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SOME INEQUALITIES OF JENSEN TYPE FOR ARG-SQUARE CONVEX FUNCTIONS OF UNITARY OPERATORS IN HILBERT SPACES

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Abstract

Some inequalities of Jensen type for Arg-square-convex functions of unitary operators in Hilbert spaces are given.

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1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. We recall that the bounded linear operator $U : H \to H$ on the Hilbert space H is *unitary* if and only if $U^* = U^{-1}$.

It is well known that (see for instance [4, pages 275–276]) if U is a unitary operator then there exists a family of *projections* $\{E_\lambda\}_{\lambda \in [0,2\pi]}$, called the *spectral family* of U, with the following properties:

(a) $E_\lambda \le E_\mu$ for $0 \le \lambda \le \mu \le 2\pi$;
(b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the *identity operator* on H);
(c) $E_{\lambda+0} = E_\lambda$ for $0 \le \lambda < 2\pi$;
(d) $U = \int_0^{2\pi} e^{i\lambda}\, dE_\lambda$, where the integral is of *Riemann–Stieltjes* type.

Moreover, if $\{F_\lambda\}_{\lambda \in [0,2\pi]}$ is a family of projections satisfying the requirements (a)–(d) for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex-valued function $f : C(0, 1) \to \mathbb{C}$ on the complex unit circle $C(0, 1)$,

$$f(U) = \int_0^{2\pi} f(e^{i\lambda})\, dE_\lambda,$$

where the integral is taken in the Riemann–Stieltjes sense.

In particular, we have the equalities

$$f(U)x = \int_0^{2\pi} f(e^{i\lambda})\, dE_\lambda x,$$
$$\langle f(U)x, y\rangle = \int_0^{2\pi} f(e^{i\lambda})\, d\langle E_\lambda x, y\rangle$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} |f(e^{i\lambda})|^2\, d\|E_\lambda x\|^2, \tag{1.1}$$

for any $x, y \in H$.

For $z \in \mathbb{C}\backslash\{0\}$ we call the *principal value* of $\log(z)$ the complex number

$$\mathrm{Log}(z) := \ln|z| + i\,\mathrm{Arg}(z),$$

where $0 \le \mathrm{Arg}(z) < 2\pi$.

We observe that for $t \in [0, 2\pi)$ we have $\mathrm{Log}(e^{it}) = it$.

If we consider the continuous function $g : [0, 2\pi] \to \mathbb{C}$,

$$g(t) := \begin{cases} \mathrm{Log}(e^{it}) = it & \text{if } t \in [0, 2\pi), \\ 2\pi i & \text{if } t = 2\pi, \end{cases}$$

then we can define a bounded linear operator denoted by $\mathrm{Log}(U) : H \to H$ as follows:

$$\mathrm{Log}(U)x := \int_0^{2\pi} g(\lambda)\, dE_\lambda x = \int_0^{2\pi} (i\lambda)\, dE_\lambda x, \;\; x \in H.$$

In what follows we establish some results connecting this operator with the function of operator $f(U)$ for a class of function we call *Arg-square-convex* such that a Jensen type inequality and related results can be derived.

2. The results

The function $f : C(0, 1) \to \mathbb{C}$ will be called *Arg-square-convex* if the composite function $\varphi : [0, 2\pi] \to [0, \infty)$,

$$\varphi(t) = \begin{cases} |f(e^{it})|^2 & t \in [0, 2\pi), \\ \lim_{s\to 2\pi-} |f(e^{is})|^2 & t = 2\pi, \end{cases}$$

is continuous and convex on $[0, 2\pi]$.

To make the distinction between the value $\varphi(0) = |f(e^{i0})|^2 = |f(1)|^2$ and the value $\varphi(2\pi) = \lim_{s\to 2\pi-} |f(e^{is})|^2$, we write by $f_c(1) := \lim_{s\to 2\pi-} f(e^{is})$. With this notation, $\varphi(2\pi) = |f_c(1)|^2$.

The function $f_n : C(0, 1) \to \mathbb{C}$, $f_n(z) = (\mathrm{Log}(z))^n$, where n is a positive integer, is Arg-square-convex. We have

$$\varphi_n(t) = |f_n(e^{it})|^2 = |(\mathrm{Log}(e^{it}))^n|^2 = |it|^{2n} = t^{2n}, \quad t \in [0, 2\pi),$$

and

$$\varphi_n(2\pi) = \lim_{s\to 2\pi-} |f_n(e^{is})|^2 = |f_{n,c}(1)|^2 = (2\pi)^{2n}.$$

For $q \geq \frac{1}{2}$, define the function $f_q : C(0,1) \to [0,\infty)$ by $f_q(z) = |\mathrm{Log}(z)|^q$. We have

$$\varphi_q(t) = |f_q(e^{it})|^2 = |\mathrm{Log}(e^{it})|^{2q} = |it|^{2q} = t^{2q}, \quad t \in [0,2\pi),$$

and

$$\varphi_q(2\pi) = \lim_{s\to 2\pi-} |f_q(e^{is})|^2 = |f_{q,c}(1)|^2 = (2\pi)^{2q}.$$

The function f_q for $q \geq \frac{1}{2}$ is an Arg-square-convex function.

If $g : [0,2\pi] \to [0,\infty)$ is continuous and convex on $[0,2\pi]$, then the composite function $f : C(0,1) \to [0,\infty)$ defined by

$$f(z) := (g(|\mathrm{Log}(z)|))^{1/2}$$

is an Arg-square-convex function on $C(0,1)$.

The following Jensen's type result holds.

THEOREM 2.1. *Let $U \in B(H)$ be a unitary operator on the Hilbert space H and $f : C(0,1) \to \mathbb{C}$ an Arg-square-convex function on $C(0,1)$. Then*

$$\begin{aligned}\Big(\frac{|f(1)|^2(\langle(2\pi 1_H - |\mathrm{Log}(U)|)x,x\rangle) + |f_c(1)|^2\langle|\mathrm{Log}(U)|x,x\rangle}{2\pi}\Big)^{1/2}&\\ \geq \|f(U)x\| \geq |f(e^{\langle \mathrm{Log}\,Ux,x\rangle})|,&\end{aligned} \tag{2.1}$$

for any $x \in H$, $\|x\| = 1$, where $f_c(1) := \lim_{s\to 2\pi-} f(e^{is})$.

PROOF. Since f is continuous on $C(0,1)$ and U is a unitary operator, then, by (1.1),

$$\|f(U)x\|^2 = \int_0^{2\pi} |f(e^{i\lambda})|^2\, d\|E_\lambda x\|^2 = \int_0^{2\pi} |f(e^{i\lambda})|^2\, d\langle E_\lambda x, x\rangle$$

for any $x \in H$, $\|x\| = 1$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

Now, since $|f(e^{i\cdot})|^2$ is continuous convex on $[0,2\pi]$, then, by Jensen's integral inequality for the Riemann–Stieltjes integral with monotonic nondecreasing integrators,

$$\frac{\int_0^{2\pi} |f(e^{i\lambda})|^2\, d\langle E_\lambda x,x\rangle}{\int_0^{2\pi} d\langle E_\lambda x,x\rangle} \geq \Big|f\Big(\exp\Big(i\frac{\int_0^{2\pi} \lambda\, d\langle E_\lambda x,x\rangle}{\int_0^{2\pi} d\langle E_\lambda x,x\rangle}\Big)\Big)\Big|^2 \tag{2.2}$$

for any $x \in H$, $\|x\| = 1$.

Since

$$\int_0^{2\pi} d\langle E_\lambda x,x\rangle = \|x\|^2 = 1$$

and

$$\int_0^{2\pi} (i\lambda)\, d\langle E_\lambda x, x\rangle = \int_0^{2\pi} \operatorname{Log}(e^{i\lambda})\, d\langle E_\lambda x, x\rangle = \langle \operatorname{Log} U x, x\rangle$$

for any $x \in H$, $\|x\| = 1$, then we get from (2.2) the second inequality in (2.1).

Now, if $\varphi : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$ then for any $\lambda \in [a, b]$ we have the inequality

$$\frac{(b-\lambda)\varphi(a) + (\lambda - a)\varphi(b)}{b-a} \geq \varphi(\lambda).$$

If we write this inequality for the continuous convex function $\varphi(t) = |f(e^{it})|^2$ on the interval $[0, 2\pi]$, then

$$\frac{(2\pi-\lambda)|f(1)|^2 + \lambda |f_c(1)|^2}{2\pi} \geq |f(e^{i\lambda})|^2$$

for any $\lambda \in [0, 2\pi]$.

Integrating on $[0, 2\pi]$ over the monotonic nondecreasing integrator $\langle E_\lambda x, x\rangle$,

$$\frac{|f(1)|^2(2\pi - \int_0^{2\pi} \lambda\, d\langle E_\lambda x, x\rangle) + |f_c(1)|^2 \int_0^{2\pi} \lambda\, d\langle E_\lambda x, x\rangle}{2\pi} \geq \int_0^{2\pi} |f(e^{i\lambda})|^2\, d\langle E_\lambda x, x\rangle$$

for any $x \in H$, $\|x\| = 1$.

Now observe that the Riemann–Stieltjes integral $\int_0^{2\pi} \lambda\, d\langle E_\lambda x, x\rangle$ exists and can be written as

$$\int_0^{2\pi} \lambda\, d\langle E_\lambda x, x\rangle = \int_0^{2\pi} |\operatorname{Log}(e^{i\lambda})|\, d\langle E_\lambda x, x\rangle = \langle |\operatorname{Log}(U)| x, x\rangle$$

for any $x \in H$, $\|x\| = 1$.

The proof is complete. □

The following result also holds.

THEOREM 2.2. *Let $U \in B(H)$ be a unitary operator on the Hilbert space H and $f : C(0, 1) \to \mathbb{C}$ an Arg-square-convex function on $C(0, 1)$. Then*

$$\begin{aligned} &\frac{1}{\pi}\Big(\frac{|f(1)|^2 + |f_c(1)|^2}{2} - |f(-1)|^2\Big)\langle(\pi 1_H - |\operatorname{Log}(U) - i\pi 1_H|)x, x\rangle \\ &\quad\leq \frac{|f(1)|^2(\langle(2\pi 1_H - |\operatorname{Log}(U)|)x, x\rangle) + |f_c(1)|^2\langle|\operatorname{Log}(U)|x, x\rangle}{2\pi} - \|f(U)x\|^2 \qquad (2.3)\\ &\quad\leq \frac{1}{\pi}\Big(\frac{|f(1)|^2 + |f_c(1)|^2}{2} - |f(-1)|^2\Big)\langle(\pi 1_H + |\operatorname{Log}(U) - i\pi 1_H|)x, x\rangle, \end{aligned}$$

for any $x \in H$, $\|x\| = 1$.

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PROOF. First, we recall the following result obtained by the author in [1] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} n \min_{i\in\{1,\dots,n\}}\{p_i\}\Big(\frac{1}{n}\sum_{i=1}^n \Phi(x_i) - \Phi\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big)\Big) &\\ \leq \frac{1}{P_n}\sum_{i=1}^n p_i\Phi(x_i) - \Phi\Big(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\Big) &\\ \leq n \max_{i\in\{1,\dots,n\}}\{p_i\}\Big(\frac{1}{n}\sum_{i=1}^n \Phi(x_i) - \Phi\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big)\Big), & \end{aligned} \tag{2.4}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^n p_i > 0$.

For $n = 2$ we deduce from (2.4) that

$$\begin{aligned} 2\min\{t, 1-t\}\Big(\frac{\Phi(x)+\Phi(y)}{2} - \Phi\Big(\frac{x+y}{2}\Big)\Big) &\\ \leq t\Phi(x) + (1-t)\Phi(y) - \Phi(tx + (1-t)y) &\\ \leq 2\max\{t, 1-t\}\Big(\frac{\Phi(x)+\Phi(y)}{2} - \Phi\Big(\frac{x+y}{2}\Big)\Big) & \end{aligned}$$

for any $x, y \in C$ and $t \in [0, 1]$.

Now, if $\varphi : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$, then, for any $\lambda \in [a, b]$,

$$\begin{aligned} 2\min\Big\{\frac{b-\lambda}{b-a}, \frac{\lambda-a}{b-a}\Big\}\Big(\frac{\varphi(a)+\varphi(b)}{2} - \varphi\Big(\frac{a+b}{2}\Big)\Big) &\\ \leq \frac{(b-\lambda)\varphi(a) + (\lambda-a)\varphi(b)}{b-a} - \varphi(\lambda) &\\ \leq 2\max\Big\{\frac{b-\lambda}{b-a}, \frac{\lambda-a}{b-a}\Big\}\Big(\frac{\varphi(a)+\varphi(b)}{2} - \varphi\Big(\frac{a+b}{2}\Big)\Big). & \end{aligned} \tag{2.5}$$

If we write (2.5) for the continuous convex function $\varphi(t) = |f(e^{it})|^2$ on the interval $[0, 2\pi]$, then

$$\begin{aligned} \frac{1}{\pi}\min\{2\pi-\lambda, \lambda\}\Big(\frac{|f(1)|^2 + |f_c(1)|^2}{2} - |f(-1)|^2\Big) &\\ \leq \frac{(2\pi-\lambda)|f(1)|^2 + \lambda|f_c(1)|^2}{2\pi} - |f(e^{i\lambda})|^2 &\\ \leq \frac{1}{\pi}\max\{2\pi-\lambda, \lambda\}\Big(\frac{|f(1)|^2 + |f_c(1)|^2}{2} - |f(-1)|^2\Big) & \end{aligned}$$

for any $\lambda \in [0, 2\pi]$.

Let $x \in H$ with $\|x\| = 1$. Integrating on $[0, 2\pi]$ over the monotonic nondecreasing integrator $\langle E_\lambda x, x\rangle$,

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$$
\begin{aligned}
\frac{1}{\pi}\Big(&\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\Big)\int_0^{2\pi}\min\{2\pi-\lambda,\lambda\}\,d\langle E_\lambda x,x\rangle \\
&\le \frac{|f(1)|^2(2\pi-\int_0^{2\pi}\lambda\,d\langle E_\lambda x,x\rangle)+|f_c(1)|^2\int_0^{2\pi}\lambda\,d\langle E_\lambda x,x\rangle}{2\pi} \\
&\quad -\int_0^{2\pi}|f(e^{i\lambda})|^2\,d\langle E_\lambda x,x\rangle \\
&\le \frac{1}{\pi}\Big(\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\Big)\int_0^{2\pi}\max\{2\pi-\lambda,\lambda\}\,d\langle E_\lambda x,x\rangle
\end{aligned}
\tag{2.6}
$$

and since

$$
\begin{aligned}
\int_0^{2\pi}&\min\{2\pi-\lambda,\lambda\}\,d\langle E_\lambda x,x\rangle \\
&=\int_0^{2\pi}(\pi-|\lambda-\pi|)\,d\langle E_\lambda x,x\rangle=\pi-\int_0^{2\pi}|\lambda-\pi|\,d\langle E_\lambda x,x\rangle \\
&=\pi-\int_0^{2\pi}|\lambda-\pi|\,d\langle E_\lambda x,x\rangle=\pi-\int_0^{2\pi}|i\lambda-i\pi|\,d\langle E_\lambda x,x\rangle \\
&=\pi-\int_0^{2\pi}|\mathrm{Log}(e^{it})-i\pi|\,d\langle E_\lambda x,x\rangle=\pi-\langle|\mathrm{Log}(U)-i\pi 1_H|x,x\rangle \\
&=\langle(\pi 1_H-|\mathrm{Log}(U)-i\pi 1_H|)x,x\rangle
\end{aligned}
$$

and similarly

$$
\int_0^{2\pi}\max\{2\pi-\lambda,\lambda\}\,d\langle E_\lambda x,x\rangle=\langle(\pi 1_H+|\mathrm{Log}(U)-i\pi 1_H|)x,x\rangle,
\tag{2.7}
$$

then by (2.6)–(2.7) we get the desired result (2.3). □

In the following, an upper bound for the nonnegative difference

$$
\|f(U)x\|^2-|f(e^{\langle \mathrm{Log}\,Ux,x\rangle})|^2,
$$

where $x\in H$ with $\|x\|=1$, is also provided.

THEOREM 2.3. *Let $U\in B(H)$ be a unitary operator on the Hilbert space H and $f:C(0,1)\to\mathbb{C}$ an Arg-square-convex function on $C(0,1)$. Then*

$$
\begin{aligned}
0&\le\|f(U)x\|^2-|f(e^{\langle \mathrm{Log}\,Ux,x\rangle})|^2 \\
&\le\frac{1}{\pi}\max\{\langle(2\pi 1_H-|\mathrm{Log}(U)|)x,x\rangle,\langle|\mathrm{Log}(U)|x,x\rangle\} \\
&\quad\times\Big(\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\Big) \\
&\le 2\Big(\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\Big)
\end{aligned}
\tag{2.8}
$$

for any $x\in H$, $\|x\|=1$.

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PROOF. By the convexity of the function $\varphi(t) = |f(e^{it})|^2$ on the interval $[0, 2\pi]$,

$$\begin{aligned}
&\|f(U)x\|^2 - |f(e^{\langle \operatorname{Log} U x, x\rangle})|^2 \\
&\quad = \int_0^{2\pi} |f(e^{i\lambda})|^2 \, d\langle E_\lambda x, x\rangle - \left| f\left(\exp\left(i \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle\right)\right)\right|^2 \\
&\quad = \int_0^{2\pi} |f(e^{i(((2\pi-\lambda)/2\pi)\cdot 0 + (\lambda/2\pi)\cdot 2\pi)})|^2 \, d\langle E_\lambda x, x\rangle \\
&\qquad - \left| f\left(\exp\left(i \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle\right)\right)\right|^2 \\
&\quad \le \int_0^{2\pi} \left(\frac{2\pi - \lambda}{2\pi}|f(1)|^2 + \frac{\lambda}{2\pi}|f_c(1)|^2\right) d\langle E_\lambda x, x\rangle \\
&\qquad - \left| f\left(\exp\left(i \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle\right)\right)\right|^2 \\
&\quad = \frac{|f(1)|^2 (2\pi - \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle) + |f_c(1)|^2 \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle}{2\pi} \\
&\qquad - \left| f\left(\exp\left(i \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle\right)\right)\right|^2
\end{aligned} \tag{2.9}$$

for any $x \in H$, $\|x\| = 1$.

Applying the second inequality from (2.5) for the convex function $\varphi(t) = |f(e^{it})|^2$ on the interval $[0, 2\pi]$ and for the intermediate point $\int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle \in [0, 2\pi]$ we can write that

$$\begin{aligned}
&\frac{|f(1)|^2 (2\pi - \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle) + |f_c(1)|^2 \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle}{2\pi} \\
&\qquad - \left| f\left(\exp\left(i \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle\right)\right)\right|^2 \\
&\quad \le 2 \max\left\{ \frac{2\pi - \int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle}{2\pi}, \frac{\int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle}{2\pi} \right\} \\
&\qquad \times \left(\frac{|f(1)|^2 + |f_c(1)|^2}{2} - |f(-1)|^2 \right)
\end{aligned} \tag{2.10}$$

for any $x \in H$, $\|x\| = 1$.

Since, as above,

$$\int_0^{2\pi} \lambda \, d\langle E_\lambda x, x\rangle = \langle |\operatorname{Log}(U)| x, x\rangle,$$

for any $x \in H$, $\|x\| = 1$, then we deduce from (2.9) and (2.10) the desired result (2.8). □

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3. Examples

Let $U \in B(H)$ be a unitary operator on the Hilbert space H. Then, for a natural number $n \geq 1$,

$$\begin{aligned}(2\pi)^{n-1/2}\langle|\mathrm{Log}(U)|x, x\rangle^{1/2} &\geq \|(\mathrm{Log}(U))^n x\| \\ &\geq |\ln|\langle \mathrm{Log}\, Ux, x\rangle| + i\,\mathrm{Arg}(\langle \mathrm{Log}\, Ux, x\rangle)|^n,\end{aligned}$$

for any $x \in H$, $\|x\| = 1$. This follows from (2.1) applied for the function $f_n : C(0, 1) \to \mathbb{C}$, $f_n(z) = (\mathrm{Log}(z))^n$.

If we apply the same inequality for $f_q : C(0, 1) \to [0, \infty)$, $f_q(z) = |\mathrm{Log}(z)|^q$, then

$$\begin{aligned}(2\pi)^{q-1/2}\langle|\mathrm{Log}(U)|x, x\rangle^{1/2} &\geq \|\,|\mathrm{Log}(U)|^q x\| \\ &\geq |\ln|\langle \mathrm{Log}\, Ux, x\rangle| + i\,\mathrm{Arg}(\langle \mathrm{Log}\, Ux, x\rangle)|^q,\end{aligned}$$

for any $x \in H$, $\|x\| = 1$ and $q \geq \frac{1}{2}$.

Now, if we use the inequality (2.3) for the function $f_n(z) = (\mathrm{Log}(z))^n$, then

$$\begin{aligned}&(2^{2n-1} - 1)\pi^{2n-1}\langle(\pi 1_H - |\mathrm{Log}(U) - i\pi 1_H|)x, x\rangle \\ &\quad\leq (2\pi)^{2n-1}\langle|\mathrm{Log}(U)|x, x\rangle - \|(\mathrm{Log}(U))^n x\|^2 \\ &\quad\leq (2^{2n-1} - 1)\pi^{2n-1}\langle(\pi 1_H + |\mathrm{Log}(U) - i\pi 1_H|)x, x\rangle,\end{aligned}$$

for any $x \in H$, $\|x\| = 1$, where n is a natural number with $n \geq 1$.

The same inequality applied for $f_q(z) = |\mathrm{Log}(z)|^q$ provides

$$\begin{aligned}&(2^{2q-1} - 1)\pi^{2q-1}\langle(\pi 1_H - |\mathrm{Log}(U) - i\pi 1_H|)x, x\rangle \\ &\quad\leq (2\pi)^{2q-1}\langle|\mathrm{Log}(U)|x, x\rangle - \|\,|\mathrm{Log}(U)|^q x\|^2 \\ &\quad\leq (2^{2q-1} - 1)\pi^{2q-1}\langle(\pi 1_H + |\mathrm{Log}(U) - i\pi 1_H|)x, x\rangle,\end{aligned}$$

for any $x \in H$, $\|x\| = 1$ and $q \geq \frac{1}{2}$.

Finally, if we use the first inequality from (2.8), we also get

$$\begin{aligned}0 &\leq \|(\mathrm{Log}(U))^n x\|^2 - |\ln|\langle \mathrm{Log}\, Ux, x\rangle| + i\,\mathrm{Arg}(\langle \mathrm{Log}\, Ux, x\rangle)|^n \\ &\leq (2^{2n-1} - 1)\pi^{2n-1}\max\{\langle(2\pi 1_H - |\mathrm{Log}(U)|)x, x\rangle, \langle|\mathrm{Log}(U)|x, x\rangle\} \\ &\leq 2(2^{2n-1} - 1)\pi^{2n}\end{aligned}$$

for any $x \in H$, $\|x\| = 1$, where n is a natural number with $n \geq 1$.

If $q \geq \frac{1}{2}$, then

$$\begin{aligned}0 &\leq \|\,|\mathrm{Log}(U)|^q x\|^2 - |\ln|\langle \mathrm{Log}\, Ux, x\rangle| + i\,\mathrm{Arg}(\langle \mathrm{Log}\, Ux, x\rangle)|^{2q} \\ &\leq (2^{2q-1} - 1)\pi^{2q-1}\max\{\langle(2\pi 1_H - |\mathrm{Log}(U)|)x, x\rangle, \langle|\mathrm{Log}(U)|x, x\rangle\} \\ &\leq 2(2^{2q-1} - 1)\pi^{2q}\end{aligned}$$

for any $x \in H$, $\|x\| = 1$.

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If $g : [0, 2\pi] \to [0, \infty)$ is continuous and convex on $[0, 2\pi]$, then the composite function $f : C(0, 1) \to [0, \infty)$ defined by

$$f(z) := (g(|\mathrm{Log}(z)|))^{1/2}$$

is an Arg-square-convex function on $C(0, 1)$.

As examples of such functions we have

$$f_\alpha(z) := \exp(\alpha|\mathrm{Log}(z)|);$$

these are Arg-square-convex functions on $C(0, 1)$ for any real number $\alpha \neq 0$.

We also notice that the functions $f_{m,n} : C(0, 1) \to \mathbb{C}$, $f_{m,n}(z) = z^m(\mathrm{Log}(z))^n$, where $m \neq 0$ is an integer and n is a positive integer, are Arg-square-convex functions.

The reader may apply the above inequalities for these functions as well; the details are omitted here.

For Jensen's type inequalities for functions of selfadjoint operators see the recent book [2]. For related results, see [3].

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