

GENERALIZED HILBERT–KUNZ FUNCTION IN GRADED DIMENSION 2

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Abstract. We prove that the generalized Hilbert–Kunz function of a graded module M over a two-dimensional standard graded normal K -domain over an algebraically closed field K of prime characteristic p has the form $gHK(M, q) = e_{gHK}(M)q^2 + \gamma(q)$, with rational generalized Hilbert–Kunz multiplicity $e_{gHK}(M)$ and a bounded function $\gamma(q)$. Moreover, we prove that if R is a \mathbb{Z} -algebra, the limit for $p \rightarrow +\infty$ of the generalized Hilbert–Kunz multiplicity $e_{gHK}^{R_p}(M_p)$ over the fibers R_p exists, and it is a rational number.

Introduction

Let R be a d -dimensional standard graded K -domain over a perfect field K of characteristic $p > 0$ that is F -finite. For every finitely generated R -module M and every natural number e , we denote by $F^{e*}(M) = M \otimes_R {}^e R$ the e th iteration of the Frobenius functor given by base change along the Frobenius homomorphism. In particular, if I is an ideal of R , we have that $F^{e*}(R/I) \cong R/I^{[p^e]}$. We denote by $q = p^e$ a power of the characteristic. Let M be a graded R -module. The function

$$gHK(M, q) := l_R(H_{R_+}^0(F^{e*}(M)))$$

and the limit

$$e_{gHK}(M) := \lim_{e \rightarrow +\infty} \frac{l_R(H_{R_+}^0(F^{e*}(M)))}{q^d}$$

are called the *generalized Hilbert–Kunz function* and the *generalized Hilbert–Kunz multiplicity* of M , respectively. If I is an R_+ -primary ideal, then $gHK(R/I, q)$ and $e_{gHK}(R/I)$ coincide with the classical Hilbert–Kunz function and multiplicity. For a survey on the classical Hilbert–Kunz function and multiplicity, see [10].

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The generalized Hilbert–Kunz function and multiplicity were first introduced, under a different name and notation, by Epstein and Yao in [8], and studied in detail by Dao and Smirnov in [6], where they proved the existence of $e_{gHK}(M)$ under some assumptions, for example if M is a module over a Cohen–Macaulay isolated singularity. In the same paper, they studied the behavior of the function $gHK(M, q)$ and compared it with the classical Hilbert–Kunz function. Further study of the generalized Hilbert–Kunz function and multiplicity has been made by Dao and Watanabe in [7], where they computed $e_{gHK}(M)$ if M is a module over a ring of finite Cohen–Macaulay type, or it is an ideal of a normal toric singularity.

In this paper, we study the function $gHK(M, q)$ for a graded module M over a two-dimensional standard graded normal domain over an algebraically closed field. In [3], Brenner proved that if I is a homogeneous R_+ -primary ideal, then the Hilbert–Kunz function of I has the following form:

$$HK(I, q) = e_{HK}(I)q^2 + \gamma(q),$$

where $e_{HK}(I)$ is a rational number and $\gamma(q)$ is a bounded function, which is eventually periodic if K is the algebraic closure of a finite field. In [6, Example 6.2], Dao and Smirnov exhibited numerical evidence that in this setting also the generalized Hilbert–Kunz function has the same form. Using an extension of the methods of Brenner, we are able to prove their claim. In fact, we obtain in Theorem 3.2 that the generalized Hilbert–Kunz function of a graded module M has the form

$$gHK(M, q) = e_{gHK}(M)q^2 + \gamma(q),$$

where $\gamma(q)$ is a bounded function, which is eventually periodic if K is the algebraic closure of a finite field. Moreover, we give an explicit formula for $e_{gHK}(M)$ in terms of the Hilbert–Kunz slope of certain locally free sheaves on the projective curve $Y = \text{Proj}R$. As a consequence of this fact, we obtain that the generalized Hilbert–Kunz multiplicity $e_{gHK}(M)$ exists, and it is a rational number.

Furthermore, in the last section of the paper, we consider the following problem. Assume that R is a standard graded \mathbb{Z} -domain of relative dimension 2, and M is a graded R -module. For each prime number p , we may consider the reduction R_p of R mod p and the extended module $M_p := M \otimes_R R_p$. For this module, we compute the generalized Hilbert–Kunz

multiplicity $e_{gHK}^{R_p}(M_p)$, and we ask whether the limit

$$\lim_{p \rightarrow +\infty} e_{gHK}^{R_p}(M_p)$$

exists. Using a result of Trivedi [12], we are able to prove (Theorem 4.4) that the previous limit exists, and it is in fact a rational number, assuming that the rings R_p are normal two-dimensional domains for almost all prime numbers.

After submitting the first version of this paper, the referee and Asgharzadeh pointed us to a recent paper of Vraciu [13]. There, she provides another method to prove Theorem 3.2 for ideals by showing that under suitable conditions, which are fulfilled in our situation, the generalized Hilbert–Kunz function of a homogeneous ideal can be expressed as a \mathbb{Z} -linear combination of the classical Hilbert–Kunz function of R_+ -primary ideals. The relevant condition is called the *(LC)* property, and was introduced by Hochster and Huneke in [9]. This condition is known to hold in some special cases, but it is an open problem whether it holds in a more general setting (see [1]).

§1. Reflexive modules

We recall some preliminary facts concerning reflexive modules. Let R be a two-dimensional normal domain with homogeneous maximal ideal \mathfrak{m} , and let U be the punctured spectrum of R ; that is, $U = \text{Spec}R \setminus \{\mathfrak{m}\}$.

We denote by $(-)^*$ the functor $\text{Hom}_R(-, R)$. If M is an R -module, then the module M^{**} is called the *reflexive hull* of M . There is a canonical map

$$\lambda : M \rightarrow M^{**}.$$

If λ is injective, M is said to be *torsionless*; if λ is an isomorphism, then M is called *reflexive*. Finitely generated projective modules are reflexive, but the converse does not hold in general. We recall the following geometric characterization of the reflexive hull in the normal situation (cf. [5, Proposition 3.10]):

$$(1) \quad M^{**} \cong \Gamma(U, \widetilde{M}),$$

where \widetilde{M} denotes the coherent sheaf associated to the module M . It follows that the restriction of this sheaf to the punctured spectrum $\widetilde{M}|_U$ coincides with the sheaf $\widetilde{M^{**}}|_U$ on U . Moreover, if M is reflexive, the sheaf $\widetilde{M}|_U$ is locally free.

The following lemma is a well-known fact (see [4, Proposition 1.4.1]). We give a proof here for the sake of completeness.

LEMMA 1.1. *Let R be a normal domain of dimension at least 2 with homogeneous maximal ideal \mathfrak{m} , and let I be a reflexive submodule of R^n . Then,*

$$H_{\mathfrak{m}}^0(R^n/I) = 0.$$

Proof. We consider the short exact sequence $0 \rightarrow I \rightarrow R^n \rightarrow R^n/I \rightarrow 0$, and we apply the local cohomology functor $H_{\mathfrak{m}}^0(-)$. We obtain a long exact sequence

$$\cdots \rightarrow H_{\mathfrak{m}}^0(R^n) \rightarrow H_{\mathfrak{m}}^0(R^n/I) \rightarrow H_{\mathfrak{m}}^1(I) \rightarrow \cdots.$$

Since R^n and I are reflexive modules over a normal domain, they have depth at least 2. It follows that $H_{\mathfrak{m}}^0(R^n) = H_{\mathfrak{m}}^1(I) = 0$, hence $H_{\mathfrak{m}}^0(R^n/I) = 0$ too. \square

We mention also the following result (cf. [7, Proposition 2.2]), concerning the generalized Hilbert–Kunz multiplicity of reflexive ideals.

PROPOSITION 1.2. *Let R be a standard graded domain of dimension 2, and let I be a homogeneous reflexive ideal of R . Then, $e_{gHK}(R/I) = 0$ if and only if I is principal.*

The fact that principal reflexive ideals have generalized Hilbert–Kunz multiplicity 0 holds also in dimension ≥ 2 , and is a consequence of Lemma 1.1. In fact, if I is a principal ideal, then $I^{[p^e]}$ is again principal, and in particular reflexive. It follows that $H_{R_+}^0(F^{e*}(R/I)) \cong H_{R_+}^0(R/I^{[p^e]}) = 0$, so $e_{gHK}(R/I) = 0$.

LEMMA 1.3. *Let R be a normal K -domain of dimension $d \geq 2$ over an algebraically closed field K of prime characteristic p . Let I be a nonzero homogeneous ideal of R such that $e_{gHK}(R/I)$ exists, and let $f \neq 0$ be a homogeneous element of R . Then,*

$$e_{gHK}(R/fI) = e_{gHK}(R/I).$$

Proof. From the short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ and the corresponding long exact sequence of local cohomology modules with support in $\mathfrak{m} := R_+$, we obtain that $H_{\mathfrak{m}}^0(R/I) \cong H_{\mathfrak{m}}^1(I)$. It follows that the generalized Hilbert–Kunz multiplicity can be seen as

$$e_{gHK}(R/I) = \lim_{e \rightarrow +\infty} \frac{l_R(H_{\mathfrak{m}}^1(I^{[q]}))}{q^d}.$$

Then, the R -module isomorphism $f^q I^{[q]} \cong I^{[q]}$ implies the claim. \square

REMARK 1.4. Let $[I]$ be an element of the divisor class group $\text{Cl}(R)$ of R , and let I be a homogeneous reflexive ideal representative of this element. If R is a standard graded normal K -domain of dimension 2, with K algebraically closed and of positive characteristic, we obtain a function $e_{gHK}(-) : \text{Cl}(R) \rightarrow \mathbb{Q}$, $[I] \mapsto e_{gHK}(R/I)$. Thanks to Theorem 3.2 and Lemma 1.3, this function is well defined, and we have $e_{gHK}([R]) = 0$. This does not mean that the generalized Hilbert–Kunz multiplicity for all ideals I that are invertible on the punctured spectrum depends only on $[I]$. For example, the homogeneous maximal ideal \mathfrak{m} and its reflexive hull $\mathfrak{m}^{**} = R$ define the same element in the class group, but $e_{gHK}(R/\mathfrak{m}) = e_{HK}(\mathfrak{m}) \neq 0$ in general, while $e_{gHK}(R/R) = 0$. Moreover, Proposition 1.2 implies that the preimage of 0 is trivial. This does not mean that the function $e_{gHK}(-)$ is injective, since in general it is not a group homomorphism as in the case of Example 3.6.

Therefore, the following question makes sense.

QUESTION 1.5. Given two homogeneous reflexive ideals I and J , is there a formula for $e_{gHK}([IJ])$ in terms of $e_{gHK}([I])$ and $e_{gHK}([J])$?

§2. The Hilbert–Kunz slope

Let Y be a smooth projective curve over an algebraically closed field with a very ample invertible sheaf of degree $\deg \mathcal{O}_Y(1) = \deg Y$. We recall some classical notions of vector bundles, and some definitions from [2] and [3]. We refer to these papers for further details and explanations.

Let \mathcal{S} be a locally free sheaf of rank r over X . The degree of \mathcal{S} is defined as the degree of the corresponding determinant line bundle $\deg \mathcal{S} = \deg \bigwedge^r \mathcal{S}$. The slope of \mathcal{S} is $\mu(\mathcal{S}) = \deg \mathcal{S}/r$. The degree is additive on short exact sequences, and moreover $\mu(\mathcal{S} \otimes \mathcal{T}) = \mu(\mathcal{S}) + \mu(\mathcal{T})$.

The sheaf \mathcal{S} is called *semistable* if for every locally free subsheaf $\mathcal{T} \subseteq \mathcal{S}$, the inequality $\mu(\mathcal{T}) \leq \mu(\mathcal{S})$ holds. If the strict inequality $\mu(\mathcal{T}) < \mu(\mathcal{S})$ holds for every proper subsheaf $\mathcal{T} \subset \mathcal{S}$, then \mathcal{S} is called *stable*.

For any locally free sheaf \mathcal{S} on Y , there exists a unique filtration, called *Harder–Narasimhan filtration*, $\mathcal{S}_1 \subseteq \dots \subseteq \mathcal{S}_t = \mathcal{S}$, with the following properties:

- \mathcal{S}_k is locally free;
- $\mathcal{S}_k/\mathcal{S}_{k-1}$ is semistable;
- $\mu(\mathcal{S}_k/\mathcal{S}_{k-1}) > \mu(\mathcal{S}_{k+1}/\mathcal{S}_k)$.

If the base field has positive characteristic, we can consider the absolute Frobenius morphism $F : Y \rightarrow Y$ on the curve and its iterates F^e . In general, the pullback via F^e of the Harder–Narasimhan filtration of \mathcal{S} is not the Harder–Narasimhan filtration of $F^{e*}\mathcal{S}$, since the quotients $F^{e*}(\mathcal{S}_k)/F^{e*}(\mathcal{S}_{k-1})$ need not be semistable.

In [11], Langer proved that for $q \gg 0$, there exists the so-called *strong Harder–Narasimhan* filtration of $F^{e*}(\mathcal{S})$. In fact, there exists a natural number e_0 such that the Harder–Narasimhan filtration of $F^{e_0*}(\mathcal{S})$,

$$0 \subseteq \mathcal{S}_{e_0,1} \subseteq \dots \subseteq \mathcal{S}_{e_0,t} = F^{e_0*}(\mathcal{S}),$$

has the property that the quotients $F^{e*}(\mathcal{S}_{e_0,k})/F^{e*}(\mathcal{S}_{e_0,k-1})$ of the pullback along F^e are semistable. Thus, for $e \geq e_0$, we have $F^{e*}(\mathcal{S}) = F^{(e-e_0)*}(F^{e_0*}(\mathcal{S}))$, and the Harder–Narasimhan filtration of $F^{e*}(\mathcal{S})$ is given by

$$F^{(e-e_0)*}(\mathcal{S}_{e_0,1}) \subseteq \dots \subseteq F^{(e-e_0)*}(\mathcal{S}_{e_0,t}) = F^{e*}(\mathcal{S}).$$

For ease of notation, we put $\mathcal{S}_{e,k} := F^{(e-e_0)*}(\mathcal{S}_{e_0,k})$ for every $e \geq e_0$ and $0 \leq k \leq t$. The length t of such a sequence and the ranks of the quotients $\mathcal{S}_{e,k}/\mathcal{S}_{e,k-1}$ are independent of e , while the degrees are not. We define the following rational numbers:

- $\bar{\mu}_k = \bar{\mu}_k(\mathcal{S}) = \frac{\mu(\mathcal{S}_{e,k}/\mathcal{S}_{e,k-1})}{p^e}$, where $\mu(-)$ denotes the usual slope of the bundle;
- $r_k = \text{rank}(\mathcal{S}_{e,k}/\mathcal{S}_{e,k-1})$;
- $\nu_k = -\frac{\bar{\mu}_k}{\text{deg} Y}$.

REMARK 2.1. We point out that the numbers $\bar{\mu}$, r_k and ν_k are rational and independent from e for $e \gg 0$. In fact, we have that $\sum_{k=1}^t r_k \mu(\mathcal{S}_{e,k}/\mathcal{S}_{e,k-1}) = \text{deg}(F^{e*}\mathcal{S}) = p^e \text{deg } \mathcal{S}$, which implies the relation

$$\sum_{k=1}^t r_k \bar{\mu}_k = \text{deg } \mathcal{S}.$$

DEFINITION 2.2. Let \mathcal{S} be a locally free sheaf over a projective curve over an algebraically closed field of prime characteristic, and let $\bar{\mu}$ and r_k be as above. The *Hilbert–Kunz slope* of \mathcal{S} is the rational number

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^t r_k \bar{\mu}_k^2.$$

This notion was introduced by the first author in [3], where he also proved Theorem 2.5 below.

EXAMPLE 2.3. Let \mathcal{L} be a line bundle, then \mathcal{L} is semistable of slope $\mu(\mathcal{L}) = \deg \mathcal{L}$. The pullback along Frobenius is again a line bundle, $F^{e*}\mathcal{L} = \mathcal{L}^q = \mathcal{L}^{\otimes q}$, with $q = p^e$. It follows that $0 \subseteq \mathcal{L}$ is the strong Harder–Narasimhan filtration of \mathcal{L} , and the Hilbert–Kunz slope is just

$$\mu_{HK}(\mathcal{L}) = (\deg \mathcal{L})^2.$$

EXAMPLE 2.4. Let $d_1 < d_2 < \dots < d_m$ be nonnegative integers, and let $\mathcal{T} := \bigoplus_{i=1}^m \mathcal{O}(-d_i)^{\oplus r_i}$, where $\mathcal{O} := \mathcal{O}_Y$ and $r_i \in \mathbb{N}$. The Harder–Narasimhan filtration of \mathcal{T} is

$$0 \subseteq \mathcal{O}(-d_1)^{\oplus r_1} \subseteq \mathcal{O}(-d_1)^{\oplus r_1} \oplus \mathcal{O}(-d_2)^{\oplus r_2} \subseteq \dots \subseteq \bigoplus_{i=1}^m \mathcal{O}(-d_i)^{\oplus r_i}.$$

The quotients are direct sums of line bundles of the same degree, so their pullbacks under Frobenius are semistable. Hence, this is also the strong Harder–Narasimhan filtration of \mathcal{T} with invariants r_k , and $\bar{\mu}_k = \deg \mathcal{O}(-d_k) = -d_k \deg \mathcal{O}_Y(1) = -d_k \deg Y$. Then, the Hilbert–Kunz slope of \mathcal{T} is

$$\mu_{HK}(\mathcal{T}) = (\deg Y)^2 \sum_{k=1}^m r_k d_k^2.$$

THEOREM 2.5. (Brenner [3]) *Let Y denote a smooth projective curve of genus g over an algebraically closed field of positive characteristic p , and let $q = p^e$ for a nonnegative integer e . Let $0 \rightarrow \mathcal{S} \rightarrow \mathcal{T} \rightarrow \mathcal{Q} \rightarrow 0$ denote a short exact sequence of locally free sheaves on Y . Then, the following hold.*

- (1) *For every nonnegative integer e , the alternating sum of the dimensions of the global sections is*

$$\begin{aligned} & \sum_{m \in \mathbb{Z}} (h^0(F^{e*}\mathcal{S}(m)) - h^0(F^{e*}\mathcal{T}(m)) + h^0(F^{e*}\mathcal{Q}(m))) \\ &= \frac{q^2}{2 \deg Y} (\mu_{HK}(\mathcal{S}) - \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{Q})) + O(q^0). \end{aligned}$$

- (2) *If the field is the algebraic closure of a finite field, then the $O(q^0)$ -term is eventually periodic.*

The alternating sum in Theorem 2.5 is, in fact, a finite sum for every q . For $m \ll 0$, the locally free sheaves have no global sections, so all of the

terms are 0, and for $m \gg 0$, we have $H^1(Y, F^{e*}\mathcal{S}(m)) = 0$, and the sum is 0. Moreover, the sum is the dimension of the cokernel

$$\sum_{m \in \mathbb{Z}} \dim(\Gamma(Y, F^{e*}\mathcal{Q}(m))) / \text{im}(\Gamma(Y, F^{e*}\mathcal{T}(m))).$$

In [3], Brenner uses Theorem 2.5 to prove that the Hilbert–Kunz function of a homogeneous R_+ -primary ideal I in a normal two-dimensional standard graded K -domain R has the following form:

$$HK(I, q) = e_{HK}(I)q^2 + \gamma(q),$$

where $e_{HK}(I)$ is a rational number and $\gamma(q)$ is a bounded function, which is eventually periodic if K is the algebraic closure of a finite field. In particular, if I is generated by homogeneous elements f_1, \dots, f_n of degrees d_1, \dots, d_n , and $r_k, \bar{\mu}_k$ denote the numerical invariants of the strong Harder–Narasimhan filtration of the locally free sheaf $\text{Syz}(f_1, \dots, f_n)$ on the curve $Y = \text{Proj}R$, then the Hilbert–Kunz multiplicity of I is given by

$$e_{HK}(I) = \frac{1}{2 \deg Y} \left(\sum_{k=1}^t r_k \bar{\mu}_k^2 - (\deg Y)^2 \sum_{i=1}^n d_i^2 \right).$$

In Section 3, we apply this method to deduce a similar result for the generalized Hilbert–Kunz function, and answer a question of Dao and Smirnov [6, Example 6.2].

§3. The generalized Hilbert–Kunz function in dimension 2

LEMMA 3.1. *Let R be a two-dimensional normal K -domain of positive characteristic p with homogeneous maximal ideal \mathfrak{m} . We denote by $U = \text{Spec}R \setminus \{\mathfrak{m}\}$ the punctured spectrum. Let M be a finitely generated graded R -module with a presentation*

$$(2) \quad 0 \rightarrow I \rightarrow R^n \rightarrow M \rightarrow 0.$$

*Let $J = I^{**}$ be the reflexive hull of I (considered inside R^n), and let \mathcal{L} be the coherent sheaf corresponding to J on U , that is, $\mathcal{L} = \tilde{J}|_U$, then*

$$(3) \quad \begin{aligned} gHK(M, q) &= l_R(\Gamma(U, F^{e*}\mathcal{L}) / \text{im}F^{e*}I) \\ &= l_R((F^{e*}J)^{**} / \text{im}F^{e*}I), \end{aligned}$$

where $q = p^e$, and $\text{im}F^{e}I$ denotes the image of the map $F^{e*}I \rightarrow F^{e*}R^n \cong R^n$.*

Before proving the lemma, we explain the right-hand side of the equality (3).

First of all, in virtue of (1), we have $\Gamma(U, F^{e*}\mathcal{L}) = (F^{e*}J)^{**}$, so the second equality is clear. Then, the inclusion $I \hookrightarrow R^n$ factors through the reflexive module J . Applying the Frobenius functor to these maps, we get a commutative diagram

$$(4) \quad \begin{array}{ccc} F^{e*}I & \longrightarrow & F^{e*}R^n \cong R^n \\ \downarrow & \nearrow & \\ F^{e*}J & & \end{array}$$

Since the functor F^{e*} is not left exact in general, the maps in (4) are not injective. For this reason, we consider the image $\text{im}F^{e*}I \subseteq R^n$.

Since R is normal, U is smooth, and the absolute Frobenius morphism $F^e : U \rightarrow U$ is exact on U . Therefore, we pull back along F^e the inclusion $\mathcal{L} \hookrightarrow \mathcal{O}_U^n$, and we take sections on U , obtaining the inclusion

$$\Gamma(U, F^{e*}\mathcal{L}) \hookrightarrow \Gamma(U, F^{e*}\mathcal{O}_U^n) \cong \Gamma(U, \mathcal{O}_U^n) = R^n.$$

Therefore, the quotient $\Gamma(U, F^{e*}\mathcal{L})/\text{im}F^{e*}I$ is a quotient of submodules of R^n .

Proof. We apply the functor F^{e*} to the short exact sequence (2), and we get $F^{e*}I \rightarrow R^n \rightarrow F^{e*}M \rightarrow 0$. Therefore, we have

$$(5) \quad F^{e*}M = R^n/\text{im}F^{e*}I.$$

Then, we consider the short exact sequence

$$0 \rightarrow \Gamma(U, F^{e*}\mathcal{L})/\text{im}F^{e*}I \rightarrow R^n/\text{im}F^{e*}I \rightarrow R^n/\Gamma(U, F^{e*}\mathcal{L}) \rightarrow 0.$$

Taking local cohomology yields

$$0 \rightarrow H_m^0(\Gamma(U, F^{e*}\mathcal{L})/\text{im}F^{e*}I) \rightarrow H_m^0(R^n/\text{im}F^{e*}I) \rightarrow H_m^0(R^n/\Gamma(U, F^{e*}\mathcal{L})).$$

The module $\Gamma(U, F^{e*}\mathcal{L})$ is reflexive by (1), then by Lemma 1.1 the last module of the previous sequence is 0. Therefore, we get the following isomorphism:

$$(6) \quad \begin{aligned} H_m^0(R^n/\text{im}F^{e*}I) &\cong H_m^0(\Gamma(U, F^{e*}\mathcal{L})/\text{im}F^{e*}I) \\ &= \Gamma(U, F^{e*}\mathcal{L})/\text{im}F^{e*}I. \end{aligned}$$

The last equality holds because the module $\Gamma(U, F^{e*}\mathcal{L})/\text{im}F^{e*}I$ has support in \mathfrak{m} , since the sheaves \mathcal{L} and \tilde{I} coincide on U . Then, the desired formula follows from (5) and (6). \square

THEOREM 3.2. *Let R be a two-dimensional normal standard graded K -domain over an algebraically closed field K of prime characteristic p , and let M be a finitely generated graded R -module. Then, the generalized Hilbert–Kunz function of M has the form*

$$g_{HK}(M, q) = e_{g_{HK}}(M)q^2 + \gamma(q),$$

where $e_{g_{HK}}(M)$ is a rational number and $\gamma(q)$ is a bounded function.

Moreover, if K is the algebraic closure of a finite field, then $\gamma(q)$ is an eventually periodic function. In particular, given a graded presentation of M

$$\bigoplus_{i=1}^n R(-d_i) \xrightarrow{\psi} \bigoplus_{j=1}^m R(-e_j) \rightarrow M \rightarrow 0$$

and the corresponding short exact sequence of locally free sheaves on the curve $Y = \text{Proj}R$

$$(7) \quad 0 \rightarrow \mathcal{S} := \widetilde{\ker \psi} \rightarrow \mathcal{T} := \bigoplus_{i=1}^n \mathcal{O}_Y(-d_i) \rightarrow \mathcal{Q} := \widetilde{\text{im} \psi} \rightarrow 0,$$

then the generalized Hilbert–Kunz multiplicity of M is

$$e_{g_{HK}}(M) = \frac{1}{2 \deg Y} \left(\mu_{HK}(\mathcal{S}) - (\deg Y)^2 \sum_{i=1}^n d_i^2 + \mu_{HK}(\mathcal{Q}) \right).$$

Proof. Let u_1, \dots, u_m be homogeneous generators of M of degrees e_1, \dots, e_m , respectively, and let

$$0 \rightarrow I \rightarrow \bigoplus_{j=1}^m R(-e_j) \xrightarrow{u_1, \dots, u_m} M \rightarrow 0$$

be the corresponding short exact sequence. Let f_1, \dots, f_n be homogeneous generators of I of degrees d_1, \dots, d_n respectively and let

$$0 \rightarrow N \rightarrow \bigoplus_{i=1}^n R(-d_i) \xrightarrow{f_1, \dots, f_n} I \rightarrow 0$$

be the corresponding graded short exact sequence. This last sequence induces the short exact sequence (7) on Y , and the short exact sequence

$$(8) \quad 0 \rightarrow \mathcal{E} := \tilde{N}|_U \rightarrow \mathcal{F} := \bigoplus_{i=1}^n \mathcal{O}_U(-d_i) \rightarrow \tilde{I}|_U \rightarrow 0$$

on the punctured spectrum U . The modules N and I are submodules of finite free R -modules, so they are torsion-free. It follows that the corresponding sheaves \mathcal{E} and $\tilde{I}|_U$ on U are locally free, since U is regular. Moreover, if $J = I^{**}$ is the reflexive hull of I , and \mathcal{L} is the coherent sheaf corresponding to J on U , we have that $\mathcal{L} = \tilde{I}|_U$ as sheaves on U .

From Lemma 3.1, the generalized Hilbert–Kunz function of M is given by

$$gHK(M, q) = l_R(\Gamma(U, F^{e*}\mathcal{L})/\text{im}F^e I) = \sum_{m \in \mathbb{Z}} l_R((\Gamma(U, F^{e*}\mathcal{L})/\text{im}F^e I)_m).$$

To compute the last sum, we consider the sequence (7), we pull it back along the e th absolute Frobenius morphism on Y and we tensor with $\mathcal{O}_Y(m)$, for an integer m . We obtain an exact sequence

$$0 \rightarrow F^{e*}\mathcal{S}(m) \rightarrow F^{e*}\mathcal{T}(m) \rightarrow F^{e*}\mathcal{Q}(m) \rightarrow 0.$$

Then, we take global sections $\Gamma(Y, -)$ of the last sequence and we get

$$0 \rightarrow \Gamma(Y, F^{e*}\mathcal{S}(m)) \rightarrow \Gamma(Y, F^{e*}\mathcal{T}(m)) \xrightarrow{\varphi_m} \Gamma(Y, F^{e*}\mathcal{Q}(m)) \rightarrow \dots$$

We are interested in the cokernel of the map φ_m . Its image is clearly $(\text{im}F^e I)_m$. For the evaluation of the sheaf $F^{e*}\mathcal{Q}(m)$ on Y , we consider the sequences (7) and (8), and we obtain

$$\Gamma(Y, F^{e*}\mathcal{Q}(m)) \cong \Gamma(U, F^{e*}\mathcal{L})_m.$$

Therefore, we get

$$\text{Coker}(\varphi_m) = \Gamma(U, F^{e*}\mathcal{L})_m / (\text{im}F^e I)_m = (\Gamma(U, F^{e*}\mathcal{L})/\text{im}F^e I)_m.$$

It follows that

$$gHK(M, q) = \sum_{m \in \mathbb{Z}} \text{Coker}(\varphi_m).$$

We compute the last sum with Theorem 2.5, and we obtain the desired formula for the generalized Hilbert–Kunz function.

For the generalized Hilbert–Kunz multiplicity, it is enough to notice that $\mu_{HK}(\mathcal{T}) = (\text{deg } Y)^2 \sum_{i=1}^n d_i^2$, by Example 2.4. □

COROLLARY 3.3. *Let I be a nonzero ideal generated by homogeneous elements f_1, \dots, f_n of degrees d_1, \dots, d_n , respectively, and let d be the degree of the ideal sheaf associated to I on $Y = \text{Proj}R$. Then, the generalized Hilbert–Kunz multiplicity of R/I is given by*

$$e_{gHK}(R/I) = \frac{1}{2 \deg Y} \left(\sum_{k=1}^t r_k \bar{\mu}_k^2 - (\deg Y)^2 \sum_{i=1}^n d_i^2 + d^2 \right),$$

where $r_k, \bar{\mu}_k$ and t are the numerical invariants of the strong Harder–Narasimhan filtration of the syzygy bundle $\text{Syz}(f_1, \dots, f_n)$.

Proof. In this case, the presenting sequence of R/I is just $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$, and the sequence (7) is then

$$0 \rightarrow \text{Syz}(f_1, \dots, f_n) \rightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(-d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{Q} \rightarrow 0.$$

Therefore, by Theorem 3.2, the generalized Hilbert–Kunz multiplicity of R/I is given by

$$\frac{1}{2 \deg Y} \left(\mu_{HK}(\text{Syz}(f_1, \dots, f_n)) - (\deg Y)^2 \sum_{i=1}^n d_i^2 + \mu_{HK}(\mathcal{Q}) \right).$$

In this situation, \mathcal{Q} is a line bundle, so by Example 2.3, $\mu_{HK}(\mathcal{Q}) = (\deg \mathcal{Q})^2 = d^2$, and by definition, $\mu_{HK}(\text{Syz}(f_1, \dots, f_n)) = \sum_{k=1}^t r_k \bar{\mu}_k^2$. \square

EXAMPLE 3.4. Let h be a homogeneous element of degree $a > 0$, and let $I = (h)$. Then, we have $0 \rightarrow \mathcal{O}_Y(-a) \xrightarrow{\sim} \mathcal{L} \rightarrow 0$; hence, $\text{Syz}(h) = 0$. Since I is principal, the degree of the ideal sheaf associated to I is $a \cdot \deg Y$. The generalized Hilbert–Kunz multiplicity is then

$$e_{gHK}(R/I) = \frac{1}{2 \deg Y} \left(-(\deg Y)^2 a^2 + (\deg Y)^2 a^2 \right) = 0,$$

in accordance with Proposition 1.2.

EXAMPLE 3.5. Let I be a prime ideal of height 1 generated by two homogeneous elements f and g of degrees a and b , respectively. Then, the syzygy sequence is

$$0 \rightarrow \text{Syz}(f, g) \rightarrow \mathcal{O}_Y(-a) \oplus \mathcal{O}_Y(-b) \rightarrow \mathcal{Q} \rightarrow 0,$$

with \mathcal{Q} the line bundle of, say, degree d associated to the ideal I . From this sequence, we see that the syzygy bundle has rank 1 and degree $\deg Y(-a - b) - d$. Therefore, we have

$$\begin{aligned} e_{gHK}(R/I) &= \frac{1}{2 \deg Y} \left((\deg Y(-a - b) - d)^2 - (\deg Y)^2(a^2 + b^2) + d^2 \right) \\ &= \frac{1}{2 \deg Y} \left(2d^2 + (\deg Y)^2(a^2 + b^2) + 2ab(\deg Y)^2 \right. \\ &\quad \left. + 2d \deg Y(a + b) - (\deg Y)^2(a^2 + b^2) \right) \\ &= \frac{1}{\deg Y} \left(d^2 + ab(\deg Y)^2 + d(a + b) \deg Y \right) \\ &= \frac{d^2}{\deg Y} + ab \deg Y + d(a + b). \end{aligned}$$

EXAMPLE 3.6. Let P be a point of the smooth projective curve $Y = \text{Proj}R$, and let $I \subseteq R$ be the corresponding homogeneous prime ideal of height 1. The ideal I is minimally generated by two linear forms f and g . In fact, f and g correspond to two hyperplanes in the projective space where Y is embedded which meet transversally in P . Then, using the notations of Example 3.5, we have $a = b = 1$, and $d = -1$, since the line bundle \mathcal{Q} associated to the ideal I is a subsheaf of \mathcal{O}_Y . Therefore, we obtain

$$e_{gHK}(R/I) = \frac{1}{\deg Y} + \deg Y - 2 = \frac{(\deg Y - 1)^2}{\deg Y}.$$

§4. The limit of generalized Hilbert–Kunz multiplicity

Let R be a standard graded domain flat over \mathbb{Z} such that almost all fiber rings $R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ are geometrically normal two-dimensional domains. We define $R_0 := R \otimes_{\mathbb{Z}} \mathbb{Q}$ and the corresponding projective curve $Y_0 := \text{Proj}R_0$ over the generic point. We denote by $Y_p := \text{Proj}R_p$ the projective curve over the prime number p . This is a smooth projective curve for almost all primes. If \mathcal{S} is a sheaf over the curve $Y := \text{Proj}R$, we denote by \mathcal{S}_p (resp. \mathcal{S}_0) the corresponding restriction to the curves Y_p (resp. Y_0).

REMARK 4.1. In our setting, the curves Y_0 and Y_p are not defined over an algebraically closed field. However, we may consider the curves $\bar{Y}_0 := Y_0 \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ and $\bar{Y}_p := Y_p \times_{\mathbb{Z}/p\mathbb{Z}} \bar{\mathbb{Z}/p\mathbb{Z}}$, which are smooth projective curves over the algebraic closures. We can consider the definitions of degree, slope,

semistable, HN filtration and strong HN filtration for these curves, and transfer them to the original curves Y_0 and Y_p . Therefore, we move to the algebraic closure and back whenever this is convenient.

Let M be a graded R -module. For every prime p , we can consider the reduction to characteristic p , $M_p := M \otimes_R R_p \cong M \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, and compute the generalized Hilbert–Kunz multiplicity $e_{gHK}^{R_p}(M_p)$ of the R_p -module M_p . Since the projective curve Y_p is smooth for almost all primes p , by Theorem 3.2, we know that $e_{gHK}^{R_p}(M_p)$ exists, and that it is rational for these primes. We are interested in the behavior of $e_{gHK}^{R_p}(M_p)$ for $p \rightarrow +\infty$.

We introduce the following characteristic zero version of the Hilbert–Kunz slope.

DEFINITION 4.2. Let \mathcal{S} be a locally free sheaf over a projective curve over an algebraically closed field of characteristic zero, and let $\mathcal{S}_1 \subseteq \dots \subseteq \mathcal{S}_t = \mathcal{S}$ be the Harder–Narasimhan filtration of \mathcal{S} . For every $k = 1, \dots, t$, we set $\bar{\mu}_k = \bar{\mu}_k(\mathcal{S}) = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $r_k = \text{rank}(\mathcal{S}_k/\mathcal{S}_{k-1})$. The *Hilbert–Kunz slope* of \mathcal{S} is the rational number

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^t r_k \bar{\mu}_k^2.$$

The name Hilbert–Kunz slope is justified by the following result of Trivedi (cf. [12, Lemma 1.14]).

LEMMA 4.3. (Trivedi) *Let $h \in \mathbb{Z}_+$, let Y be a smooth projective curve over $\text{Spec}\mathbb{Z}_h$, and let \mathcal{S} be a locally free sheaf over Y . We denote by \mathcal{S}_0 and \mathcal{S}_p the restrictions of \mathcal{S} to Y_0 and Y_p , for $p \nmid h$. Then,*

$$\lim_{\substack{p \rightarrow +\infty \\ p \nmid h}} \mu_{HK}(\mathcal{S}_p) = \mu_{HK}(\mathcal{S}_0).$$

THEOREM 4.4. *Let R be a standard graded domain flat over \mathbb{Z} such that almost all fiber rings $R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ are geometrically normal two-dimensional domains. Let M be a graded R -module with a graded presentation*

$$\bigoplus_{i=1}^n R(-d_i) \xrightarrow{\psi} \bigoplus_{j=1}^m R(-e_j) \rightarrow M \rightarrow 0,$$

and corresponding short exact sequence of locally free sheaves $0 \rightarrow \mathcal{S}_0 \rightarrow \mathcal{T}_0 \rightarrow \mathcal{Q}_0 \rightarrow 0$ on the generic fiber $Y_0 = \text{Proj}R_0$, with notations as above.

Then, the limit

$$\lim_{p \rightarrow +\infty} e_{gHK}^{R_p}(M_p)$$

exists, and it is equal to the rational number

$$\frac{1}{2 \deg Y_0} (\mu_{HK}(\mathcal{S}_0) - \mu_{HK}(\mathcal{T}_0) + \mu_{HK}(\mathcal{Q}_0)).$$

Proof. Let u_1, \dots, u_m be homogeneous generators of M as R -module, and let f_1, \dots, f_n be homogeneous generators of $I := \text{Syz}(u_1, \dots, u_m)$. We obtain two short exact sequences

$$\begin{aligned} 0 \rightarrow I \rightarrow \bigoplus_{j=1}^m R(-e_j) \xrightarrow{u_1, \dots, u_m} M \rightarrow 0 \quad \text{and} \\ (9) \quad 0 \rightarrow N \rightarrow \bigoplus_{i=1}^n R(-d_i) \xrightarrow{f_1, \dots, f_n} I \rightarrow 0. \end{aligned}$$

Tensoring these sequences with the flat \mathbb{Z} -module \mathbb{Q} , we obtain exact sequences of R_0 -modules. On the other hand, if we apply the functor $-\otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ to the sequences (9), exactness is preserved for all primes except for a finite number of them. Let h be the product of those primes, and we consider the smooth projective curve $Y = \text{Proj}R_h$ over $\text{Spec}\mathbb{Z}_h$.

Let $U = D(R_{h+})$ denote the relative punctured spectrum. The sheaf $\tilde{I}|_U$ restricts to $U_0 = U \cap \text{Spec}R_0$ as a locally free sheaf. By possibly shrinking the set $D(h)$, we may assume that $\tilde{I}|_U$ is locally free. By further shrinking, we may assume that $\mathcal{E} := \tilde{N}|_U$ is also locally free. Then, for almost all p , I_p and N_p are locally free on U_p .

Let $\mathcal{S}, \mathcal{T}, \mathcal{Q}$ be the locally free sheaves on Y corresponding to $\mathcal{E}, \bigoplus_{i=1}^n R(-d_i), \tilde{I}|_U$, which, by the second sequence of (9), form an exact sequence $0 \rightarrow \mathcal{S} \rightarrow \mathcal{T} \rightarrow \mathcal{Q} \rightarrow 0$ on Y . Its restrictions give the short exact sequences $0 \rightarrow \mathcal{S}_0 \rightarrow \mathcal{T}_0 \rightarrow \mathcal{Q}_0 \rightarrow 0$ on the generic fiber Y_0 , and $0 \rightarrow \mathcal{S}_p \rightarrow \mathcal{T}_p \rightarrow \mathcal{Q}_p \rightarrow 0$ on the fiber Y_p , for $p \nmid h$.

Let p be a prime number not dividing h , then we are in the situation of Theorem 3.2, so we obtain

$$e_{gHK}^{R_p}(M_p) = \frac{1}{2 \deg Y_p} (\mu_{HK}(\mathcal{S}_p) - \mu_{HK}(\mathcal{T}_p) + \mu_{HK}(\mathcal{Q}_p)).$$

Then, taking the limit for $p \rightarrow +\infty$ and applying Lemma 4.3, we conclude the proof. □

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