5
Fermions

In this chapter we turn to a subject that is still not completely understood, the lattice formulation of fermionic fields. These complications with spinor particles are already present at the free-field level; a straightforward generalization of the ideas in the last chapter does not give a simple particle spectrum. The action needs additional terms which vanish in the naive continuum limit. These terms, needed to eliminate certain lattice artifacts, tend to mutilate the classical symmetries of the theory. The extent to which this is necessary is still an open question.

Before proceeding to these topics, we must introduce the concepts of anticommuting numbers and integration over these variables. The path integral is no longer a sum, but a particular linear operation from functions of anticommuting variables into the complex numbers. We will also introduce anticommuting sources for the dynamical fields. A differentiation with respect to these sources gives the Green's functions, as in the last chapter. Both integration and differentiation with anticommuting variables have useful analogous properties to ordinary integrals and derivatives; however, there are some amusing distinctions. In particular, fermionic integrals and derivatives involve essentially the same operation.

As in the previous chapter, we begin with the continuum Lagrangian density for a free field, in this case a four-component Dirac spinor

$$\mathscr{L} = \bar{\psi}(\partial\!\!\!/ + m)\psi. \tag{5.1}$$

A slash through a four-vector represents the usual sum

$$p\!\!\!/ = p_\mu \gamma_\mu, \tag{5.2}$$

where the γ_μ are a set of four-by-four Euclidian Dirac matrices satisfying the algebra

$$[\gamma_\mu, \gamma_\nu]_+ = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \tag{5.3}$$

$$\gamma_\mu^\dagger = \gamma_\mu. \tag{5.4}$$

As usual, we define

$$\bar{\psi} = \psi^\dagger \gamma_4. \tag{5.5}$$

For future use we also introduce

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \gamma_5^\dagger. \tag{5.6}$$

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A convenient representation for these matrices is

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad i = 1, 2, 3, \tag{5.7}$$

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.8}$$

$$\gamma_5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{5.9}$$

The matrix elements here are themselves two-by-two matrices and the σ_i are the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.10}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{5.11}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.12}$$

Note that this Lagrangian is invariant under the substitution

$$\psi \to e^{i\theta}\psi. \tag{5.13}$$

This symmetry is directly related to the conservation of fermion number. When the mass m vanishes, the theory also has a 'chiral' or 'γ_5' symmetry under

$$\psi \to e^{i\theta\gamma_5}\psi. \tag{5.14}$$

In a naive canonical treatment, these symmetries are generated by the currents

$$j_\mu = \bar{\psi}\gamma_\mu \psi \tag{5.15}$$

and

$$j_\mu^5 = \bar{\psi}\gamma_\mu \gamma_5 \psi. \tag{5.16}$$

Careful perturbative analysis (Adler, 1969; Bell and Jackiw, 1969) indicates the impossibility of maintaining conservation of both these currents in the four-dimensional quantum theory. This 'anomaly' will not be derived here; we only note that it is deeply related to the difficulties encountered in the lattice formulation, which naively preserves these symmetries (Chodos and Healy, 1977; Nielsen and Ninomiya, 1981*a*, *b*; Kerler, 1981*a*; Becher and Joos, 1982; Rabin, 1982).

As in the previous chapter, we introduce a four-dimensional hypercubic lattice of N^4 sites. With each site m we associate an independent four-component spinor variable ψ_m. To keep the lattice action simple we define the derivative symmetrically

$$\partial_\mu \psi \to \frac{1}{2a}(\psi_{m_\nu+\delta_{\mu\nu}} - \psi_{m_\nu-\delta_{\mu\nu}}). \tag{5.17}$$

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Summing the Lagrangian over all sites gives the lattice action

$$S = \sum_{m,n} \bar{\psi}_m M_{mn} \psi_n, \tag{5.18}$$

where
$$M_{mn} = \tfrac{1}{2}a^3 \sum_\mu \gamma_\mu (\delta^4_{m_\nu+\delta_{\mu\nu}, n_\nu} - \delta^4_{m_\nu-\delta_{\mu\nu}, n_\nu}) + a^4 m \delta^4_{mn}. \tag{5.19}$$

Note that the symmetries of eq. (5.13) and, when $m = 0$, eq. (5.14) are still manifest. We now put this action into a path integral

$$Z_0 = \int [\mathrm{d}\psi\, \mathrm{d}\bar{\psi}]\, \mathrm{e}^{-S}. \tag{5.20}$$

Unlike in the scalar case, this is not an ordinary integral, and needs further definition. We will first discuss such integrals for quadratic actions of the form of eq. (5.18) with an arbitrary matrix M. Later we will return to the specific theory in eq. (5.19).

We begin by requiring the integration variables to anticommute

$$[\psi^\alpha_m, \psi^\beta_n]_+ = [\psi^{\alpha\dagger}_m, \psi^\beta_n]_+ = [\psi^{\alpha\dagger}_m, \psi^{\beta\dagger}_n]_+ = 0, \tag{5.21}$$

where α and β are the usually suppressed spinor indices. This equation contrasts sharply with the canonical relations for Dirac operators in Hilbert space. In the path integral ψ and $\psi^\dagger$ are independent fermionic objects. As in the previous chapter, our integral is of an exponentiated quadratic form. We will see that its evaluation again reduces to knowing the determinant of M. Before proceeding, however, we find it advantageous at this point to introduce the concept of sources for the fermionic fields.

As our fields anticommute, any sources coupled to them should behave similarly. We consider separate sources b^α_m and c^α_m for ψ^α_m and $\bar{\psi}^\alpha_m$, respectively. Suppressing repeated site and spinor indices, we generalize the action to

$$S = \bar{\psi} M \psi + b\psi - \bar{\psi} c. \tag{5.22}$$

We adopt the convention that all the spinor quantities ψ, $\bar{\psi}$, b and c anticommute with themselves and each other. We wish to define the fermionic path integral such that the linear source terms can be eliminated by a simple completion of the square and a shift of the integration variables, in analogy to an ordinary integral. Thus we demand

$$Z = Z_0 \exp(-bM^{-1}c), \tag{5.23}$$

where Z_0 is the sourceless integral from eq. (5.20). For the free field considered here, the overall factor of Z_0 is irrelevant to the evaluation of Green's functions. In particular, the fermion propagator is given, as for scalar fields, by the inverse of the kinetic matrix M. However, in more general applications, i.e. with gauge fields, one may wish to have M to depend on other interacting fields. In this case we need the explicit

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functional dependence of Z_0 on this matrix. We will now demonstrate that Z_0 is simply the determinant of M (Matthews and Salam, 1954).

To proceed, we need the concept of derivatives with respect to our fermionic sources. Such derivatives should satisfy

$$\left[\frac{\mathrm{d}}{\mathrm{d}b_m^\alpha}, b_n^\beta\right]_+ = \delta_{mn}\,\delta^{\alpha\beta} \tag{5.24}$$

and a corresponding equation for the c's. This generalizes ordinary differentiation, where one would have a commutator. Note that these anticommutation relations are precisely those of the creation and annihilation operators for fermions on the sites of our lattice

$$[b_m^{\alpha\dagger}, b_n^\beta]_+ = \delta_{mn}\,\delta^{\alpha\beta}. \tag{5.25}$$

We can realize these relations on a Fock space of states created by

$$\frac{\mathrm{d}}{\mathrm{d}b_m^\alpha} \leftrightarrow b_m^{\alpha\dagger} \tag{5.26}$$

on a 'vacuum' satisfying

$$b_m^\alpha\,|0) = c_m^\alpha\,|0) = 0. \tag{5.27}$$

Operating on this vacuum is equivalent to turning off the sources. This 'Euclidian vacuum' should not be confused with the conventional Hilbert space state in Minkowski space, as found in the transfer matrix formalism discussed in the first chapter. We use a creation operator notation here for compactness and to avoid confusion from the fact that $\mathrm{d}/\mathrm{d}b$ is not really a usual derivative.

Our path integral with sources is an operator in this space. Operating on the vacuum from the right, we define the useful state

$$(Z\,| = (0\,|\,Z(b,c). \tag{5.28}$$

Fermion Green's functions are matrix elements between this state and the vacuum

$$\int [\mathrm{d}\psi\,\mathrm{d}\bar{\psi}]\,\mathrm{e}^{-S}\bar{\psi}_{i_1}\ldots\bar{\psi}_{i_n}\psi_{j_1}\ldots\psi_{j_n} = (Z\,|\,c_{i_1}^\dagger\ldots c_{i_n}^\dagger\, b_{j_1}^\dagger\ldots b_{j_n}^\dagger\,|\,0). \tag{5.29}$$

Our creation operators produce the ends of the external lines in a general correlation function.

Continuing toward our goal of evaluating Z_0, we now present a useful identity on exponentials of quadratic forms in creation and annihilation operators. Let F and G be N^4-by-N^4 symmetric matrices. We would like to take the expression

$$(\psi(\lambda)\,| = (0\,|\,\mathrm{e}^{bFc}\,\mathrm{e}^{\lambda b^\dagger G c^\dagger} \tag{5.30}$$

and manipulate the creation operators to the left to obtain a single exponential of a quadratic form in the annihilation operators alone.

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Straightforward manipulations, which we invite the reader to perform, yield the identities

$$(\psi(\lambda)\,|\,b^\dagger = -(\psi(\lambda)\,|\,(F^{-1}-\lambda G)^{-1}c, \tag{5.31}$$

$$(\psi(\lambda)\,|\,c^\dagger = +(\psi(\lambda)\,|\,(F^{-1}-\lambda G)^{-1}b. \tag{5.32}$$

Using these in the derivative of expression (5.30) with respect to the parameter λ gives a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\psi\,| = (\psi\,|\,[-\mathrm{Tr}\,(G(F^{-1}-\lambda G)^{-1})+b(F^{-1}-\lambda G)^{-1}G(F^{-1}-\lambda G)^{-1}c]. \tag{5.33}$$

With the initial condition

$$(\psi(\lambda = 0)\,| = (0\,|\,\mathrm{e}^{bFc}, \tag{5.34}$$

we can integrate to obtain

$$(\psi\,| = |\,I-\lambda FG\,|\,(0\,|\exp{(b(F^{-1}-\lambda G)^{-1}c)}. \tag{5.35}$$

To verify that this is indeed a solution of eq. (5.33), make use of the well-known identity

$$|\,F\,| = \exp{[\mathrm{Tr}\,(\ln F)]}. \tag{5.36}$$

With eq. (5.33) in hand, we return to our path integral and write

$$M = I+(M-I), \tag{5.37}$$

where I is the identity matrix. Treating $M-I$ as a perturbation, we have

$$(Z\,| = (0\,|\int[\mathrm{d}\psi\,\mathrm{d}\bar{\psi}]\,\mathrm{e}^{-\bar{\psi}\psi-b\psi+\bar{\psi}c}\exp{(c^\dagger(I-M)\,b^\dagger)}. \tag{5.38}$$

As before, the integral can be done by completing the square; however, now the normalization is truly arbitrary. We define

$$\int[\mathrm{d}\psi\,\mathrm{d}\bar{\psi}]\,\mathrm{e}^{-\bar{\psi}\psi} = 1. \tag{5.39}$$

Thus we have

$$(Z\,| = (0\,|\,\mathrm{e}^{-bc}\,\mathrm{e}^{-b^\dagger(I-M)\,c^\dagger}. \tag{5.40}$$

This is exactly in the form needed for the identity in eq. (5.33), which gives the final result

$$(Z\,| = |\,M\,|\,(0\,|\,\mathrm{e}^{-bM^{-1}c}. \tag{5.41}$$

Turning off the sources, we see that Z_0 is precisely the determinant of M

$$|\,M\,| = \int[\mathrm{d}\psi\,\mathrm{d}\bar{\psi}]\,\mathrm{e}^{-\bar{\psi}M\psi}. \tag{5.42}$$

Note the similarity of this with the boson result in eq. (4.22). The anticommuting fermion fields have moved the determinant from the denominator to the numerator. For scalar fields there is also an operator formalism parallel to that presented here. We invite the reader to work out the Bose analog of eq. (5.35).

This discussion paid no attention to the precise form of the kinetic matrix

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M; we only used the quadratic nature of the fermion action. We now return to the specific case in eq. (5.19) and study the propagator

$$S_{mn} = Z_0^{-1} \int [\mathrm{d}\psi\, \mathrm{d}\bar{\psi}]\, \mathrm{e}^{-\bar{\psi} M \psi}\, \psi_m \bar{\psi}_n. \tag{5.43}$$

In our operator formalism we have

$$S_{mn} = Z_0^{-1} (Z | b_m^\dagger c_n^\dagger | 0) = (M^{-1})_{mn}. \tag{5.44}$$

As in the last chapter, M is diagonalized and inverted with a Fourier transform. This gives

$$(M^{-1})_{mn} = a^{-4} N^{-4} \sum_k \tilde{M}_k^{-1}\, \mathrm{e}^{2\pi \mathrm{i} k \cdot (m-n)/N}, \tag{5.45}$$

where
$$\tilde{M}_k = m + \mathrm{i} a^{-1} \sum_\mu \gamma_\mu \sin(2\pi k/N). \tag{5.46}$$

Still following the last chapter, we let our lattice size become large and replace sums over k with integrals

$$q_\mu = 2\pi k_\mu/(Na), \tag{5.47}$$

$$a^{-4} N^{-4} \sum_k \to \int \mathrm{d}^4 q/(2\pi)^4, \tag{5.48}$$

$$\tilde{M}_k = m + \mathrm{i} a^{-1} \sum_\mu \gamma_\mu \sin(a q_\mu). \tag{5.49}$$

If we now consider small lattice spacing and expand in powers of a, we find
$$\tilde{M}_k = m + \mathrm{i}\not{q} + O(a^2). \tag{5.50}$$

It thus appears that we have recovered the usual continuum fermion propagator. Unfortunately, more care is needed at the upper limits of the momentum integrals. When q_μ is π/a, the periodic sine function in eq. (5.49) vanishes. Here the $O(a^2)$ terms cannot be neglected. Indeed, the propagator has no supression of momentum values near π/a; therefore we must expect rapid variations in the fields from site to neighboring site. This precludes the above simple continuum limit and will also destroy any attempt to formulate a transfer matrix along the lines of chapter 3.

To isolate the large momentum region, consider one component of q_μ and replace it with
$$\tilde{q}_\mu = q_\mu - \pi/a \tag{5.51}$$
over half the integration region

$$\int_{-\pi/a}^{\pi/a} \mathrm{d}q_\mu\, \tilde{M}_k^{-1} = \int_{-\pi/2a}^{\pi/2a} (\mathrm{d}q_\mu + \mathrm{d}\tilde{q}_\mu)\, \tilde{M}_k^{-1}. \tag{5.52}$$

For small lattice spacing, a finite range of the integration variables dominates each of these terms. Now an approximation along the lines of eq. (5.50) is valid. For each space-time dimension we have two independent regions where the theory gives a free fermion propagator in the continuum

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limit. We actually have $2^4 = 16$ independent fermion species, even though we initially seemed to have but one.

This multiplicity in the spectrum arose because we have implemented a regularization scheme that, when $m = 0$, keeps an exact γ_5 symmetry at all stages. It therefore cannot possess the known chiral anomaly. The theory has created extra species which cancel this phenomenon. Note that the new fermions use different γ matrices; i.e. when we shift as in eq. (5.51), the sine function gives an extra negative sign

$$\gamma_\mu \sin(q_\mu a) = -\gamma_\mu \sin(\tilde{q}_\mu a). \tag{5.53}$$

This sign is absorbed by redefining γ_μ and therefore γ_5 as well. Those fermions associated with an odd number of components of q being shifted by π/a will transform under the conjugate of the rotation in eq. (5.14). We have equal numbers of states with each chirality.

Several solutions exist for this 'doubling' problem. Perhaps the simplest is to ignore it and say that the theory is automatically generating a large number of fermion 'flavors.' Indeed, real quarks do appear to come in several species. Nonetheless, it seems a bit far fetched to use an artifact of the lattice formulation to explain this degeneracy.

Observing that the problem only occurs when the magnitude of q is large, one might try artificially to exclude large components. In general this is dangerous because of completeness in the Fourier transform. Here, however, we can use the spinor index to partially do precisely this. By associating only a single spinor component with each site and putting different components on separate classes of sites, one effectively puts the components on smaller sublattices. This reduces the effective upper limit of the momentum integrals and thereby reduces some of the unwanted degeneracy. Such techniques have had considerable success in a Hamiltonian formulation of the lattice theory, where continuous time removes half of the unwanted states (Kogut and Susskind, 1975; Banks *et al.*, 1977).

The multiplicity problem arises from the periodic nature of the sine function appearing in the Fourier transform of the nearest-neighbor form for the lattice derivative. In a continuum theory, a derivative is simply a factor of the momentum in Fourier space. Thus another solution to the lattice degeneracy is to replace the sin of the momentum with the momentum itself. This defines a new lattice derivative which immediately kills the extra states. On returning to position space, this derivative no longer involves just nearby sites, but includes products of site variables with arbitrary separations. This keeps an apparent chiral symmetry; to see the anomaly requires a careful and somewhat controversial treatment of limits (Drell, Weinstein and Yankielowicz, 1976; Karsten and Smit, 1981).

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A utilitarian approach to the doubling problem is to add to the naive action new terms which suppress the extra states while vanishing in a continuum limit with the desired fermion species. To keep the action as local as possible, we require the new terms to involve nearest-neighbor pairs of lattice sites. This means that in momentum space these terms will involve only simple trigonometric functions of the momentum. An addition which accomplishes our needs replaces $\tilde{M}_k$ with

$$\tilde{M}_k = m + \mathrm{i}a^{-1}\sum_{\mu}\gamma_\mu \sin(aq_\mu) + ra^{-1}\sum_{\mu}(1-\cos(aq_\mu)). \tag{5.54}$$

Here r is an arbitrary parameter. Note that for small momentum the new term is of order the cutoff and thus drops out. However, when a component of q is near π/a, the addition increases the mass of the unwanted state by $2r/a$. In the continuum limit all the extra states go to infinite mass and only one species of mass m survives. Setting r to unity (Wilson, 1977), we obtain the position space form

$$\begin{aligned} M_{mn} &= (a^4 m + 4a^3)\,\delta^4_{mn} \\ &\quad + \tfrac{1}{2}a^3\sum_{\mu}[(1+\gamma_\mu)\,\delta^4_{m_\nu+\delta_{\mu\nu},\, n_\nu} + (1-\gamma_\mu)\,\delta^4_{m_\nu-\delta_{\mu\nu},\, n_\nu}]. \end{aligned} \tag{5.55}$$

Whenever a quark moves from one site to the next, its wave function picks up a factor of $1 \pm \gamma_\mu$ rather than the γ_μ from eq. (5.19). Note that $(1 \pm \gamma_\mu)/2$ is a rank two projection

$$(\tfrac{1}{2}(1 \pm \gamma_\mu))^2 = \tfrac{1}{2}(1 \pm \gamma_\mu), \tag{5.56}$$

$$\mathrm{Tr}\,(\tfrac{1}{2}(1 \pm \gamma_\mu)) = 2. \tag{5.57}$$

Thus part of the spinor field no longer propagates. This reduces the degeneracy by a factor of two for each dimension, exactly as needed to remove the extra states. This method is referred to as the projection operator technique of Wilson.

The simplicity of this method is convenient for calculation. However, it totally mutilates the chiral symmetry of the theory because the added piece is like a mass term for the unwanted fermions. This is probably more of a mutilation than necessary; with several flavors not all currents need to have an anomaly. Consequences of the related symmetries, such as Goldstone bosons, are masked in the projection operator formalism until one reaches the continuum. The extent to which these latent symmetries can survive in a lattice theory is still unclear.

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Problems

1. Derive the analog of eq. (5.35) for bosonic operators.
2. For a single pair of fermionic variables ψ and $\overline{\psi}$, derive the formulae

$$\int d\psi\, d\overline{\psi}\, 1 = \int d\psi\, d\overline{\psi}\, \psi =$$
$$\int d\psi\, d\overline{\psi}\, \overline{\psi} = 0$$
$$\int d\psi\, d\overline{\psi}\, \psi\overline{\psi} = 1.$$

3. Rescale the fields to put eq. (5.55) in the form

$$M_{mn} = \delta^4_{mn} + K \sum_\mu ((1+\gamma_\mu)\, \delta^4_{m_\nu+\delta_{\mu\nu},\, n_\nu} + (1-\gamma_\mu)\, \delta^4_{m_\nu-\delta_{\mu\nu},\, n_\nu}),$$

where the 'hopping constant' K approaches 1/8 for a continuum limit. This represents a critical point where the correlation length diverges when expressed in units of the lattice spacing.

4. We have discussed periodic boundary conditions. Is there any motivation for antiperiodic boundary conditions for fermionic fields? (Hint: what sign does a fermion loop wrapping around the lattice give to eq. (3.32)?)

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