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ON THE TOPOLOGICAL STRUCTURE OF AFFINELY CONNECTED MANIFOLDS

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Introduction

The purpose of the present paper is to investigate the relationship between the topological structure and differential geometric objects for affinely connected manifolds.

Let M be a compact, connected and oriented Riemannian manifold, $P^r(M)$ the vector space of all parallel r-forms on M and $b_r(M)$ the r-th Betti number of M. Since every parallel form is harmonic, it follows from the Hodge—de Rham theory that the inequality $\dim P^r(M) \leqq b_r(M)$ holds for all $r = 1, \cdots, \dim M$ (cf. [3], [5]). We shall generalize these inequalities to compact affinely connected manifolds.

Next, let M be a non-compact manifold. A connected submanifold N of M is called a *soul* if $\dim N < \dim M$ and if the inclusion $i: N \to M$ is a homotopy equivalence. J. Cheeger and D. Gromoll proved the following remarkable theorem. If M is a complete Riemannian manifold with non-negative sectional curvature then M has a compact soul (see [1] Theorem 1.11 and 2.1). We shall give another kind of sufficient conditions for M to have a (compact) soul.

Finally, a connected manifold M is said to be *reducible* if M is diffeomorphic to a product manifold $M_1 \times M_2$ with $\dim M_i \geqq 1$, $i = 1, 2$. Otherwise, M is said to be *irreducible*. We shall find a differential geometric condition for M to be reducible. We note that de Rham's Decomposition Theorem ([2]) furnishes a prototype for this condition (for irreducible manifolds, see [8]).

In order to obtain our results in a unified manner, we introduce certain family of functions on a connected manifold M with an affine connection Γ. A function f on M is called an *affine function* if, for every geodesic $c(t)$ with an affine parameter t, there are real constants a and b

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such that $f(c(t)) = at + b$ for all t. We can regard each affine function as a "harmonic mapping". In fact, if Γ is symmetric, then every affine function satisfies formally the defining equation of harmonic mappings (see [4] p. 116).

It is shown that the set $A(M, \Gamma)$ of all affine functions on M is a finite-dimensional real vector space and satisfies $1 \leqq \dim A(M, \Gamma) \leqq \dim M + 1$. By making use of $A(M, \Gamma)$, we shall define another finite-dimensional real vector space $V(M, \Gamma)$. Let $P^1(M, \Gamma)$ be the vector space of all parallel 1-forms on M and $k(M)$ the non-negative integer defined to be the largest k such that $H_k(M, \boldsymbol{Z}_2) \neq 0$, where $H_k(M, \boldsymbol{Z}_2)$ denotes the singular homology group of M with coefficient $\boldsymbol{Z}_2$. For simplicity, we suppose that Γ is complete and symmetric. Then we can state our main results as follows.

A. *If M is compact, then* $\dim P^1(M, \Gamma) \leqq b_1(M)$.

B. *If* $\dim A(M, \Gamma) > 1$, *then M has a soul. Moreover, we have* $k(M) \leqq \dim M - \dim A(M, \Gamma) + 1$. *The equality holds if and only if M has a compact soul N with* $\dim N = \dim M - \dim A(M, \Gamma) + 1$.

C. *If* $m = \dim V(M, \Gamma) > 0$, *then there exists a totally geodesic submanifold M' of M such that M is diffeomorphic to $\boldsymbol{R}^m \times M'$.*

We remark that M is not always affinely isomorphic to the product affinely connected manifold $\boldsymbol{R}^m \times M'$. However, in the Riemannian case, we can prove more (see [7]).

In Section 1 and Section 2, we shall study some basic properties of affine functions. In Section 3, we shall state our main theorems in a rigorous form. The proof of the theorems will be given in Section 4. The crucial point of the proof lies in a careful use of geodesics. In the last section, we shall consider affine symmetric spaces and prove the following result. Let $M = G/H$ be an affine symmetric space with the canonical connection Γ on G/H (see [10] Chap. III). If G is solvable and M is simply connected, then $\dim A(M, \Gamma) > 1$.

Throughout this paper, all manifolds and differential geometric objects on them are assumed to be differentiable of class C^∞. For brevity's sake, we shall often use the adjective "smooth" instead of "differentiable".

§ 1. Affine functions

Let M be a connected smooth manifold with an affine connection Γ. For a smooth curve $c(t)$ in M, we denote by $\dot{c}(t)$ the tangent vector to

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the curve at $c(t)$ and by $D\dot{c}(t)/dt$ the covariant derivative of $\dot{c}(t)$. A smooth curve $c(t)$ in M defined on an open interval I is called a *geodesic* if $D\dot{c}(t)/dt = 0$ on I. If c is a geodesic (as a point set), any parameter t with respect to which $c = c(t)$ is a geodesic is called an *affine parameter* of c. In this paper, all geodesics under consideration are assumed to be parametrized by affine parameter. The connection Γ is said to be *complete* if every geodesic can be extended to a geodesic $c(t)$ defined for all $t \in \boldsymbol{R}$, where $\boldsymbol{R}$ denotes the field of real numbers.

DEFINITION 1.1. A smooth function f on M is called an *affine function* on M if, for every geodesic $c(t)$, there are real constants a and b such that $f(c(t)) = at + b$ for any t whenever it is defined.

This definition does not depend on the choice of an affine parameter t because any other affine parameter t' is given by an affine transformation $t' = ct + d$, where $c \neq 0$ and d are real constants.

PROPOSITION 1.1. *Let f be an affine function on M. If the differential $(df)_x$ of f at some point $x \in M$ vanishes, then f is a constant function on M.*

Proof. Let N denote the subset of M consisting of all points y such that $(df)_y = 0$. Clearly, N is non-empty and closed in M. Let us take any $y \in N$ and any geodesic $c(t)$ with $c(0) = y$. Then we can put $f(c(t)) = at + b$ $(a, b \in \boldsymbol{R})$. Hence we have

$$a = \frac{d}{dt} f(c(t))|_{t=0} = (df)_y(\dot{c}(0)) = 0 \,.$$

This means that f is constant on every geodesic starting from the point y. Let U be a convex neighborhood of y (see [9] vol. 1, p. 149). Since every point of U can be joined to y by a geodesic segment, f is constant on U. Thus $U \subset N$ and hence N is open in M. Since M is connected, we can conclude that f is constant on M.

Let $A(M, \Gamma)$ denote the set of all affine functions of M. Then it is clear that $A(M, \Gamma)$ is a linear subspace of the real vector space of all smooth functions on M. Every real number can be identified with a constant function on M, so we get the natural inclusion $i : \boldsymbol{R} \to A(M, \Gamma)$.

PROPOSITION 1.2. *$A(M, \Gamma)$ is finite-dimensional and satisfies* $1 \leqq \dim A(M, \Gamma) \leqq \dim M + 1$. *Moreover, if M is compact, then* $\dim A(M, \Gamma) = 1$.

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Proof. Let us fix a point x of M. Let $F: A(M, \Gamma) \to T_x^*(M)$ denote the linear mapping given by $F(f) = (df)_x$ $(f \in A(M, \Gamma))$, where $T_x^*(M)$ is the cotangent space to M at x. Then Proposition 1.1 implies that the sequence $0 \to \boldsymbol{R} \xrightarrow{i} A(M, \Gamma) \xrightarrow{F} T_x^*(M)$ is exact. This proves the first and second assertions. The last assertion follows easily from Proposition 1.1 and the fact that every smooth function on a compact manifold has a critical point.

PROPOSITION 1.3. *If Γ is complete, then every bounded affine function f on M is a constant function on M.*

Proof. Let $x \in M$ and let $c(t)$ $(t \in \boldsymbol{R})$ be any geodesic with $c(0) = x$. Then we can put $f(c(t)) = at + b$ $(a, b \in \boldsymbol{R})$. Now the function $t \mapsto |at|$ on $\boldsymbol{R}$ is bounded, so $a = 0$. Thus we have $(df)_x(\dot{c}(0)) = 0$ and hence $(df)_x = 0$. Therefore the assertion follows immediately from Proposition 1.1.

PROPOSITION 1.4. *Let $1, f_1, \cdots, f_n$ be elements of $A(M, \Gamma)$. Then the following two statements are equivalent*:

1) *$1, f_1, \cdots, f_n$ are linearly independent in $A(M, \Gamma)$;*
2) *$df_1, \cdots, df_n$ are linearly independent at each point of M.*

Proof. Suppose 1). Let x be any point of M and assume that $\sum_{i=1}^n a_i(df_i)_x = 0$ for real constants $a_1, \cdots, a_n$. Then we have $(d(\sum_{i=1}^n a_i f_i))_x = 0$, so by Proposition 1.1 there is a constant b such that $\sum_{i=1}^n a_i f_i + b = 0$. Hence we get $a_i = 0$ for all i, which implies 2). The converse is obvious.

Now we set $a(M, \Gamma) = \dim A(M, \Gamma) - 1$.

PROPOSITION 1.5. *Let $\boldsymbol{R}^n$ be the n-dimensional affine space with the standard flat affine connection Γ_0. Then we have $a(\boldsymbol{R}^n, \Gamma_0) = n$.*

Proof. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system on $\boldsymbol{R}^n$. Then the coordinate functions $x_1, \cdots, x_n$ belong to $A(\boldsymbol{R}^n, \Gamma_0)$. Moreover, it follows from Propositions 1.2 and 1.4 that $1, x_1, \cdots, x_n$ form a basis of $A(\boldsymbol{R}^n, \Gamma_0)$. Hence we have $a(\boldsymbol{R}^n, \Gamma_0) = n$.

Let M' be another connected smooth manifold with an affine connection Γ'.

DEFINITION 1.2 (cf. [11]). A smooth mapping $h: M \to M'$ is said to be *totally geodesic* if, for every geodesic $c(t)$ of M, $h(c(t))$ is a geodesic of M'.

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For a smooth mapping $h: M \to M'$ and a smooth function f on M', we denote by $h^*(f)$ the smooth function on M given by $h^*(f) = f \circ h$. Then we have immediately the following proposition.

PROPOSITION 1.6. *If $h: M \to M'$ is a totally geodesic mapping, then $h^*(A(M', \Gamma')) \subset A(M, \Gamma)$ and $h^*: A(M', \Gamma') \to A(M, \Gamma)$ is a linear homomorphism. If moreover h is surjective, then $h^*: A(M', \Gamma) \to A(M, \Gamma)$ is injective.*

PROPOSITION 1.7. *Let M_i be a connected smooth manifold with an affine connection Γ_i $(i = 1, 2)$. Let $\Gamma_1 \times \Gamma_2$ denote the product affine connection on $M_1 \times M_2$. Then we have*

$$a(M_1 \times M_2, \Gamma_1 \times \Gamma_2) = a(M_1, \Gamma_1) + a(M_2, \Gamma_2)\,.$$

Proof. For simplicity, we write $A = A(M_1 \times M_2, \Gamma_1 \times \Gamma_2)$ and $A_i = A(M_i, \Gamma_i)$, $i = 1, 2$. Since the natural projection $p_i: M_1 \times M_2 \to M_i$ is totally geodesic, $p_i^*: A_i \to A$ is an injective homomorphism $(i = 1, 2)$. Let us fix a point (x_0, y_0) of $M_1 \times M_2$ $(x_0 \in M_1, y_0 \in M_2)$. Let $h_i: M_i \to M_1 \times M_2$, $i = 1, 2$, denote the smooth mappings given by $h_1(x) = (x, y_0)$ $(x \in M_1)$ and $h_2(y) = (x_0, y)$ $(y \in M_2)$, respectively. Clearly, we have $h_i^*(A) \subset A_i$, $i = 1, 2$. For any $f \in A$, we set

$$f = f - p_1^*(h_1^*(f)) - p_2^*(h_2^*(f)) + f(x_0, y_0)\,.$$

Then $\tilde{f}$ lies in A and satisfies $\tilde{f}(x_0, y_0) = 0$. It is not hard to verify that $d\tilde{f}$ vanishes at (x_0, y_0). It follows from Proposition 1.1 that $\tilde{f}$ vanishes identically on $M_1 \times M_2$. Hence,

$$f = p_1^*(h_1^*(f)) + p_2^*(h_2^*(f)) - f(x_0, y_0)\,.$$

This formula means that $A = p_1^*(A_1) + p_2^*(A_2)$. Since $p_1^*(A_1) \cap p_2^*(A_2)$ consists of all constant functions on $M_1 \times M_2$, it follows that $\dim A = \dim A_1 + \dim A_2 - 1$. This proves Proposition 1.7.

§ 2. Parallel 1-forms and affine functions

Let M be a connected smooth manifold with an affine connection Γ. Let T be the torsion tensor, R the curvature tensor and ∇ the covariant differentiation of Γ. Γ is said to be *symmetric* if T vanishes identically on M. Let f be any smooth function on M. We set

$$H_f(X, Y) = (\nabla_X df)(Y) + \tfrac{1}{2} df(T(X, Y))$$

for all vector fields X and Y on M. Then it is easy to see that H_f is a symmetric covariant 2-tensor on M.

LEMMA 2.1. *For any smooth function f on M and any smooth curve $c(t)$ in M, we have*

$$\frac{d^2}{dt^2}f(c(t)) = H_f(\dot{c}(t), \dot{c}(t)) + df\left(\frac{D\dot{c}(t)}{dt}\right).$$

Proof. For simplicity, let us denote by $F(t)$ the second derivative of $f(c(t))$ and set $H_f = H_f(\dot{c}(t), \dot{c}(t))$. We can assume that the curve $c(t)$ lies in a coordinate chart $(U, (y_1, \cdots, y_m))$ of M ($m = \dim M$). Let Γ_{ij}^k, $i, j, k = 1, \cdots, m$, denote the components of Γ with respect to the coordinate system and set $c^i(t) = y_i \circ c(t)$, $i = 1, \cdots, m$. Then we have

$$F(t) = \sum_{i,j=1}^m \frac{\partial^2 f}{\partial y_i \partial y_j} \cdot \frac{dc^i}{dt} \cdot \frac{dc^j}{dt} + \sum_{k=1}^m \frac{\partial f}{\partial y_k} \cdot \frac{d^2 c^k}{dt^2}$$

and

$$H_f = \sum_{i,j=1}^m \left(\frac{\partial^2 f}{\partial y_i \partial y_j} - \frac{1}{2}\sum_{k=1}^m (\Gamma_{ij}^k + \Gamma_{ji}^k)\frac{\partial f}{\partial y_k}\right) \cdot \frac{dc^i}{dt} \cdot \frac{dc^j}{dt}.$$

Hence $F(t) - H_f$ is given by

$$\sum_{k=1}^m \frac{\partial f}{\partial y_k}\left(\frac{d^2 c^k}{dt^2} + \sum_{i,j=1}^m \Gamma_{ij}^k \frac{dc^i}{dt} \cdot \frac{dc^j}{dt}\right),$$

which proves the formula.

LEMMA 2.2. *Let f be a smooth function on M. If df is parallel, then $df(T(X, Y)) = 0$ and $df(R(X, Y)Z) = 0$ hold for all vector fields X, Y and Z on M.*

Proof. This can be obtained from the following simple calculations:

1) $df(T(X, Y)) = df(\nabla_X Y) - df(\nabla_Y X) - df([X, Y])$
$= X(df(Y)) - Y(df(X)) - df([X, Y]) = 0;$

2) $df(R(X, Y)Z) = df(\nabla_X \nabla_Y Z) - df(\nabla_Y \nabla_X Z) - df(\nabla_{[X,Y]} Z)$
$= XY(df(Z)) - YX(df(Z)) - [X, Y](df(Z)) = 0.$

PROPOSITION 2.1. *Let f be a smooth function on M. Then:*

(1) *f is an affine function on M if and only if H_f vanishes identically on M;*

(2) *If df is a parallel 1-form, then f is an affine function;*

(3) *If Γ is symmetric, then, for every $f \in A(M, \Gamma)$, df is a parallel 1-form.*

Proof. (1) Suppose that f is an affine function. Let x be any point of M and let $c(t)$ be any geodesic with $c(0) = x$. From Lemma 2.1, we have

$$H_f(\dot{c}(0), \dot{c}(0)) = \frac{d^2}{dt^2} f(c(t))|_{t=0} = 0$$

and hence $H_f(u, u) = 0$ for any $u \in T_x(M)$, where $T_x(M)$ denotes the tangent space to M at x. Since H_f is symmetric, we finally have $H_f(u, v) = 0$ for all $u, v \in T_x(M)$. This implies that H_f vanishes on M. In a similar way, we can prove the converse.

(2) Since df is parallel, it follows from Lemma 2.2 that H_f vanishes on M. Hence f is an affine function.

(3) Since Γ is symmetric, we have $(\nabla_X df)(Y) = H_f(X, Y) = 0$ for all vector fields X and Y on M, so df is parallel.

Remark. Let M be a connected Riemannian manifold and let f be a smooth function on M. Then H_f is the Hessian of f and the Laplace-Beltrami operator Δ is given by $\Delta f =$ Trace of H_f. f is said to be harmonic if $\Delta f = 0$. By Proposition 2.1(1), we can assert that *every affine function on M is harmonic.*

Now we prove an inequality which gives a relation between $A(M, \Gamma)$ and the curvature of M. For any $x \in M$, let $\mathfrak{P}_x$ denote the linear subspace of $T_x^*(M)$ consisting of all covectors ω such that $\omega(R(X, Y)Z) = 0$ for all $X, Y, Z \in T_x(M)$.

PROPOSITION 2.2. *If Γ is symmetric, then we have $a(M, \Gamma) \leqq \dim \mathfrak{P}_x$ for any $x \in M$.*

Proof. Let $F_x : A(M, \Gamma) \to T_x^*(M)$ denote the linear mapping given by $F_x(f) = (df)_x$ ($f \in A(M, \Gamma)$). Then, by Proposition 2.1(3) and Lemma 2.2, the image $F_x(A(M, \Gamma))$ is contained in $\mathfrak{P}_x$. Hence the sequence

$$0 \longrightarrow \boldsymbol{R} \xrightarrow{i} A(M, \Gamma) \xrightarrow{F_x} \mathfrak{P}_x$$

is exact. This proves Proposition 2.2.

PROPOSITION 2.3. *For a given affine connection Γ on M, there exists an affine connection $\tilde{\Gamma}$ on M such that $\tilde{\Gamma}$ is symmetric and $A(M, \tilde{\Gamma}) =$*

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$A(M, \Gamma)$. *Moreover, if* Γ *is complete, then* $\tilde{\Gamma}$ *can be taken so that it is complete.*

Proof. For all vector fields X and Y on M, we set $\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}T(X, Y)$. Then it is easy to see that $\tilde{\nabla}$ defines a desired affine connection on M (cf. [9] vol. 1, p. 146).

§ 3. The main theorems

Let M be a connected smooth manifold with an affine connection Γ, $A(M, \Gamma)$ the vector space of all affine functions on M and $P^1(M, \Gamma)$ the vector space of all parallel 1-form of M. As before, we set $a(M, \Gamma) = \dim A(M, \Gamma) - 1$. Let $\tilde{W}$ denote the set of all vector fields X on M such that, for every $f \in A(M, \Gamma)$, Xf is a constant function on M and that $\nabla_X X = 0$. We set

$$W(M, \Gamma) = \{X \in \tilde{W};\ \nabla_Y X = 0 \text{ for all } Y \in \tilde{W}\}$$

and

$$W_0(M, \Gamma) = \{X \in W(M, \Gamma);\ Xf = 0 \text{ for all } f \in A(M, \Gamma)\}\,.$$

Then it is easy to see that $W(M, \Gamma)$ is a linear subspace of the real vector space of all vector fields on M and that $W_0(M, \Gamma)$ is a linear subspace of $W(M, \Gamma)$. Hence we can define a real vector space $V(M, \Gamma)$ by $V(M, \Gamma) = W(M, \Gamma)/W_0(M, \Gamma)$.

PROPOSITION 3.1. *If* Γ *is symmetric, then every parallel vector field* X *on* M *belongs to* $W(M, \Gamma)$.

Proof. Let f be any element of $A(M, \Gamma)$. By Proposition 2.1(3), df is parallel. Hence Xf is a constant function on M. Since X is parallel, we have $\nabla_Y X = 0$ for *all* vector fields Y on M. Therefore, X belongs to $W(M, \Gamma)$.

PROPOSITION 3.2. $V(M, \Gamma)$ *is finite-dimensional and satisfies* $\dim V(M, \Gamma) \leqq a(M, \Gamma)$.

Proof. We can assume that $n = a(M, \Gamma) > 0$. Let $1, f_1, \cdots, f_n$ be a basis of $A(M, \Gamma)$ and let F denote the linear mapping of $W(M, \Gamma)$ into $\boldsymbol{R}^n$ defined by $F(X) = (Xf_1, \cdots, Xf_n)(X \in W(M, \Gamma))$. Then it is easy to verify that the kernel of F coincides with $W_0(M, \Gamma)$. Hence we get $\dim V(M, \Gamma) \leqq n$.

PROPOSITION 3.3. *Let M be a connected Riemannian manifold with metric tensor g and Γ the Riemannian connection of M. Then we have* $\dim V(M, \Gamma) = a(M, \Gamma)$.

Proof. For any smooth function f on M, we denote by $\operatorname{grad} f$ the gradient of f. Namely, $\operatorname{grad} f$ is a unique vector field on M such that $g(\operatorname{grad} f, X) = df(X)$ for any vector field X on M. Let $f \in A(M, \Gamma)$. For all vector fileds X and Y on M, we have

$$\begin{aligned} g(\nabla_X \operatorname{grad} f, Y) &= Xg(\operatorname{grad} f, Y) - g(\operatorname{grad} f, \nabla_X Y) \\ &= X(df(Y)) - df(\nabla_X Y) \\ &= H_f(X, Y) \end{aligned}$$

and hence $\operatorname{grad} f$ is a parallel vector field of M. For any $X \in W(M, \Gamma)$, let $[X] \in V(M, \Gamma)$ denote the coset determined by X. Let $n = a(M, \Gamma)$ and let $1, f_1, \cdots, f_n$ be a basis of $A(M, \Gamma)$. To prove Proposition 3.3, it suffices to verify that $[\operatorname{grad} f_1], \cdots, [\operatorname{grad} f_n]$ are linearly independent in $V(M, \Gamma)$. Assume now that $\sum_{i=1}^n a_i [\operatorname{grad} f_i] = 0$ for real constants $a_1, \cdots, a_n$. If we set $f = \sum_{i=1}^n a_i f_i$, then $\operatorname{grad} f$ belongs to $W_0(M, \Gamma)$. Let $x \in M$ and let $\|v\|$ denote the norm of $v \in T_x(M)$. Then we have

$$\|(\operatorname{grad} f)_x\|^2 = (df(\operatorname{grad} f))(x) = 0$$

and hence $(\operatorname{grad} f)_x = 0$. It follows from Proposition 1.1 that there is a real constant b such that $\sum_{i=1}^n a_i f_i + b = 0$. Thus we get $a_i = 0$ for all $i = 1, \cdots, n$. This completes the proof of Proposition 3.3.

Let $H^1(M)$ be the first de Rham cohomology group of M and $H_*(M, Z_2)$ the singular homology group of M with coefficient group $Z_2 = Z/2Z$, Z being the module of all rational integers. We define a non-negative integer $k(M)$ by the following two conditions:

1) $H_i(M, Z_2) = 0$ for all $i > k(M)$;
2) $H_k(M, Z_2) \neq 0$ for $k = k(M)$.

We are now in a position to state our main theorems, which will be proved in the next section.

THEOREM 3.4. *Let M be a connected smooth manifold with a symmetric affine connection Γ. Then there exist natural linear homomorphisms $j : A(M, \Gamma) \to P^1(M, \Gamma)$ and $k : P^1(M, \Gamma) \to H^1(M)$ such that the sequence*

$$0 \longrightarrow \boldsymbol{R} \xrightarrow{i} A(M, \Gamma) \xrightarrow{j} P^1(M, \Gamma) \xrightarrow{k} H^1(M)$$

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is exact. Hence,

$$0 \leqq \dim P^1(M, \Gamma) - a(M, \Gamma) \leqq \dim H^1(M).$$

In particular, if M is compact, then $\dim P^1(M, \Gamma) \leqq b_1(M)$. *Here $b_1(M)$ denotes the first Betti number of M.*

THEOREM 3.5. *Let M be a non-compact connected smooth manifold with a complete affine connection Γ. Assume that $n = a(M, \Gamma) > 0$. Then there exists a totally geodesic surjective submersion $\pi : M \to \boldsymbol{R}^n$ with the following properties:*

(1) *every fibre $N_a = \pi^{-1}(a)$ $(a \in \boldsymbol{R}^n)$ is a connected totally geodesic submanifold of M;*

(2) *for every $a \in \boldsymbol{R}^n$, the inclusion $i_a : N_a \to M$ is a homotopy equivalence;*

(3) *if N_b is compact for some $b \in \boldsymbol{R}^n$, then so is N_a for every $a \in \boldsymbol{R}^n$;*

(4) *if M is non-orientable, then so is N_a for every $a \in \boldsymbol{R}^n$. Moreover, if $\pi' : M \to \boldsymbol{R}^n$ is another totally geodesic surjective submersion, then there exists an affine transformation T of $\boldsymbol{R}^n$ such that $\pi' = T \circ \pi$.*

We remark that if Γ is symmetric then π is an affine mapping (see [11]). Let (x, y) be the canonical coordinate system on $\boldsymbol{R}^2$ and set $M = \boldsymbol{R}^2 - \{(-1, 0), (1, 0)\}$. Then we have $H_1(M, \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$. Let $p : M \to \boldsymbol{R}$ denote the smooth function given by

$$p(x, y) = \log((x-1)^2 + y^2) - \log((x+1)^2 + y^2).$$

Then p is a surjective submersion. The fibre $p^{-1}(0)$, 0 being the origin of $\boldsymbol{R}$, is a line, while any other fibre $p^{-1}(a)$ $(a \in \boldsymbol{R}, a \neq 0)$ is a circle. Therefore, this shows that the existence of surjective submersion does not always imply (2) or (3) of Theorem 3.5. It should be also remarked that M has no soul.

THEOREM 3.6. *Let M and Γ be as in Theorem 3.5. Then we have*

$$k(M) \leqq \dim M - a(M, \Gamma).$$

The equality holds if and only if there exists a compact connected totally geodesic submanifold N of M such that

1) $\dim N = \dim M - a(M, \Gamma)$;

2) *the inclusion $i : N \to M$ is a homotopy equivalence.*

THEOREM 3.7. *Let M and Γ be as in Theorem 3.5 and let $n = \dim M$. If $a(M, \Gamma) = n$, then M is diffeomorphic to $\boldsymbol{R}^n$. Assume further that Γ is* *symmetric. Then M is affinely isomorphic to $\boldsymbol{R}^n$ if and only if $a(M, \Gamma) = n$.*

THEOREM 3.8. *Let M and Γ be as in Theorem 3.5. Assume that $m = \dim V(M, \Gamma) > 0$. Then there exists a connected totally geodesic submanifold M' of M such that M is diffeomorphic to the product manifold $\boldsymbol{R}^m \times M'$. Moreover, M' is compact if and only if $k(M) = \dim M - m$.*

We remark that there is a connected manifold M with a complete and symmetric affine connection Γ satisfying the following inequalities:

$$0 < \dim V(M, \Gamma) < a(M, \Gamma) < \dim M.$$

Let M be a connected complete Riemannian manifold and Γ the Riemannian connection of M. From Proposition 3.3, we have $\dim V(M, \Gamma) = a(M, \Gamma)$. In this case, we can prove more: *There exists a connected Riemannian manifold M' such that M is isometric to the Riemannian product $\boldsymbol{R}^n \times M'$ of the standard Euclidean space $\boldsymbol{R}^n$ and M', where we put $n = a(M, \Gamma)$.* We shall prove this theorem in [7].

§ 4. Proof of the main theorems

We keep the notations in Section 3. First of all, we prove Theorem 3.4. Let Γ be a symmetric affine connection on M. Then, by Proposition 2.1(3), we can define a linear mapping $j: A(M, \Gamma) \to P^1(M, \Gamma)$ by $j(f) = df$ ($f \in A(M, \Gamma)$). Since every parallel 1-form ω is closed, it determines a cohomology class $k(\omega) \in H^1(M)$. Thus we get the linear mapping $k: P^1(M, \Gamma) \to H^1(M)$ and the sequence:

$$0 \longrightarrow \boldsymbol{R} \xrightarrow{i} A(M, \Gamma) \xrightarrow{j} P^1(M, \Gamma) \xrightarrow{k} H^1(M).$$

To prove the exactness of the sequence, it suffices to verify the relation $\operatorname{Ker} k \subset \operatorname{Im} j$. Let ω be any element of $\operatorname{Ker} k$. Then there is a smooth function f on M such that $\omega = df$. By Proposition 2.1(2), f lies in $A(M, \Gamma)$ and hence $\omega = j(f) \in \operatorname{Im} f$. If M is compact, then every $f \in A(M, \Gamma)$ is constant on M (Proposition 1.2). This means that the sequence

$$0 \longrightarrow P^1(M, \Gamma) \xrightarrow{k} H^1(M)$$

is exact. Hence we have $\dim P^1(M, \Gamma) \leqq b_1(M)$. We have thereby proved Theorem 3.4.

To prove Theorem 3.5, we need some lemmas.

Let M be a non-compact connected manifold with a complete affine connection Γ. Assume that $n = a(M, \Gamma) > 0$. Let us fix a basis $1, f_1, \cdots, f_n$ of $A(M, \Gamma)$ and define a smooth mapping $\pi: M \to \boldsymbol{R}^n$ by $\pi(x) = (f_1(x), \cdots, f_n(x))$ $(x \in M)$. Then it is clear that, for every geodesic $c(t)$ $(t \in \boldsymbol{R})$, there are two elements a and b of $\boldsymbol{R}^n$ such that $\pi(c(t)) = at + b$ for all $t \in \boldsymbol{R}$. This shows that π is a totally geodesic mapping. Let us fix a Riemannian metric g^0 on M and consider the vector fields $\operatorname{grad} f_1, \cdots, \operatorname{grad} f_n$ on M. Let A_{ij} denote the function on M given by $A_{ij} = g^0$ $(\operatorname{grad} f_i, \operatorname{grad} f_j)$ $(i, j = 1, \cdots, n)$. From Proposition 1.4, it is easy to see that $\operatorname{grad} f_1, \cdots, \operatorname{grad} f_n$ are linearly independent at each point of M. Hence the $n \times n$ matrix $(A_{ij}(x))$ is non-singular for every $x \in M$. Let (B_{ij}) be the inverse matrix of (A_{ij}). For any $a = (a_1, \cdots, a_n) \in \boldsymbol{R}^n$, we set

$$X(a) = \sum_{i,j=1}^{n} a_i B_{ij} \operatorname{grad} f_j \,.$$

Then $X(a)$ is a smooth vector field on M. As usual, we identify $\boldsymbol{R}^n$ with the tangent space $T_a(\boldsymbol{R}^n)$, $a \in \boldsymbol{R}^n$, by the canonical absolute parallelism on $\boldsymbol{R}^n$.

LEMMA 4.1. *For any $a = (a_1, \cdots, a_n) \in \boldsymbol{R}^n$ and any $x \in M$, we have $df_i(X(a)_x) = a_i$, $i = 1, \cdots, n$ and $\pi_*(X(a)_x) = a$.*

Proof. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system on $\boldsymbol{R}^n$. Then we have

$$\begin{aligned}(\pi_*(X(a)_x))(x_i) &= df_i(X(a)_x) \\ &= df_i\left(\sum_{j,k=1}^{n} a_j B_{jk}(x)(\operatorname{grad} f_k)_x\right) \\ &= \sum_{j,k=1}^{n} a_j B_{jk}(x) A_{ki}(x) \\ &= a_i\end{aligned}$$

for all $i = 1, \cdots, n$, which proves the formulas.

Let TM be the tangent bundle of M and $\exp: TM \to M$ the exponential mapping of M. Let $G: \boldsymbol{R} \times \boldsymbol{R}^n \times M \to M$ denote the mapping given by $G(t, a, x) = \exp tX(a)_x$ $(t \in \boldsymbol{R}, a \in \boldsymbol{R}^n, x \in M)$.

LEMMA 4.2. *$G: \boldsymbol{R} \times \boldsymbol{R}^n \times M \to M$ is smooth and satisfies*

$$\pi(G(t, a, x)) = at + \pi(x)$$

for all $t \in \boldsymbol{R}$, $a \in \boldsymbol{R}^n$ and $x \in M$.

Proof. We define a mapping $G_0: \boldsymbol{R} \times \boldsymbol{R}^n \times M \to TM$ by $G_0(t, a, x) = tX(a)_x$ $(t \in \boldsymbol{R}, a \in \boldsymbol{R}^n, x \in M)$, so that $G = \exp \circ G_0$. Therefore, it suffices to prove that G_0 is smooth. But this can be easily checked by taking suitable local coordinate systems. Now we shall prove the second assertion. Since the curve $\boldsymbol{R} \ni t \mapsto G(t, a, x) \in M$ is a geodesic, there is an element $b = b(a, x)$ of $\boldsymbol{R}^n$ such that $\pi(G(t, a, x)) = bt + \pi(x)$. Differentiating this with respect to t at $t = 0$, we have $b = \pi_*(X(a)_x)$ and hence $a = b$ by Lemma 4.1.

LEMMA 4.3. *$\pi: M \to \boldsymbol{R}^n$ is a surjective submersion.*

Proof. It is clear from Proposition 1.4 that the rank of π is equal to n at each point of M. Let x_0 be a point of M and let a be any point of $\boldsymbol{R}^n$. Then, from Lemma 4.2, we have

$$\pi(G(1, a - \pi(x_0), x_0) = a ,$$

so π is surjective.

Proof of Theorem 3.5. Consider the totally geodesic surjective submersion $\pi: M \to \boldsymbol{R}^n$ and set $N_a = \pi^{-1}(a)$ $(a \in \boldsymbol{R}^n)$. For any $x \in N_a$ and any $v \in T_x(N_a)$, let $c(t)$ $(t \in \boldsymbol{R}^n)$ be the geodesic determined by (x, v). Then we can put $\pi(c(t)) = bt + a$ $(b \in \boldsymbol{R}^n)$. Now we have

$$b = \frac{d}{dt}\pi(c(t))|_{t=0} = \pi_*(v) = 0$$

and hence $\pi(c(t)) = a$ for all $t \in \boldsymbol{R}$. Thus N_a is a totally geodesic submanifold of M.

By Lemma 4.2, $\pi(G(1, a - \pi(x), x) = a$ holds for all $a \in \boldsymbol{R}^n$ and $x \in M$, so we can define a smooth mapping $r_a: M \to N_a$ by $r_a(x) = G(1, a - \pi(x), x)$ $(x \in M)$. We have easily $r_a \circ i_a(x) = x$ for any $x \in N_a$, where $i_a: N_a \to M$ is the inclusion. It follows that N_a is connected. Let $H_a: \boldsymbol{R} \times M \to M$ denote the smooth mapping given by $H_a(t, x) = G(t, a - \pi(x), x)$ $(t \in \boldsymbol{R}, x \in M)$. Then we have $H_a(0, x) = x$ and $H_a(1, x) = i_a \circ r_a(x)$ for all $x \in M$. Hence the mapping $i_a \circ r_a: M \to M$ is homotopic to the identity mapping of M. Therefore $i_a: N_a \to M$ is a homotopy equivalence. (More precisely, N_a is a strong deformation retract of M.)

Suppose now that N_a is orientable for some $a \in \boldsymbol{R}^n$. Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ be the canonical orthonormal basis of $\boldsymbol{R}^n$ and set $X_i = X(\boldsymbol{e}_i)$, $i = 1, \cdots, n$. Then, from Lemma 4.1, it is easy to see that, for any $x \in N_a$ and any basis $v_1, \cdots, v_p$ of $T_x(N_a)$ $(p = \dim N_a)$, $v_1, \cdots, v_p, (X_1)_x, \cdots, (X_n)_x$ form a basis of $T_x(M)$. Let ω be a non-vanishing continuous p-form on N_a and let $m = \dim M$. For any $x \in N_a$, let Ω_x^0 denote the m-covector of $T_x(M)$ defined by

$$\Omega_x^0(v_1 \wedge \cdots \wedge v_p \wedge (X_1)_x \wedge \cdots \wedge (X_n)_x) = \omega_x(v_1 \wedge \cdots \wedge v_p)$$

for all vectors $v_1, \cdots, v_p$ of $T_x(N_a)$. Let E be the pull-back of TM by the inclusion $i_a : N_a \to M$, i.e., $E = i_a^\# TM$. Then $\Omega_x^0 (x \in N_a)$ defines a non-vanishing continuous cross section of $\Lambda^m E^*$, where E^* is the dual bundle of E and $\Lambda^m E^*$ the exterior product bundle of E^*. For any $y \in M$, we set $c_y(t) = H_a(t, y)$ $(t \in \boldsymbol{R})$, so that $c_y(0) = y$ and $c_x(1) = r_a(y) \in N_a$. Let $p(c_y)$ denote the parallel translation along the curve $c_y(t)$ $(0 \leqq t \leqq 1)$. Thus $p(c_y)$ is a linear isomorphism of $T_y(M)$ onto E_x $(x = r_a(y))$. Then $p(c_y)$ can be canonically extended to a smooth vector bundle homomorphism $p^m(c) : \Lambda^m TM \to \Lambda^m E$. Now we define the m-covector Ω_y on $T_y(M)$ by $\Omega_y(V) = \Omega_x^0\ (p^m(c)(V))$ $(V \in \Lambda^m T_y(M),\ x = r_a(y))$. Then it can be easily seen that Ω_y $(y \in M)$ defines a non-vanishing continuous m-form on M. Hence M is orientable. We have thereby proved (4) of Theorem 3.5.

Let $\pi' : M \to \boldsymbol{R}^n$ be another totally geodesic surjective submersion. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system on $\boldsymbol{R}^n$ and set $f_i' = x_i \circ \pi'$, $i = 1, \cdots, n$. Then $f_1', \cdots, f_n'$ are affine functions on M and linearly independent in $A(M, \Gamma)$. Hence there are a non-singular $n \times n$ matrix (a_{ij}) and real numbers $b_1, \cdots, b_n$ such that $f_i' = \sum_{j=1}^n a_{ij} f_j + b_i$ for all $i = 1, \cdots, n$. This proves the last assertion of Theorem 3.5.

To prove (3) of Theorem 3.5, we need the following lemma.

LEMMA 4.4. *Let N be a connected smooth manifold. Then we have* $0 \leqq k(N) \leqq \dim N$. *Moreover, N is compact if and only if* $k(N) = \dim N$.

Proof of Lemma 4.4. It is well-known that the singular homology group $H_*(N, Z_2)$ has the following properties:

1) $H_q(N, Z_2) = 0$ for all $q > \dim N$;
2) If N is non-compact, then $H_p(N, Z_2) = 0$ for $p = \dim N$;
3) If N is compact, then $H_p(N, Z_2) \cong Z_2$ for $p = \dim N$.

(For more details, see for example [6]). Now the lemma follows immediately from these properties.

We return to the proof of Theorem 3.5(3). Suppose that N_b $(b \in \boldsymbol{R}^n)$ is compact. Let a be any point of $\boldsymbol{R}^n$. Then N_a is homotopy equivalent to N_b. From Lemma 4.4, we have

$$k(N_a) = k(N_b) = \dim N_b = \dim N_a\,.$$

Hence N_a is compact. This completes the proof of Theorem 3.5.

Proof of Theorem 3.6. By Theorem 3.5, there exists a connected totally geodesic submanifold N of M such that a) the inclusion $i: N \to M$ is a homotopy equivalence and b) $\dim N = \dim M - a(M, \Gamma)$. From a), b) and Lemma 4.4, we have

$$k(M) = k(N) \leqq \dim N = \dim M - a(M, \Gamma)\,.$$

Moreover, N is compact if and only if $k(M) = \dim M - a(M, \Gamma)$. We have thereby proved Theorem 3.5.

Proof of Theorem 3.7. In view of Propositions 1.5 and 2.3, it will be sufficient to prove that if Γ is symmetric and $a(M, \Gamma) = \dim M$ then M is affinely isomorphic to $\boldsymbol{R}^n$. Accordingly, we assume that Γ is symmetric and $a(M, \Gamma) = \dim M$. Let $n = \dim M$ and consider the surjective submersion $\pi: M \to \boldsymbol{R}^n$ (Lemma 4.3). In this case, π is an immersion, so we can define a Riemannian metric g on M by $g = \pi^* ds^2$, where ds^2 denotes the standard Euclidean metric on $\boldsymbol{R}^n$ and π^* the codifferential of π. As before, let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ be the canonical orthonormal basis of $\boldsymbol{R}^n$ and set $X_i = X(\boldsymbol{e}_i)$, $i = 1, \cdots, n$. By Lemma 4.1, $df_i(X_j)$ is constant on M $(i, j = 1, \cdots, n)$. On the other hand, by Proposition 2.1(3), df_i is a parallel 1-form of M $(i = 1, \cdots, n)$. Hence, for any vector field X on M, we have $df_i(\nabla_X X_j) = X(df_i(X_j)) = 0$ for all $i, j = 1, \cdots, n$. It follows that $X_1, \cdots, X_n$ are parallel vector fields. Hence,

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0 \qquad (i, j = 1, \cdots, n)\,.$$

Let $\tilde{\nabla}$ denote the covariant differentiation of the Riemannian connection of (M, g). Since $g(X_i, X_j)$ is constant on M for all $i, j = 1, \cdots, n$ we have $g(\tilde{\nabla}_{X_i} X_j, X_k) = 0$, $i, j, k = 1, \cdots, n$, and hence $\tilde{\nabla}_{X_i} X_j = 0$, $i, j = 1, \cdots, n$ (cf. [9] vol. 1 p. 160). This means that Γ coincides with the Riemannian connection of (M, g). As Γ is complete, (M, g) is a complete Riemannian manifold. It therefore follows from a well-known theorem in [9] (vol. 1 p. 176, Theorem 4.6) that π is an isometry of M onto $\boldsymbol{R}^n$. This completes the proof of Theorem 3.7.

Proof of Theorem 3.8. We begin with the following lemma.

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LEMMA 4.5. *Let X be any element of $W(M, \Gamma)$. Then:*
1) *Every integral curve of X is a geodesic of M;*
2) *X is a complete vector field on M.*

Proof. 1) follows immediately from the condition $\nabla_X X = 0$.

2) Let $x(t)$ ($|t| < \varepsilon, \varepsilon > 0$) be an integral curve of X. Since Γ is complete, $x(t)$ can be extended to a geodesic $c(t)$ defined for all $t \in \boldsymbol{R}$. Let I denote the subset of $\boldsymbol{R}$ consisting of all points t such that $\dot{c}(t) = X_{c(t)}$. Clearly, I is non-empty and closed in $\boldsymbol{R}$. Let t_0 be any point of I and let $y(t)$ ($|t - t_0| < \varepsilon', \varepsilon' > 0$) be an integral curve of X with $y(t_0) = c(t_0)$. Then $y(t)$ is a geodesic with the initial condition $(c(t_0), X_{c(t_0)})$. Hence $c(t)$ must coincide with $y(t)$ on a small open neighborhood of t_0. This shows that I is open in $\boldsymbol{R}$ and hence $I = \boldsymbol{R}$. Therefore every integral curve of X can be extended to an integral curve defined for all $t \in \boldsymbol{R}$. Hence X is complete.

Let us set $m = \dim V(M, \Gamma)$ and $n = a(M, \Gamma)$. Then we have $m \leqq n$ (Proposition 3.2).

LEMMA 4.6. *We can choose $Y_1, \cdots, Y_m \in W(M, \Gamma)$ and $f_1, \cdots, f_m \in A(M, \Gamma)$ in such a way that*

1) *$1, f_1, \cdots, f_m$ are linearly independent in $A(M, \Gamma)$;*

2) *$df_i(Y_j) = \delta_{ij}$ for all $i, j = 1, \cdots, m$, where δ_{ij} denotes Kronecker's delta.*

Proof. For any $Y \in W(M, \Gamma)$, let $[Y] \in V(M, \Gamma)$ denote the coset determined by Y. Then we can choose $Y_1, \cdots, Y_m \in W(M, \Gamma)$ so that $[Y_1], \cdots, [Y_m]$ form a basis of $V(M, \Gamma)$. Let $1, g_1, \cdots, g_n$ be a basis of $A(M, \Gamma)$ and set $A_{ij} = dg_i(Y_j)$, $i = 1, \cdots, n$, $j = 1, \cdots, m$. Then, by definition, A_{ij}'s are constants. Assume that $\sum_{j=1}^m a_j A_{ij} = 0$ for real constants $a_1, \cdots, a_m$ $(i = 1, \cdots, n)$. Then we have easily $\sum_{j=1}^m a_j [Y_j] = 0$ and hence $a_j = 0$ for all $j = 1, \cdots, m$. This means that the rank of the $n \times m$ matrix (A_{ij}) is equal to m. We can assume that the $m \times m$ matrix $(A_{ij})_{1 \leq i, j \leq m}$ is non-singular. Let $(B_{ij})_{1 \leq i, j \leq m}$ be the inverse matrix of (A_{ij}) and set $f_i = \sum_{j=1}^m B_{ij} g_j$, $i = 1, \cdots, m$. Then $1, f_1, \cdots, f_m$ are linearly independent. Moreover, we have

$$df_i(Y_j) = \sum_{k=1}^m B_{ik}\, dg_k(Y_j) = \sum_{k=1}^m B_{ij}\, A_{kj} = \delta_{ij}$$

for all $i, j = 1, \cdots, m$. This proves Lemma 4.6.

From now on, we fix $Y_1, \cdots, Y_m \in W(M, \Gamma)$ and $f_1, \cdots, f_m \in A(M, \Gamma)$ with the properties listed in Lemma 4.6. Let $p: M \to \boldsymbol{R}^m$ denote the smooth mapping given by $p(x) = (f_1(x), \cdots, f_m(x))$ $(x \in M)$. For any $a = (a_1, \cdots, a_m) \in \boldsymbol{R}^m$, let Y^a denote the vector field given by $Y^a = \sum_{i=1}^m a_i Y_i$. By Lemma 4.5, Y^a is a complete vector field on M. We denote by F_t^a the 1-parameter family of diffeomorphisms generated by Y^a. Let $F: \boldsymbol{R} \times \boldsymbol{R}^m \times M \to M$ denote the mapping defined by $F(t, a, x) = F_t^a(x)$ $(t \in \boldsymbol{R}, a \in \boldsymbol{R}^m, x \in M)$.

LEMMA 4.7. *$F: \boldsymbol{R} \times \boldsymbol{R}^m \times M \to M$ is smooth and satisfies*

$$p(F(t, a, x)) = at + p(x)$$

for all $t \in \boldsymbol{R}$, $a \in \boldsymbol{R}^m$ and $x \in M$.

Proof. By Lemma 4.5, the curve $\boldsymbol{R} \ni t \mapsto F(t, a, x) \in M$ is a geodesic, so we can write $F(t, a, x) = \exp tY_x^a$. Therefore, we can prove Lemma 4.7 by the same reasoning as in Lemma 4.2.

Now we set $M' = p^{-1}(0)$, 0 being the origin of $\boldsymbol{R}^m$. Then M' is a closed totally geodesic submanifold of M. Let h denote the smooth mapping $\boldsymbol{R}^m \times M' \to M$ given by $h(a, x) = F(1, a, x) = F_1^a(x)$ $(a \in \boldsymbol{R}^m, x \in M')$. From Lemma 4.7, we have $p(F(-1, p(x), x)) = 0$ for any $x \in M$, so we can define the smooth mapping $q: M \to M'$ by $q(x) = F(-1, p(x), x)$ $(x \in M)$. For any $a \in \boldsymbol{R}^m$ and any $x \in M'$, we have $p(h(a, x)) = a$ and hence

$$\begin{aligned}(p \times q) \circ h(a, x) &= (p(h(a, x)), q(h(a, x))) \\ &= (a, F_{-1}^a \circ F_1^a(x)) \\ &= (a, x).\end{aligned}$$

On the other hand, we have for any $y \in M$

$$\begin{aligned}h \circ (p \times q)(y) &= h(p(y), q(y)) \\ &= F_1^a \circ F_{-1}^a(y) \\ &= y,\end{aligned}$$

where we put $a = p(y)$. These results show that h is a diffeomorphism of $\boldsymbol{R}^m \times M'$ onto M. The last assertion follows easily from Lemma 4.4. This completes the proof of Theorem 3.8.

§ 5. Affine symmetric spaces

A *symmetric space* is a triple (G, H, s) consisting of a connected Lie group G, a closed subgroup H of G and an involutive automorphism s of G

such that H lies between the closed subgroup G_s of all fixed points of s and the identity component of G_s. Let us consider the coset space G/H and let s_0 denote the diffeomorphism $gH \to s(g)H$ of G/H onto itself. Then G/H has a unique affine connection Γ invariant under s_0 and under the natural left action of G (see [10] § 15). Γ is called the *canonical affine connection* on G/H. Then G/H turns out to be an affine symmetric space with respect to the canonical affine connection. Conversely, every affine symmetric space is expressed in this form. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Let $\mathfrak{m}$ denote the (-1)-eigenspace of the differential of s. Then we have the canonical decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \qquad \text{(direct sum)}.$$

THEOREM 5.1. *Let (G, H, s) be a symmetric space, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the canonical decomposition of the Lie algebra of G and Γ the canonical affine connection on $M = G/H$. Then:*

1) $a(M, \Gamma) \leqq \dim \mathfrak{m} - \dim [\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]]$;
2) *The equality holds if M is simply connected.*

Proof. We denote by $\mathfrak{gl}(\mathfrak{m})$ the Lie algebra of all linear endomorphisms of $\mathfrak{m}$ and by $\mathfrak{m}^*$ the dual vector space of $\mathfrak{m}$. Let $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$ denote the linear isotropy representation given by $\rho(X)(Y) = [X, Y]$ $(X \in \mathfrak{h}, Y \in \mathfrak{m})$. If we set $\mathfrak{h}' = [\mathfrak{m}, \mathfrak{m}]$, then $\mathfrak{h}'$ is an ideal of $\mathfrak{h}$. We remark here that the Lie subalgebra $\rho(\mathfrak{h}')$ of $\mathfrak{gl}(\mathfrak{m})$ can be identified with the Lie algebra of the linear holonomy group L_0 at the origin $0 = H \in G/H$ (see [9] vol. 2 p. 232). Moreover, if $M = G/H$ is simply connected, L_0 is connected. Let $\rho^* : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m}^*)$ denote the representation defined by

$$(\rho^*(X)\omega)(Y) = -\omega(\rho(X)(Y)) \qquad (X \in \mathfrak{h}, Y \in \mathfrak{m}, \omega \in \mathfrak{m}^*).$$

We set

$$\tilde{\mathfrak{a}} = \{\omega \in \mathfrak{m}^* ; \rho^*(X)\omega = 0 \quad \text{for all } X \in \mathfrak{h}'\}.$$

Let $P^1(M, \Gamma)$ be the vector space of all parallel 1-form of M. From the above remarks, it can be easily seen that $\dim P^1(M, \Gamma) \leqq \dim \tilde{\mathfrak{a}}$ and that if M is simply connected then $\dim P^1(M, \Gamma) = \dim \tilde{\mathfrak{a}}$. Let $\mathfrak{a}$ denote the linear subspace of $\mathfrak{m}$ consisting of all vectors Y such that $\omega(Y) = 0$ for all $\omega \in \tilde{\mathfrak{a}}$. Then we have $\dim \tilde{\mathfrak{a}} = \dim \mathfrak{m} - \dim \mathfrak{a}$. For simplicity, we write $\mathfrak{b} = \rho(\mathfrak{h}')(\mathfrak{m})$. Let $X \in \mathfrak{h}'$ and $Y \in \mathfrak{m}$. For any $\omega \in \tilde{\mathfrak{a}}$, we have

$$\omega(\rho(X)(Y)) = -(\rho^*(X)\omega)(Y) = 0,$$

which implies that $\mathfrak{b}$ is a linear subspace of $\mathfrak{a}$. Hence,

$$\dim \tilde{\mathfrak{a}} \leqq \dim \mathfrak{m} - \dim \mathfrak{b}\,.$$

On the other hand, let $\tilde{\mathfrak{b}}$ denote the linear subspace of $\mathfrak{m}^*$ consisting of all $\omega \in \mathfrak{m}^*$ such that $\omega(Z) = 0$ for any $Z \in \mathfrak{b}$. Then, as in the above case, we have $\tilde{\mathfrak{b}} \subset \tilde{\mathfrak{a}}$. Hence,

$$\dim \tilde{\mathfrak{a}} \geqq \dim \mathfrak{m} - \dim \mathfrak{b}\,.$$

We have thereby proved the formula: $\dim \tilde{\mathfrak{a}} = \dim \mathfrak{m} - \dim \mathfrak{b}$. Now Theorem 5.1 follows easily from Theorem 3.4.

COROLLARY 5.2. *Let (G, H, s) be a symmetric space and Γ the canonical affine connection on $M = G/H$.*

1) *If G is semisimple, then $a(M, \Gamma) = 0$.*

2) *If G is solvable and if M is simply connected, then $a(M, \Gamma) > 0$.*

Proof. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition. If we set $\mathfrak{g}' = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$, then $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$. We have easily

$$[\mathfrak{g}', \mathfrak{g}'] = [\mathfrak{m}, \mathfrak{m}] + [\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \qquad \text{(direct sum)}\,.$$

Suppose first that $\mathfrak{g}$ is semisimple. Then $\mathfrak{g}'$ is also semisimple. Thus we get $[\mathfrak{g}', \mathfrak{g}'] = \mathfrak{g}'$ and hence $[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] = \mathfrak{m}$. Suppose now that $\mathfrak{g}$ is solvable. Then $\mathfrak{g}'$ is also solvable. Thus $[\mathfrak{g}', \mathfrak{g}'] \subsetneqq \mathfrak{g}'$ and hence $[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \subsetneqq \mathfrak{m}$. Therefore the assertions 1) and 2) follow from Theorem 5.1.

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