G\rangle$ for all $d \in D - N_2$. It follows that $F \leq \operatorname{Im}\bar{\sigma}$, that $d \in M_2 \cdot d\psi\bar{\sigma} \subseteq K \cdot d\psi\bar{\sigma}$ for all $d \in D$, and that $K \cap \operatorname{Im}\bar{\sigma} \leq N_1 \cap \operatorname{Im}\bar{\sigma} = 1$, whence $\operatorname{Im}\bar{\sigma}$ is the desired complement.

The above results can be used to characterize the a.c. $\mathfrak{X}$-groups as in [**8**]. This removes the countability assumption from [**8**, Theorems C(c) and D(c)].

THEOREM 4.3. ([**5**, Theorem 4.3(a)]). *Let G be an e.c. $\mathfrak{X}$-group.*

(a) *If $K \ntrianglelefteq G$ such that K does not occur in any chief factor of G, then G/K is e.c. in $\mathfrak{X}$.*

(b) *If M/N is a chief factor in G such that $N = \Omega(M)$ for some verbal subgroup $\Omega(M)$ of M, then G/N is a.c. in $\mathfrak{X}$, but G/M is not a.c. in $\mathfrak{X}$.*

Proof. (a) Let $\mathcal{S}$ be a finite system of equations and inequalities with coefficients $Kg_1, \ldots, Kg_r \in G/K$ and a solution in some $\mathfrak{X}$-supergroup H of G/K. By Theorem 4.2 there exist $c_i \in Kg_i$ such that $\langle c_1, \ldots, c_r\rangle \cap K = 1$. Let $\mathcal{T}$ be the system obtained from replacing Kg_i by c_i in $\mathcal{S}$. Choose $U = \langle c_1, \ldots, c_r\rangle = R$, and apply the construction of Section 2 to the canonical epimorphism $\bar{\ } : G \to G/K \leq H$. This yields embeddings $\sigma : H \to W_0$ and $\tau : G \to W_0$ (where $W_0 \in \mathfrak{X}$ is as in Theorem 2.1) with $(Kg_i)\sigma = \bar{c}_i\sigma = c_i\tau$ for $1 \leq i \leq r$. Hence W_0 contains a solution to $\mathcal{T}\tau$. Since G is e.c. in $\mathfrak{X}$, there already exists a solution to $\mathcal{T}$ in G. Now it follows from Theorem 3.2 that $\mathcal{S} = \mathcal{T}K/K$ has a solution in G/K.

(b) Let $\mathcal{S}$ be a finite system of equations with coefficients $Ng_1, \ldots, Ng_r \in G/N$ and a solution in some $\mathfrak{X}$-supergroup H of G/N. By Theorem 4.1 there exist $c_i \in Ng_i$ such that $\langle c_1, \ldots, c_r\rangle \cap N = 1$. Let $\mathcal{T}$ be the system obtained from replacing Ng_i by c_i in $\mathcal{S}$. Proceed as in (a) to find a solution $h_1, \ldots, h_s$ to $\mathcal{T}$ in G. Because $\mathcal{T}$ consists of equations only, and because $\langle c_1, \ldots, c_r\rangle \cap N = 1$, the system $\mathcal{S} = \mathcal{T}N/N$ has the solution $Nh_1, \ldots, Nh_s$ in G/N. This shows that G/N is a.c. in $\mathfrak{X}$. The remaining assertion follows as in the proof of [**5**, Theorem 4.3(b)] from the remark at the end of Section 5.

THEOREM 4.4. ([**8**, Theorem C(c)]). *If the non-trivial a.c. $\mathfrak{X}$-group G has no minimal normal subgroups, then G is e.c. in $\mathfrak{X}$.*

Proof. Let $\mathscr{S}$ be a finite system of equations and inequalities with coefficients from G and a solution in some $\mathfrak{X}$-supergroup H of G. We may assume that H is e.c. in $\mathfrak{X}$. Let $K = \bigcup \{N \mid N \trianglelefteq H \text{ and } N \cap G = 1\}$. Assume that there exists a minimal normal subgroup M/K in H/K. If $g \in (M \cap G) - 1$, then $M = \langle g^H \rangle$ by [5, Theorem 2.3]; and since G is a.c. in $\mathfrak{X}$, we conclude that $M \cap G = \langle g^G \rangle$. Thus, $M \cap G$ is a minimal normal subgroup in G, in contradiction to our assumption. Therefore, H/K has no minimal normal subgroup. Because of $G \cap K = 1$ we may identify G canonically with $GK/K \leq H/K$. Then the system $\mathscr{S} = \mathscr{S}K/K$ has a solution in H/K by Theorem 3.2. But $G \cap L \neq 1$ for every non-trivial normal subgroup L of H/K, whence G is e.c. in $\mathfrak{X}$ by [8, Lemma 3].

THEOREM 4.5. ([8, Theorem D(c)]). *If every e.c. $\mathfrak{X}$-group satisfies* (3.2), *then the a.c. $\mathfrak{X}$-groups are precisely the factor groups H/N of the e.c. $\mathfrak{X}$-groups H by their normal subgroups N satisfying $N \neq \langle h^H \rangle$ for all $h \in H - 1$.*

Proof. By Theorem 4.3 it suffices to show that every a.c. $\mathfrak{X}$-group G occurs as a suitable factor of some e.c. $\mathfrak{X}$-group H. If G has no minimal normal subgroup, we may apply Theorem 4.4 and choose $H = G$. If there exists a minimal normal subgroup M of G, we may follow the proof of [8, Theorem D(a)], provided that the following can be shown.

If $G \leq \tilde{H} \in \mathfrak{X}$, and if $\tilde{K}/\tilde{L}$ is a chief factor in $\tilde{H}$ such that $\tilde{K} - \tilde{L}$ contains some $c \in M$, then $\tilde{L} \cap G = 1 < M \leq \tilde{K} \cap G$. (4.1)

However, since the normal subgroups of G are totally ordered under inclusion by [8, Proposition (b)], we immediately obtain that $\tilde{L} \cap G \lneqq M$, whence $\tilde{L} \cap G = 1$. Moreover, $M = \langle c^G \rangle \leq \tilde{K} \cap G$ (thus (4.1) obtains and this completes the proof of Theorem 4.5).

5. Amalgamation in $L(\mathfrak{F}_\pi \cap \mathfrak{S})$. In [7] we gave a necessary and sufficient condition for an amalgam of finite soluble π-groups to be contained in a finite soluble π-group. Combining the construction of Section 2 with the technique of [7] we are able to extend this result to amalgams of $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-groups over a finite common subgroup.

THEOREM 5.1. ([7, Theorem 2], [5, Theorem 2.1]). *An amalgam $G \cup H \mid U$ of $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-groups G and H over a finite common subgroup U is contained in an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-group, if and only if there exist normal series Σ_G in G and Σ_H in H with elementary-abelian factors, such that $\Sigma_G \cap U = \Sigma_H \cap U$, and such that the following condition holds:*

(*) *whenever M/N and K/L are factors of Σ_G resp. Σ_H satisfying $M \cap U = K \cap U > L \cap U = N \cap U$, then there exists an elementary-abelian group E containing the amalgam*

$$M/N \cup K/L \mid (U \cap M)N/N = (U \cap K)L/L$$

(where $(U \cap M)N/N$ and $(U \cap K)L/L$ are identified via $uN = uL$ for all $u \in U \cap M = U \cap K$), and there exist homomorphisms $\alpha: G/M \to \mathrm{Aut}(E)$ and $\beta: H/K \to \mathrm{Aut}(E)$ such that every $(Mg)\alpha$ acts on UN/N as conjugation with Ng, such that every $(Kh)\beta$ acts on UL/L as conjugation with Lh, and such that $A = \langle \mathrm{Im}\,\alpha, \mathrm{Im}\,\beta \rangle$ is an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-subgroup of $\mathrm{Aut}(E)$.

Proof. If $G \cup H \mid U$ is contained in an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-group W, then any chief series Σ_W in W induces normal series Σ_G and Σ_H in G resp. H with elementary-abelian factors, and

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such that $\Sigma_G \cap U = \Sigma_H \cap U$. Moreover (*) is satisfied, if we choose for E the corresponding factor of Σ_W and define α and β via action by conjugation.

For the proof of the converse, we proceed by induction over the length of $\Sigma_G \cap U$. In the case when $U = 1$, the amalgam is contained in $G \times H \in L(\mathfrak{F}_\pi \cap \mathfrak{S})$. Assume now, that there exist elementary-abelian factors M/N in G and K/L in H such that $M \cap U = K \cap U > L \cap U = N \cap U = 1$, and such that the amalgam

$$G/M \cup H/K \mid UM/M = UK/K$$

is contained in an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-group D. Let E, A, α and β be as in (*). Denote epimorphic images modulo N resp. L by bars. As in the proof of [7, Theorem 2], we can define an embedding η of the amalgam $\bar{G} \cup \bar{H} \mid \bar{U}$ into $(E \rtimes A)\ Wr\ D$. Observe that $\bar{g}\eta = f_g \cdot Mg$ and $\bar{h}\eta = f_h \cdot Kh$ for all $g \in G$, $h \in H$, where $\operatorname{Im} f_g \subseteq E \rtimes (\langle g, U\rangle M/M)\alpha$ and $\operatorname{Im} f_h \subseteq E \rtimes (\langle h, U\rangle M/M)\beta$. Thus,

$$\operatorname{Im} \eta \subseteq Z = \bigcup \{(E \rtimes A_0) Wr\, D \mid A_0 \leq A \text{ finite}\} \in L(\mathfrak{F}_\pi \cap \mathfrak{S}).$$

In the following, we suppress η and regard Z as an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-supergroup of $\bar{G} \cup \bar{H} \mid \bar{U}$.

Now, apply the construction of Section 2 to $^{-}: G \rightarrow G/N \leq Z$ with $R = U$. This yields embeddings $\sigma: Z \rightarrow W_0$ and $\tau: G \rightarrow W_0$, where W_0 is an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-subgroup of $G\ Wr\ Z$ as in Theorem 2.1. From our choice of R, we have that $\sigma \mid U = \tau \mid U$. Hence the above gives an embedding of $G \cup \bar{H} \mid U = \bar{U}$ into W_0. A further application of the construction of Section 2 yields an embedding of $G \cup H \mid U$ into an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-group.

As in the proof of [7, Theorem 5] it can be deduced from Theorem 5.1, that an amalgam $G \cup H \mid U$ of $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-groups G and H over a finite supersoluble common subgroup U is contained in an $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-group, if there exist chief series in G and H which induce a common chief series in U. (This also generalizes [6, Theorem 3.1].) This allows us to construct all kinds of embeddings of countable locally supersoluble π-groups into e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-groups (as in [5, §3]). Moreover, it can be shown as in [7, Theorem 6], that a finite supersoluble π-group is an amalgamation base in $L(\mathfrak{F}_\pi \cap \mathfrak{S})$, if and only if it is either a cyclic p-group, or the split extension of a cyclic p-group P by a cyclic q-group Q with $C_Q(P) = 1$ and $q \mid p - 1$.

Note also that the results of [5, §3] about embeddings of countable locally nilpotent π-groups into countable e.c. $\mathfrak{X}$-groups satisfying (3.2) carry over to uncountable e.c. $\mathfrak{X}$-groups. This can be proved easily with the technique of proof of Theorem 5.1.

Added October 2, 1989. It recently occurred to the author, that the construction of Section 2 is not limited to only locally finite groups. In fact, it may be used in the more general set up of [4], i.e., for the study of e.c. $L\mathfrak{X}$-groups, where the class $\mathfrak{X}$ is closed with respect to subgroups, quotients, extensions, and with respect to cartesian powers of finitely generated (f.g.) $\mathfrak{X}$-groups. This works, because the group W_0 of Theorem 2.1 is contained in the union of split extensions $W_X = \Delta_X \rtimes H\sigma$, where X ranges over all f.g. subgroups of G, and where $\Delta_X = \{f: H \rightarrow G \mid (\overline{tsr})f \in sXs^{-1} \text{ for all } t \in T, s \in S, r \in R\}$; the above assumptions ensure that $W_X \in \mathfrak{X}$. (The construction may even be simplified by deleting S and choosing T as left transversal of $\bar{U}$ in H.) This allows it to remove the countability assumptions from [4, Theorems 4.7–4.11 and 5.3]. Also, [5, Theorem 3.1] can be extended to embeddings of f.g. nilpotent $\mathfrak{X}$-groups into e.c. $L\mathfrak{X}$-groups (and correspondingly to embeddings of polycyclic $\mathfrak{X}$-groups into e.c. $L\mathfrak{X}$-groups with abelian chief factors, provided that $\mathfrak{X}$ contains all torsion-free divisible abelian groups).

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