A NOTE ON PRIME-POWER GROUPS

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In this paper an unsuccessful attempt to classify prime-power groups is described. The attempt consisted in combining some ideas of P. Hall and O. N. Golovin.

Golovin [2] defined nilpotent products of groups and showed that every nilpotent group is a factor group of a nilpotent product of cyclic groups.

DEFINITION. Let F be the free product of the cylic groups A_i, $F_1=F$, $F_{n+1}=(F_n, F) = \{u^{-1}v^{-1}uv \mid u \in F_n, v \in F\}$; then the (n-2)nd nilpotent product of the A_i is F/F_n (Golovin, [2]).

Let "nilpotent product of cyclic groups" be abbreviated npcg.

Hall [5] defined an equivalence relation among groups called isoclinism. Hall hoped that this equivalence relation might prove useful in classifying non-Abelian prime-power groups. (All Abelian groups are isoclinic to $\{1\}$.)

DEFINITION. Let G, G' be two groups with centers Z, Z' respectively. Let $H=(G,G) = \{g_1^{-1} g_2^{-1} g_1 g_2 \mid g_i \in G\}$, $H' = (G', G')$. Then according to Hall [5], G is isoclinic to G' if

1) $G/Z \cong G'/Z'$ ($\cong$ designates "is isomorphic to"),
2) $H \cong H'$,
3) the isomorphisms of 1) and 2) can be selected in such a way that whenever aZ and bZ correspond respectively to a'Z' and b'Z' under 1), then $(a,b) = a^{-1}b^{-1}ab$ corresponds to (a', b') under 2).

The following speculation occurred to the author.

PROPOSITION. Every finite nilpotent group is isoclinic to an npcg.

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This proposition appeared plausible because the two non-Abelian groups of order p^3, $p \geqslant 3$ and six groups of order p^4 are isoclinic to each other, and one of them is an npcg, the first nilpotent product of two cyclic groups of order p. It would be desirable for the proposition to be true because

1) npcgs can be defined in a simple way;
2) a unique decomposition theorem holds for npcgs ([4], 2.1, of which a simpler proof appears in [7], Theorem 6.14);
3) isoclinic groups have similar structures.

Unfortunately this proposition is false as shown by the following theorem.

THEOREM. There exists a group of order p^4, $p \geqslant 3$ which is not isoclinic to an npcg. In particular, let

$$G = \{a, b, c \mid a^{p^2} = b^p = c^p = 1,\ b^{-1}ab = a^{p+1},\ c^{-1}ac = ab,\ c^{-1}bc = b\}.$$

G is not isoclinic to an npcg.

The proof of this theorem follows from the following. (Henceforth, G stands for the group appearing in the statement of the theorem.)

1) Isoclinic groups have the same class of nilpotency; (Hall, [5]) i.e., let K be any group and let $K_1 = K$, $K_{n+1} = (K_n, K) = \{u^{-1}v^{-1}uv \mid u \in K_n,\ v \in K\}$. If $K_c \neq \{1\}$, $K_{c+1} = \{1\}$, then the class of nilpotency of K is c. Abelian groups are of class 1.

2) $G_3 \neq \{1\}$, $G_4 = \{1\}$; i.e., G is of class 3 (of nilpotency). ([1] p. 145; G is group (xi).)

3) The only npcgs which might be isoclinic to G are second nilpotent products. The smallest prime power group, $p \geq 3$, of this type is the second nilpotent product to two cyclic groups of order p. Its order is p^5 (see lemma below). Its center is contained in its commutator subgroup and hence its order is minimal among groups isoclinic to it (Hall, [5]). Since G is of order p^4, and also minimal among its isoclinic "brothers", it cannot be isoclinic to an npcg.

LEMMA. Let $A = \{a\}$, $B = \{b\}$, $a^p = b^p = 1$, p a prime $\geqslant 3$. Let $F = A*B$ be the free product of A and B, $H = (A*B)/F_4 = A(2)B$ (the second nilpotent product of A and B). Then H is of order p^5 and every element of H can be uniquely expressed as

$$a^{\alpha} b^{\beta} (a,b)^{\gamma} ((a,b),a)^{\delta} ((a,b),b)^{\varepsilon} \tag{1}$$

where α, β, γ, δ, ε are integers modulo p.

Proof. (This lemma is a special case of theorem 1 of [9].)

According to [6],

$$((a,b),b)^p \equiv ((a,b),a)^p \equiv ((a^p,b),a) \equiv 1 \text{ modulo } F_4 .$$

According to a lemma of [8],

$$(a^p,b) \equiv (a,b)^p ((a,b),a)^{\binom{p}{2}} \equiv 1 \text{ modulo } F_4 .$$

Hence (a,b) is of order p in H, and every element of H can be expressed in the form (1). If (1) is unique, the lemma is true. To show this, consider two elements of form (1):

$$X = a^{\alpha_1} b^{\beta_1} (a,b)^{\gamma_1} ((a,b),a)^{\delta_1} ((a,b),b)^{\varepsilon_1}$$

$$Y = a^{\alpha_2} b^{\beta_2} (a,b)^{\gamma_2} ((a,b),b)^{\delta_2} ((a,b),b)^{\varepsilon_2}$$

If

$$X \cdot Y = Z = a^{\alpha_3} b^{\beta_3} (a,b)^{\gamma_3} ((a,b),a)^{\delta_3} ((a,b),b)^{\varepsilon_3}$$

then

$$\begin{aligned} \alpha_3 &= \alpha_1 + \alpha_2 \\ \beta_3 &= \beta_1 + \beta_2 \\ \gamma_3 &= \gamma_1 + \gamma_2 - \beta_1 \alpha_2 \\ \delta_3 &= \delta_1 + \delta_2 + \gamma_1 \alpha_2 - \beta_1 \tbinom{\alpha_2}{2} \\ \varepsilon_3 &= \varepsilon_1 + \varepsilon_2 - \alpha_2 \tbinom{\beta_1}{2} + \gamma_1 \beta_2 - \alpha_2 \beta_1 \beta_2 . \end{aligned} \tag{2}$$

Consider the set K consisting of 5-tuples $(\alpha, \beta, \gamma, \delta, \varepsilon)$ where $\alpha, \beta, \gamma, \delta, \varepsilon$ are integers modulo p, and if $M = (\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1)$ and $N = (\alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2) \in K$, then $M \cdot N = (\alpha_3, \beta_3, \gamma_3, \delta_3, \varepsilon_3)$ is given by (2). K forms a group with $(0,0,0,0,0)$ as the identity.

That K is isomorphic to H follows from the fact that K_4 contains only (0,0,0,0,0). This is sufficient to prove the Lemma.

REMARK. Another difficulty of combining Hall's classification scheme with Golovin's npcgs is that some npcgs are isoclinic to each other, e.g., if A and B are cyclic groups of order p, and C is the cylic group of order p^2, then A(1)B (the first nilpotent product of A and B) is isoclinic to A(1)C [3]. Hence even if every finite nilpotent group were isoclinic to an npcg, the npcgs do not form a system of distinct "canonical" groups.

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