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On the Kohnen plus space for Hilbert modular forms of half-integral weight I

Kaoru Hiraga and Tamotsu Ikeda

To the memory of Masaru Ueda

Abstract

In this paper, we construct a generalization of the Kohnen plus space for Hilbert modular forms of half-integral weight. The Kohnen plus space can be characterized by the eigenspace of a certain Hecke operator. It can be also characterized by the behavior of the Fourier coefficients. For example, in the parallel weight case, a modular form of weight $\kappa + (1/2)$ with ξth Fourier coefficient $c(\xi)$ belongs to the Kohnen plus space if and only if $c(\xi) = 0$ unless $(-1)^\kappa\, \xi$ is congruent to a square modulo 4. The Kohnen subspace is isomorphic to a certain space of Jacobi forms. We also prove a generalization of the Kohnen–Zagier formula.

Introduction

This is the first part of a series of papers in which we generalize the theory of the Kohnen plus spaces to Hilbert modular forms of half-integral weight.

Let us recall the theory of the Kohnen plus space for modular forms of one variable. The automorphy factor $j^{1/2}(\gamma, z)$ for $\gamma \in \Gamma_0(4)$ and $z \in \mathfrak{h}$ is given by

$$j^{1/2}(\gamma, z) = \varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),\ z \in \mathfrak{h}$$

where (c/d) is Shimura's quadratic reciprocity symbol [Shi73] and ε_d is 1 or $\sqrt{-1}$ according as $d \equiv 1 \bmod 4$ or $d \equiv 3 \bmod 4$. Let $S_{\kappa+(1/2)}(\Gamma_0(4))$ be the space of cusp forms with respect to the automorphy factor $j^{\kappa+(1/2)}(\gamma, z) = (j^{1/2}(\gamma, z))^{2\kappa+1}$. The operators $\mathbf{U}$ and $\mathbf{W}$ on $S_{\kappa+(1/2)}(\Gamma_0(4))$ are defined by

$$\mathbf{U}h(z) = \frac{1}{4}\sum_{i \bmod 4} h\left(\frac{z+i}{4}\right),$$
$$\mathbf{W}h(z) = (2z/\sqrt{-1})^{-\kappa-(1/2)} h\left(-\frac{1}{4z}\right).$$

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The Kohnen plus space $S^+_{\kappa+(1/2)}(\Gamma_0(4)) \subset S_{\kappa+(1/2)}(\Gamma_0(4))$ consists of all $h(z) \in S_{\kappa+(1/2)}(\Gamma_0(4))$ with Fourier expansion of the form

$$h(\tau) = \sum_{\substack{N\geqslant 0\\(-1)^\kappa N\equiv 0,1(4)}} c(N)\mathbf{e}(Nz).$$

Then Kohnen [Koh80] proved the following theorems.

Theorem 1 (Kohnen). *The space $S^+_{\kappa+(1/2)}(\Gamma_0(4))$ is equal to the eigenspace of* **WU** *with eigenvalue $2^\kappa(-1)^{\kappa(\kappa+1)/2}$.*

Theorem 2 (Kohnen). *As a Hecke module, $S^+_{\kappa+(1/2)}(\Gamma_0(4))$ is isomorphic to $S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$.*

Let $h(z) = \sum_{(-1)^\kappa N\equiv 0,1(4)} c(N)\mathbf{e}(Nz) \in S^+_{\kappa+(1/2)}(\Gamma_0(4))$ be a Hecke eigenform. By Theorem 2, one can associate a normalized Hecke eigenform $f(z) = \sum_{N>0} a(N)\mathbf{e}(Nz) \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$. For each $N > 0$ such that $(-1)^\kappa N \equiv 0, 1\ (4)$, there exists a fundamental discriminant D and a natural number t such that $(-1)^\kappa N = Dt^2$. Then it is known that $h(z)$ satisfies the following formula:

$$c(N) = c(|D|)\sum_{d|t} \mu(d)\hat{\chi}_D(d)d^{\kappa-1}a(t/d). \tag{0.1}$$

Here, $\hat{\chi}_D$ is the Dirichlet character corresponding to $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. This formula can be rewritten as follows. Let $\{\alpha_p, \alpha_p^{-1}\}$ be the Satake parameter of $f(z)$ for a prime p. We define a Laurent polynomial $\Psi_p(N, X) \in \mathbb{C}[X, X^{-1}]$ by

$$\Psi_p(N, X) = \frac{X^{e+1} - X^{-e-1}}{X - X^{-1}} - p^{-1/2}\hat{\chi}_D(p)\frac{X^e - X^{-e}}{X - X^{-1}},$$

where $e = \mathrm{ord}_p(N)$. Then (0.1) is equivalent to the following formula:

$$c(N) = c(|D|)t^{\kappa-(1/2)}\prod_p \Psi_p(N, \alpha_p). \tag{0.2}$$

Kohnen and Zagier [KZ81] proved a beautiful formula

$$\frac{|c(|D|)|^2}{\langle h, h\rangle} = |D|^{\kappa-(1/2)}\frac{\Lambda(\kappa, f, \hat{\chi}_D)}{\langle f, f\rangle}, \tag{0.3}$$

where $\Lambda(s, f, \hat{\chi}_D)$ is the complete L-function

$$\Lambda(\kappa, f, \hat{\chi}_D) = 2(2\pi)^{-s}\Gamma(s)\sum_{N=1}^{\infty} a(N)\hat{\chi}_D(N)N^{-s}.$$

Representation-theoretic interpretation of the Kohnen plus space was treated by several authors: Baruch and Mao [BM07], Ichino [Ich05], and Loke and Savin [LS10].

In this series of papers, we are going to generalize these results to Hilbert modular forms.

Let F be a totally real number field with the integer ring $\mathfrak{o}_F$. The different of $F/\mathbb{Q}$ is denoted by $\mathfrak{d}_F$. We define a congruence subgroup Γ by

$$\Gamma = \Gamma[\mathfrak{d}_F^{-1}, 4\mathfrak{d}_F] = \left\{ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(F) \,\middle|\, a, d \in \mathfrak{o}_F, b \in \mathfrak{d}_F^{-1}, c \in 4\mathfrak{d}_F \right\}.$$

Let $\iota_1, \ldots, \iota_n$ be the embeddings of F into $\mathbb{R}$. For $\xi \in F$ and $z = (z_1, \ldots, z_n) \in \mathfrak{h}^n$, we set $\mathbf{e}(\xi z) = \exp(2\pi\sqrt{-1}\sum_{i=1}^n \iota_i(\xi)z_i)$. The basic theta function $\theta_0(z)$ is defined by

$$\theta_0(z) = \sum_{\xi\in\mathfrak{o}_F} \mathbf{e}(\xi^2 z).$$

Then there is an automorphy factor $j^{1/2}(\gamma, z)$ such that

$$\theta_0(\gamma(z)) = j^{1/2}(\gamma, z)\theta_0(z) \quad (\gamma \in \Gamma, z \in \mathfrak{h}^n).$$

For simplicity, we consider only parallel weight case here. The more general case will be treated in the text. Let $\kappa \geqslant 2$ be an integer. Set $j^{\kappa+(1/2)}(\gamma, z) = (j^{1/2}(\gamma, z))^{2\kappa+1}$. Let $S_{\kappa+(1/2)}(\Gamma)$ be the space of Hilbert cusp forms of weight $\kappa + (1/2)$ with respect to the automorphy factor $j^{\kappa+(1/2)}(\gamma, z)$. Then an element $h(z) \in S_{\kappa+(1/2)}(\Gamma)$ has a Fourier expansion of the form

$$h(z) = \sum_{\xi \in \mathfrak{o}_F} c(\xi)\mathbf{e}(\xi z).$$

We define the Kohnen plus space $S^+_{\kappa+(1/2)}(\Gamma)$ by the subspace of $S_{\kappa+(1/2)}(\Gamma)$ which consists of all $h(z) \in S_{\kappa+(1/2)}(\Gamma)$ with Fourier coefficient of the form

$$h(z) = \sum_{\substack{\xi \in \mathfrak{o}_F \\ (-1)^\kappa \xi \equiv \square\ (4)}} c(\xi)\mathbf{e}(\xi z).$$

Here $\xi \equiv \square\ (4)$ means that there exists $x \in \mathfrak{o}_F$ such that $\xi \equiv x^2 \bmod 4$.

Let $\mathbb{A}$ be the adele ring of F and $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ an additive character, whose component at each real place is given by $x \mapsto \mathbf{e}((-1)^\kappa x)$. Let v be a non-Archimedean place of F. We consider the Weil representation ω_{ψ_v} of $\widetilde{\mathrm{SL}_2(F_v)}$, where $\widetilde{\mathrm{SL}_2(F_v)}$ is the metaplectic double covering of $\mathrm{SL}_2(F_v)$. The representation space of ω_{ψ_v} is the Schwartz space $\mathcal{S}(F_v)$. The Weil representation ω_{ψ_v} is unitary with respect to the inner product

$$(\phi_1, \phi_2) = \int_{F_v} \phi_1(t)\overline{\phi_2(t)}\, dt \quad (\phi_1, \phi_2 \in \mathcal{S}(F_v)).$$

Here, the Haar measure dt is normalized so that the volume of the maximal order $\mathfrak{o}_v$ of F_v is 1. Let Γ_v be the compact open subgroup of $\mathrm{SL}_2(F_v)$ given by

$$\Gamma_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F_v) \,\middle|\, a, d \in \mathfrak{o}_v, b \in \mathfrak{d}_F^{-1}\mathfrak{o}_v, c \in 4\mathfrak{d}_F\mathfrak{o}_v \right\}.$$

The inverse image of Γ_v in $\widetilde{\mathrm{SL}_2(F_v)}$ is denoted by $\widetilde{\Gamma_v}$. There exists a genuine character $\varepsilon_v : \widetilde{\Gamma_v} \to \mathbb{C}^\times$ such that

$$\omega_\psi(g)\phi_{0,v} = \varepsilon_v(g)^{-1}\phi_{0,v},$$

where $\phi_{0,v} \in \mathcal{S}(F_v)$ is the characteristic function of $\mathfrak{o}_v$. The Hecke algebra $\widetilde{\mathcal{H}}_v = \widetilde{\mathcal{H}}(\widetilde{\Gamma_v}\backslash\widetilde{\mathrm{SL}_2(F_v)}/\widetilde{\Gamma_v}; \varepsilon_v)$ consists of all compactly supported functions f on $\widetilde{\mathrm{SL}_2(F_v)}$ such that $f(k_1hk_2) = \varepsilon_v(k_1)\varepsilon_v(k_2)f(h)$ for any $k_1, k_1 \in \widetilde{\Gamma_v}$ and $h \in \widetilde{\mathrm{SL}_2(F_v)}$. The multiplication of $\widetilde{\mathcal{H}}_v$ is given by the convolution. We define an idempotent $E_v^K \in \widetilde{\mathcal{H}}_v$ by

$$E_v^K(g) = \begin{cases} |2|_v^{-1}(\phi_{0,v}, \omega_{\psi_v}(g)\phi_{0,v}) & \text{if } g \in \widetilde{\Gamma'_v}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\widetilde{\Gamma'_v}$ is the inverse image of

$$\Gamma'_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, d \in \mathfrak{o}_v, b \in (4\mathfrak{d}_F)^{-1}\mathfrak{o}_v, c \in 4\mathfrak{d}_F\mathfrak{o}_v \right\}.$$

Define the Hecke algebra $\widetilde{\mathcal{H}}$ by the restricted tensor product $\widetilde{\mathcal{H}} = \bigotimes'_{v<\infty} \widetilde{\mathcal{H}}_v$. Put $E^K = \prod_{v<\infty} E_v^K \in \widetilde{\mathcal{H}}$. Then E^K is an idempotent.

An element of $S_{\kappa+(1/2)}(\Gamma)$ can be considered as an automorphic form on the adelic metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. Let $\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})};\varepsilon)$ be the space of automorphic forms on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ which come from the elements of $S_{\kappa+(1/2)}(\Gamma)$. Then the Hecke algebra acts on the space $\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})};\varepsilon)$ by the right translation ρ. Put

$$\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})};\varepsilon)^{E^K} = \{\rho(E^K)\varphi \mid \varphi \in \mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})};\varepsilon)\}.$$

Let $S_{\kappa+(1/2)}(\Gamma)^{E^K} \subset S_{\kappa+(1/2)}(\Gamma)$ be the subspace corresponding to the subspace $\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})})^{E^K}$ of $\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})})$.

Let $\mathcal{A}^{\mathrm{cusp}}_{2\kappa}(\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})/\mathcal{K}_0)$ denote the space of automorphic forms on the adele space $\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})$ of weight 2κ, which is right invariant by $\mathcal{K}_0 = \prod_{v<\infty}\mathrm{PGL}_2(\mathfrak{o}_v)$. It is well known that there exists a direct sum decomposition

$$\mathcal{A}^{\mathrm{cusp}}_{2\kappa}(\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})/\mathcal{K}_0) = \bigoplus_{i=1}^{d} \mathbb{C}\cdot f_i$$

where f_i is a Hecke eigenform for each $1 \leqslant i \leqslant d$. Then the automorphic representation τ_i of $\mathrm{PGL}_2(\mathbb{A})$ generated by f_i is irreducible. As a generalization of Theorem 2, we will prove the following theorem (Theorem 9.4).

Theorem. *There exists a direct sum decomposition*

$$S_{\kappa+(1/2)}(\Gamma)^{E^K} = \bigoplus_{i=1}^{d} \mathbb{C}\cdot h_i$$

with the following properties, (1) and (2).

(1) *The cusp form h_i is a Hecke eigenform with respect to the Hecke algebra $\widetilde{\mathcal{H}}_v$ for finite places $v \nmid 2$ for each $1 \leqslant i \leqslant d$.*

(2) *Let σ_i be the automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ generated by h_i. Then we have $\tau_i \simeq \mathrm{Wald}(\sigma_i,\psi)$ for each $1 \leqslant i \leqslant d$.*

Here, $\mathrm{Wald}(\sigma_i,\psi)$ is the Shimura correspondence between automorphic representations of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ and those of $\mathrm{PGL}_2(\mathbb{A})$ studied in detail by Waldspurger [Wal80, Wal91]. In particular, we have

$$\dim_{\mathbb{C}} S_{\kappa+(1/2)}(\Gamma)^{E^K} = \dim_{\mathbb{C}} \mathcal{A}^{\mathrm{cusp}}_{2\kappa}(\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})/\mathcal{K}_0).$$

Moreover, these spaces are isomorphic as modules over Hecke algebras for good primes. We will treat this point of view in our future work by using the stable trace formula [Li11] for $\widetilde{\mathrm{SL}_2(\mathbb{A})}$.

As for the generalization of Theorem 1, we shall prove the following theorem (Theorem 13.5).

Theorem. *We have*

$$S^+_{\kappa+(1/2)}(\Gamma) = S_{\kappa+(1/2)}(\Gamma)^{E^K}.$$

If $F = \mathbb{Q}$, then E^K is the projection operator to the eigenspace of **WU** with eigenvalue $(-1)^{(\kappa^2+\kappa)/2}2^\kappa$. Thus these theorems can be considered to be generalization of Kohnen's theorems.

We believe that our method can be applied to higher level cases (cf. Ueda [Ued93, Ued98]). It might be possible to generalize the results of Ikeda [Ike01] by using our results.

Let us explain the content of this paper. In § 1, we recall elementary properties of the metaplectic groups and Weil representations. In § 2, we calculate some local integrals. In § 3, we introduce the idempotents $\widetilde{e^K}$ and E^K in the Hecke algebra. We shall show that if an irreducible representation $(\pi, \mathcal{V})$ of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ has a non-zero E^K-invariant vector v, then π is either a principal series or an even Weil representation. We also calculate a Whittaker function associated to such a vector. In § 8, we recall how to regard a Hilbert modular form of half-integral weight as an automorphic form on the adelic metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. Then we prove Theorem 9.4 by applying Waldspurger's results. In § 10, we generalize the formula (0.2) to $S_{\kappa+(1/2)}(\Gamma)$. We review the theory of Baruch and Mao [BM07] in § 11. We will discuss a generalization (Theorem 12.3) of the Kohnen–Zagier formula (0.3) in § 12. In § 13, we give a proof of Theorem 13.5. In § 14, we discuss a relation to Jacobi forms. Finally, in § 15, we give some examples.

Notation

When X and Y are sets, we put $X \backslash Y = \{x \in X \mid x \notin Y\}$. For $z \in \mathbb{C}$, we set $\mathbf{e}(z) = \exp(2\pi\sqrt{-1}z)$. As usual, $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denotes the upper-half plane. If R is a ring, then we set $R^{\times 2} = \{r^2 \mid r \in R^\times\}$.

If F is a non-Archimedean local field, then $\mathfrak{o}$, $\mathfrak{p}$, q, and ϖ denote the ring of integers, the maximal ideal, the order of the residue field, and a prime element of F, respectively. If F is a finite extension of $\mathbb{Q}_p$, then $\mathfrak{d}$ is the different for $F/\mathbb{Q}_p$. The Haar measure dx on F is normalized so that $\mathrm{Vol}(\mathfrak{o}) = 1$. The Haar measure dg on $\mathrm{SL}_2(F)$ is normalized so that $\mathrm{Vol}(\mathrm{SL}_2(\mathfrak{o})) = 1$. If $\psi : F \to \mathbb{C}^\times$ is an additive character, then the order c_ψ is the maximal integer c such that $\psi(\mathfrak{p}^{-c}) = 1$.

If F is a totally real number field, then $\mathfrak{o}_F$, $\mathfrak{d}_F$, and $\mathfrak{D}_F$ denote the ring of integers, the different for $F/\mathbb{Q}$, and the discriminant of F, respectively. The adele ring of F is denoted by $\mathbb{A} = \mathbb{A}_F$. The finite part of $\mathbb{A}$ is denoted by $\mathbb{A}_\mathrm{f}$. The embeddings of F into $\mathbb{R}$ are denoted by $\iota_1, \ldots, \iota_n$, where $n = [F : \mathbb{Q}]$. For $\xi \in F$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we use a multi-index notation $\mathbf{e}(\xi z) = \exp(2\pi\sqrt{-1}\sum_{i=1}^n \iota_i(\xi) z_i)$.

1. The metaplectic group and the Weil representation

Let F be a local field of characteristic 0 and ψ a non-trivial additive character of F. We assume $F \not\simeq \mathbb{C}$. If $F = \mathbb{R}$, we assume $\psi(x) = \mathbf{e}(x)$. We fix an element $\boldsymbol{\delta} \in F^\times$ as follows. If $F \simeq \mathbb{R}$, then $\boldsymbol{\delta} = 1$. If F is non-Archimedean, then $\boldsymbol{\delta}$ is any element such that $\mathrm{ord}(\boldsymbol{\delta}) = c_\psi$, where c_ψ is the order of ψ. The Haar measure dx on F is the usual Lebesgue measure if F is real, and normalized so that $\mathrm{Vol}(\mathfrak{o}) = 1$ if F is non-Archimedean.

The quadratic Hilbert symbol for F is denoted by $\langle\ ,\ \rangle$. Recall that Kubota's 2-cocycle $\boldsymbol{c}(g_1, g_2)$ on $\mathrm{SL}_2(F)$ is given by

$$\boldsymbol{c}(g_1, g_2) = \left\langle \frac{\boldsymbol{x}(g_1)}{\boldsymbol{x}(g_1 g_2)}, \frac{\boldsymbol{x}(g_2)}{\boldsymbol{x}(g_1 g_2)} \right\rangle$$

where

$$\boldsymbol{x}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

We define the metaplectic covering group $\widetilde{\mathrm{SL}_2(F)}$ by Kubota's 2-cocycle $\boldsymbol{c}(g_1, g_2)$. Thus, as a set, $\widetilde{\mathrm{SL}_2(F)} = \{[g, \zeta] \mid g \in \mathrm{SL}_2(F), \zeta \in \{\pm 1\}\}$ and the multiplication law of $\widetilde{\mathrm{SL}_2(F)}$ is given by $[g_1, \zeta_1] \cdot [g_2, \zeta_2] = [g_1 g_2, \zeta_1 \zeta_2 \boldsymbol{c}(g_1, g_2)]$. For $g \in \mathrm{SL}_2(F)$, we denote $[g, 1]$ by $[g]$. We set

$$\mathbf{u}^\sharp(x) = \left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right], \quad \mathbf{u}^\flat(x) = \left[\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right] \quad (x \in F).$$

We also set

$$\mathbf{m}(a) = \left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right], \quad \mathbf{w}_a = \left[\begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix}\right] \quad (a \in F^\times).$$

For a subset H of SL_2, we denote the inverse image of H in $\widetilde{\mathrm{SL}_2(F)}$ by $\widetilde{H}$.

For each Schwartz function $\phi \in \mathcal{S}(F)$, the Fourier transform $\hat{\phi}$ is defined by

$$\hat{\phi}(x) = |\boldsymbol{\delta}|^{1/2} \int_F \phi(y) \psi(xy)\, dy.$$

Note that the Haar measure $|\boldsymbol{\delta}|^{1/2} dy$ is the self-dual Haar measure for the Fourier transform. For each $a \in F^\times$, there exists a constant $\alpha_\psi(a)$, called the Weil constant, which satisfies

$$\int_F \phi(x) \psi(ax^2)\, dx = \alpha_\psi(a) |2a|^{-1/2} \int_F \hat{\phi}(x) \psi\left(-\frac{x^2}{4a}\right) dx \tag{1.1}$$

for any $\phi \in \mathcal{S}(F)$ (cf. Weil [Wei64]). The Weil constant $\alpha_\psi(a)$ is an eighth root of unity, which depends only on the class of a in $F^\times / F^{\times 2}$. Clearly, we have $\alpha_\psi(-a) = \overline{\alpha_\psi(a)}$. For any $a, b \in F^\times$, we have

$$\frac{\alpha_\psi(a) \alpha_\psi(b)}{\alpha_\psi(1) \alpha_\psi(ab)} = \langle a, b \rangle.$$

In particular, $\alpha_\psi(a)/\alpha_\psi(1)$ is a fourth root of unity for any $a \in F^\times$. It is easy to see $\alpha_{\psi_\xi}(a) = \alpha_\psi(\xi a)$ for $\xi \in F^\times$, where $\psi_\xi(x) = \psi(\xi x)$. If F is a non-Archimedean local field with odd residual characteristic, $c_\psi = 0$, and if $a \in \mathfrak{o}^\times$, then $\alpha_\psi(a) = 1$.

The Weil representation ω_ψ of $\widetilde{\mathrm{SL}_2(F)}$ on $\mathcal{S}(F)$ is given by

$$\begin{aligned} \omega_\psi(\mathbf{m}(a))\phi(t) &= \frac{\alpha_\psi(1)}{\alpha_\psi(a)} |a|^{1/2} \phi(at), \\ \omega_\psi(\mathbf{u}^\sharp(b))\phi(t) &= \psi(bt^2)\phi(t), \\ \omega_\psi(\mathbf{w}_a)\phi(t) &= \overline{\alpha_\psi(a)} |2a^{-1}|^{1/2} \hat{\phi}(-2a^{-1}t). \end{aligned}$$

From these formulas, it is easy to see

$$\omega_\psi(\mathbf{u}^\flat(c))\phi(t) = \overline{\alpha_\psi(c)} |2\boldsymbol{\delta} c^{-1}|^{1/2} \int_{y \in F} \phi(t+y) \psi(c^{-1} y^2)\, dy \quad (c \in F^\times).$$

The Weil representation ω_ψ is unitary with inner product

$$(\phi_1, \phi_2) = \int_F \phi_1(t) \overline{\phi_2(t)}\, dt, \quad (\phi_1, \phi_2 \in \mathcal{S}(F)).$$

Suppose that F is non-Archimedean. Put $\mathfrak{c} = \mathfrak{c}_\psi = \mathfrak{p}^{c_\psi}$. For fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{o}$, we define a compact open subgroup $\Gamma[\mathfrak{a}, \mathfrak{b}] \subset \mathrm{SL}_2(F)$ by

$$\Gamma[\mathfrak{a}, \mathfrak{b}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \,\middle|\, a, d \in \mathfrak{o}, b \in \mathfrak{a}, c \in \mathfrak{b} \right\}.$$

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Set

$$\Gamma = \Gamma[\mathfrak{c}^{-1}, 4\mathfrak{c}].$$

Let $\phi_0 \in \mathcal{S}(F)$ be the characteristic function of $\mathfrak{o}$.

Lemma 1.1. *There exists a genuine character* $\varepsilon : \widetilde{\Gamma} \to \mathbb{C}^\times$ *such that*

$$\omega_\psi(g)\phi_0 = \varepsilon^{-1}(g)\phi_0$$

for any $g \in \widetilde{\Gamma}$.

Proof. We shall show that

$$\omega_\psi(\mathbf{u}^\sharp(b)\mathbf{m}(a))\phi_0 = \frac{\alpha_\psi(1)}{\alpha_\psi(a)}\phi_0 \quad (a \in \mathfrak{o}^\times, b \in \mathfrak{c}^{-1}),$$
$$\omega_\psi(\mathbf{u}^\flat(c))\phi_0 = \phi_0 \quad (c \in 4\mathfrak{c}).$$

The first equation is easy. The second equation follows from the fact that $\omega_\psi(\mathbf{w}_1)\phi_0(t) = \overline{\alpha_\psi(1)}|2|^{1/2}\phi_0(2t)$ is invariant under $\{\mathbf{u}^\sharp(c) \mid c \in 4\mathfrak{o}\}$. Note that $\phi_0(2t)$ is the characteristic function of $2^{-1}\mathfrak{o}$. Since $\widetilde{\Gamma}$ is generated by these elements (modulo the center), the lemma follows. □

If $2 \nmid q$, then Γ is perfect and there is a unique splitting $\boldsymbol{s} : \Gamma \to \widetilde{\Gamma}$. In this case, we have $\varepsilon([g, \zeta]) = \zeta \boldsymbol{s}(g)$ for $g \in \Gamma$. If $2|q$, we have

$$\varepsilon\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta\right]\right) = \begin{cases} \zeta \dfrac{\alpha_\psi(d)}{\alpha_\psi(1)} & \text{if } c = 0, \\ \zeta \dfrac{\alpha_\psi(1)}{\alpha_\psi(d)} \langle d, c\rangle & \text{if } c \neq 0. \end{cases}$$

This is obvious for $c = 0$. In the case $c \neq 0$, observe that

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \left[\begin{pmatrix} a - bcd^{-1} & b \\ 0 & d \end{pmatrix}, \langle -c, d\rangle\right] \cdot \left[\begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}\right],$$

from which the assertion follows easily.

2. Calculation of some Gauss sums

Hereafter, until the end of § 5, we assume that F is a non-Archimedean local field. The maximal order of F and its maximal ideal are denoted by $\mathfrak{o}$ and $\mathfrak{p}$, respectively. The number of elements of the residue field $\mathfrak{k} = \mathfrak{o}/\mathfrak{p}$ is denoted by q. Let e be the integer such that $|2|^{-1} = q^e$. We assume the additive character $\psi : F \to \mathbb{C}^\times$ is of order 0. The Haar measure dx is normalized so that $\int_\mathfrak{o} dx = 1$.

Let ϕ_0 be the characteristic function of $\mathfrak{o}$. By putting $\phi = \phi_0$ in (1.1), we obtain the following lemma.

Lemma 2.1. (1) *For* $a \in F^\times$, *we have*

$$\int_{x\in\mathfrak{o}} \psi\left(\frac{x^2}{a}\right) dx = \alpha_\psi(a)|2^{-1}a|^{1/2} \int_{x\in\mathfrak{o}} \psi\left(-\frac{ax^2}{4}\right) dx.$$

(2) *For* $a \in \mathfrak{o}\backslash\{0\}$, *we have*

$$\alpha_\psi(a) = |2a|^{-1/2} \int_{x\in\mathfrak{o}} \psi\left(\frac{x^2}{4a}\right) dx.$$

Definition 2.1. For a non-negative integer r, we set

$$U_r = \begin{cases} \mathfrak{o}^\times & \text{if } r = 0, \\ 1 + \mathfrak{p}^r & \text{if } r > 0, \end{cases}$$

$$\mathcal{U}_r = \begin{cases} \mathfrak{o}^{\times 2} U_{2r} & \text{if } 0 \leqslant r \leqslant e, \\ \emptyset & \text{if } r > e. \end{cases}$$

Lemma 2.2. *If $a \in F^\times$ and $u \in \mathcal{U}_e$, then $\alpha_\psi(au) = \alpha_\psi(a)\langle a, u\rangle$.*

Proof. We may assume $u \in U_{2e}$. If $a \in \mathfrak{o}^\times$, then the lemma follows from Lemma 2.1(2). In particular, we have $\alpha_\psi(1) = \alpha_\psi(u)$. Now, the lemma follows from $\alpha_\psi(u)\alpha_\psi(au) = \alpha_\psi(1)\alpha_\psi(a)\langle a, u\rangle$. □

The following two lemmas are well known (see, e.g., [O'M00, 63A], [HPS89, Proposition 1.1] and [Oka91, Proposition 3]).

Lemma 2.3. *The following assertions hold:*

(1) $\mathfrak{o}^{\times 2} U_{2r} = \mathfrak{o}^{\times 2} U_{2r+1} = \mathcal{U}_r$ *for* $0 \leqslant r < e$;

(2) $U_r \subset \mathfrak{o}^{\times 2}$ *for* $r \geqslant 2e + 1$;

(3) $[\mathcal{U}_r : \mathcal{U}_{r+1}] = q$ *for* $0 \leqslant r < e$;

(4) $[\mathcal{U}_e : \mathfrak{o}^{\times 2}] = 2$.

Lemma 2.4. *Let d_ξ be the order of the conductor of $F(\sqrt{\xi})/F$ for $\xi \in F^\times$. If $\xi \in \mathcal{U}_r \backslash \mathcal{U}_{r+1}$, $(0 \leqslant r \leqslant e)$, then $d_\xi = 2e - 2r$. If $\operatorname{ord}(\xi)$ is odd, then $d_\xi = 2e + 1$.*

For $\xi \in F^\times$, we define $\mathfrak{f}_\xi \in \mathbb{Z}$ and $\chi_\xi \in \{1, 0, -1\}$ as follows.

Definition 2.2. For $\xi \in F^\times$, put

$$\mathfrak{f}_\xi = (\operatorname{ord}(\xi) - d_\xi)/2,$$

$$\chi_\xi = \begin{cases} 1 & \text{if } \xi \in F^{\times 2}, \\ -1 & \text{if } F(\sqrt{\xi})/F \text{ is an unramified quadratic extension}, \\ 0 & \text{if } F(\sqrt{\xi})/F \text{ is a ramified quadratic extension}. \end{cases}$$

When $\xi = 0$, we formally put $\mathfrak{f}_\xi = +\infty$.

Lemma 2.5. *For $\xi \in F^\times$, we have*

$$\mathfrak{f}_\xi = \begin{cases} m - e + r & \text{if } \xi = \varpi^{2m} u, u \in \mathcal{U}_r \backslash \mathcal{U}_{r+1}, \\ m - e & \text{if } \operatorname{ord}(\xi) = 2m + 1. \end{cases}$$

For $\xi = \varpi^{2m} u$, $u \in \mathfrak{o}^\times$, we have

$$\chi_\xi = \begin{cases} \langle u, \varpi\rangle & \text{if } u \in \mathcal{U}_e, \\ 0 & \text{if } u \notin \mathcal{U}_e. \end{cases}$$

Proof. The first part follows from Lemma 2.4. The latter part is obvious. □

Note that if $\xi \in \mathfrak{o}$ and $\mathfrak{f}_\xi < 0$, then $\chi_\xi = 0$.

Lemma 2.6. *Suppose that $\xi \in \mathfrak{o}$. Then there exists $y \in \mathfrak{o}$ such that $\xi \equiv y^2 \bmod \mathfrak{p}^{2e}$ if and only if $\mathfrak{f}_\xi \geqslant 0$.*

Proof. If $\mathrm{ord}(\xi) \geqslant 2e$, then $\mathfrak{f}_\xi \geqslant 0$ by Lemma 2.5. This settles the case $\mathrm{ord}(\xi) \geqslant 2e$. We consider the case $\mathrm{ord}(\xi) < 2e$. We may assume $\mathrm{ord}(\xi)$ is even. Assume that $\xi = \varpi^{2m} u$, $u \in \mathcal{U}_r \backslash \mathcal{U}_{r+1}$. Then there exists $y \in \mathfrak{o}$ such that $\xi \equiv y^2 \bmod \mathfrak{p}^{2e}$ if and only if $u \in \mathcal{U}_{e-m}$, i.e., $r \geqslant e - m$. By Lemma 2.5, $r \geqslant e - m$ if and only if $\mathfrak{f}_\xi \geqslant 0$. □

Fix $A, B \in \mathfrak{o}$ and put $D = A^2 - 4B$. Note that $\mathfrak{f}_D \geqslant 0$ by Lemma 2.6. If $D \neq 0$, then we put $E = F[x]/(x^2 - Ax + B)$. Let $\mathfrak{o}_E$ be the maximal order of E. For an integer α, we set

$$S_\alpha = S_\alpha(A, B) = \{x \in F \mid x^2 - Ax + B \equiv 0 \bmod \mathfrak{p}^\alpha\}.$$

Note that $S_\alpha \subset \mathfrak{o}$ for $\alpha \geqslant 0$.

LEMMA 2.7. *Let dx be the Haar measure of F such that $\int_{\mathfrak{o}} dx = 1$. Then the volume of S_α with respect to dx is given by*

$$\mathrm{Vol}(S_\alpha) = \begin{cases} q^{-[(\alpha+1)/2]} & \text{if } \alpha \leqslant 2\mathfrak{f}_D, \\ (\chi_D + 1) q^{-\mathfrak{f}_D - 1} & \text{if } \alpha = 2\mathfrak{f}_D + 1, \\ \chi_D(\chi_D + 1) q^{-\alpha + \mathfrak{f}_D} & \text{if } \alpha \geqslant 2\mathfrak{f}_D + 2. \end{cases}$$

Proof. The case $D = 0$ is obvious. We assume $D \neq 0$. Suppose that $\chi_D = 1$. After some linear transform, we may assume that $x^2 - Ax + B = x(x-1)$, $D = 1$, and $\mathfrak{f}_D = 0$. If $\alpha \leqslant 0$, then the equation is reduced to $x^2 \equiv 0 \bmod \mathfrak{p}^\alpha$. Clearly, $S_\alpha = \mathfrak{p}^{[(\alpha+1)/2]}$ in this case. Assume now $\alpha > 0$. In this case, $\min\{\mathrm{ord}(x), \mathrm{ord}(x-1)\} = 0$ for $x \in S_\alpha$. It follows that $S_\alpha = \mathfrak{p}^\alpha \cup (1 + \mathfrak{p}^\alpha)$. This settles the case $\chi_D = 1$.

Next, consider the case $\chi_D = -1$. In this case, E/F is an unramified quadratic extension. Let $\{1, \varepsilon\}$ be an $\mathfrak{o}$-basis of $\mathfrak{o}_E$. Let $w = u + v\varepsilon \in \mathfrak{o}_E$, $(u, v \in \mathfrak{o})$ be a solution of $x^2 - Ax + B = 0$. By definition, $\mathrm{ord}(v)$ is the order of the conductor of the order $\mathfrak{o}[w]$, which is equal to $\mathfrak{f}_D$. Note that

$$\mathrm{ord}_E(a + b\varepsilon) = \begin{cases} \mathrm{ord}(a) & \text{if } \mathrm{ord}(a) \leqslant \mathrm{ord}(b), \\ \mathrm{ord}(b) & \text{if } \mathrm{ord}(a) \geqslant \mathrm{ord}(b), \end{cases}$$

for $a, b \in \mathfrak{o}$. It follows that

$$\begin{aligned} S_\alpha &= \{x \in F \mid 2\mathrm{ord}_E(w - x) \geqslant \alpha\} \\ &= \begin{cases} u + \mathfrak{p}^{[(\alpha+1)/2]} & \text{if } \alpha \leqslant 2\mathfrak{f}_D, \\ \emptyset & \text{if } \alpha > 2\mathfrak{f}_D. \end{cases} \end{aligned}$$

This settles the case $\chi_D = -1$.

Now, consider the case $\chi_D = 0$. In this case, E/F is a ramified quadratic extension. Let ε be a prime element of E. Then $\{1, \varepsilon\}$ is an $\mathfrak{o}$-basis of $\mathfrak{o}_E$. Let $w = u + v\varepsilon \in \mathfrak{o}_E$ be a solution of $x^2 - Ax + B = 0$. As in the previous case, we have $\mathfrak{f}_D = \mathrm{ord}(v)$. Note that

$$\mathrm{ord}_E(a + b\varepsilon) = \begin{cases} 2\mathrm{ord}(a) & \text{if } \mathrm{ord}(a) \leqslant \mathrm{ord}(b), \\ 1 + 2\mathrm{ord}(b) & \text{if } \mathrm{ord}(a) > \mathrm{ord}(b), \end{cases}$$

for $a, b \in \mathfrak{o}$. It follows that

$$\begin{aligned} S_\alpha &= \{x \in F \mid \mathrm{ord}_E(w - x) \geqslant \alpha\} \\ &= \begin{cases} u + \mathfrak{p}^{[(\alpha+1)/2]} & \text{if } \alpha \leqslant 2\mathfrak{f}_D + 1, \\ \emptyset & \text{if } \alpha > 2\mathfrak{f}_D + 1. \end{cases} \end{aligned}$$

Hence the lemma is proved. □

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DEFINITION 2.3. For an integer $\alpha \geqslant 0$ and $A, B \in \mathfrak{o}$, we put

$$T_\alpha(A, B) = \int_{x\in\mathfrak{o}^\times} \int_{y\in\mathfrak{o}} \psi(\varpi^{-\alpha}x(y^2 - Ay + B))\, dy\, dx.$$

Note that

$$T_\alpha(A, B) = \int_{x\in\mathfrak{o}^\times} \int_{y\in\mathfrak{o}} \psi(C^{-1}x(y^2 - Ay + B))\, dy\, dx$$

for any $C \in \varpi^\alpha\mathfrak{o}^\times$. Since

$$\int_{x\in\mathfrak{o}^\times} \psi(\xi x)\, dx = \begin{cases} 1 - q^{-1} & \text{if } \xi \in \mathfrak{o}, \\ -q^{-1} & \text{if } \mathrm{ord}(\xi) = -1, \\ 0 & \text{otherwise,} \end{cases}$$

we have $T_\alpha(A, B) = (1 - q^{-1})\mathrm{Vol}(S_\alpha) - q^{-1}(\mathrm{Vol}(S_{\alpha-1}) - \mathrm{Vol}(S_\alpha)) = \mathrm{Vol}(S_\alpha) - q^{-1}\mathrm{Vol}(S_{\alpha-1})$ for $\alpha > 0$. Clearly, we have $T_0(A, B) = 1 - q^{-1}$. Thus, we obtain the following proposition.

PROPOSITION 2.8. (1) *Assume that $\alpha \geqslant 0$ is even. Then we have*

$$T_\alpha(A, B) = \begin{cases} q^{-\alpha/2}(1 - q^{-1}) & \text{if } \alpha \leqslant 2\mathfrak{f}_D, \\ -q^{-(\alpha/2)-1} & \text{if } \alpha = 2\mathfrak{f}_D + 2 \text{ and } \chi_D = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *Assume that $\alpha > 0$ is odd. Then we have*

$$T_\alpha(A, B) = \begin{cases} \chi_D q^{-(\alpha+1)/2} & \text{if } \alpha = 2\mathfrak{f}_D + 1, \\ 0 & \text{if } \alpha \neq 2\mathfrak{f}_D + 1. \end{cases}$$

DEFINITION 2.4. We define a Gauss sum $\widetilde{\mathcal{G}}(a; \xi)$ by

$$\widetilde{\mathcal{G}}(a; \xi) = \int_{x\in\mathfrak{o}^\times} \alpha_\psi(ax)\overline{\psi(\xi x)}\, dx \quad (a, \xi \in F, a \neq 0).$$

It is enough to consider the case $0 \leqslant \mathrm{ord}(a) \leqslant 1$. In fact, it is enough to consider the cases $a = 1$ of $a = \varpi$ in later application. Since $\tilde{\mathcal{G}}(a, \xi) = \tilde{\mathcal{G}}(au, u\xi)$ for $u \in \mathfrak{o}^\times$, we do not lose a generality by this restriction.

PROPOSITION 2.9. *Suppose that $\xi \in F$.*

(1) *We have*

$$\widetilde{\mathcal{G}}(1; \xi) = \begin{cases} q^{-e/2}(1 - q^{-1}) & \text{if } \mathfrak{f}_\xi \geqslant -e, \\ -q^{-(e/2)-1} & \text{if } \mathfrak{f}_\xi = -e - 1 \text{ and } \chi_\xi = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *We have*

$$\widetilde{\mathcal{G}}(\varpi; \varpi^{-1}\xi) = \begin{cases} \chi_\xi q^{-(e+1)/2} & \text{if } \mathfrak{f}_\xi = -e \text{ and } \chi_\xi \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

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Proof. First we prove assertion (1). By Lemma 2.2, $\tilde{\mathcal{G}}(1,\xi)=0$ for $4\xi\notin\mathfrak{o}$, and so we may assume $4\xi\in\mathfrak{o}$. By Lemma 2.1(2), we have

$$\begin{aligned}\tilde{\mathcal{G}}(1;\xi) &= |2|^{-1/2}\int_{x\in\mathfrak{o}^\times}\int_{y\in\mathfrak{o}}\psi(xy^2/4)\psi(-\xi x)\,dy\,dx\\ &= |2|^{-1/2}\int_{x\in\mathfrak{o}^\times}\int_{y\in\mathfrak{o}}\psi(x(y^2-4\xi)/4)\,dy\,dx\\ &= q^{e/2}T_{2e}(0,-4\xi).\end{aligned}$$

Set $A=0$ and $B=-4\xi$, and apply Proposition 2.8(1). Note that $D=16\xi$ and $\mathfrak{f}_D=2e+\mathfrak{f}_\xi$ in this situation. From this, assertion (1) follows easily.

Now we consider the case $a=\varpi$. We may assume $4\xi\in\mathfrak{o}$, since otherwise $\tilde{\mathcal{G}}(\varpi,\varpi\xi)=0$ by Lemma 2.2. By a similar argument to that above, we have

$$\tilde{\mathcal{G}}(\varpi;\varpi^{-1}\xi)=q^{(e+1)/2}T_{2e+1}(0,-4\xi).$$

Set $A=0$, $B=-4\xi$ and apply Proposition 2.8(2). Then assertion (2) follows easily. □

Definition 2.5. For $z\in F^\times$ and $\lambda\in F$, we set

$$\boldsymbol{\omega}(z;\lambda)=\int_{y\in\mathfrak{o}}\psi(-z^{-1}(y+\lambda)^2)\,dy.$$

Lemma 2.10. *Put $i=\mathrm{ord}(2\lambda)$. Suppose that $0\leqslant i<e$.*

(1) *If $\mathrm{ord}(z)>2i+1$, then $\boldsymbol{\omega}(z;\lambda)=0$.*
(2) *There exists an element $z\in\varpi^{2i}\mathfrak{o}^\times$ such that $\boldsymbol{\omega}(z;\lambda)\neq 0$.*

Proof. Suppose that $\mathrm{ord}(z)>2i+1$. Put $r=[(\mathrm{ord}(z)+1)/2]$. Then we have $2i<i+r<\mathrm{ord}(z)$, and so

$$\begin{aligned}\boldsymbol{\omega}(z;\lambda) &= \int_{y\in\mathfrak{o}}\psi(-z^{-1}(y+\lambda)^2)\,dy\\ &= \int_{x\in\mathfrak{o}}\int_{y\in\mathfrak{o}}\psi(-z^{-1}(y+\varpi^r x+\lambda)^2)\,dy\,dx\\ &= \int_{y\in\mathfrak{o}}\left[\int_{x\in\mathfrak{o}}\psi(-z^{-1}2\varpi^r x(y+\lambda))\,dx\right]\psi(-z^{-1}(y+\lambda)^2)\,dy\\ &= 0.\end{aligned}$$

Hence we have assertion (1). Observe that

$$T_{2i}(-2\lambda,0)=\int_{z\in\varpi^{2i}\mathfrak{o}^\times}\boldsymbol{\omega}(z;\lambda)\psi(z^{-1}\lambda^2)\,dz.$$

By Proposition 2.8, $T_{2i}(-2\lambda,0)=q^{-i}(1-q^{-1})\neq 0$. Hence we have assertion (2). □

3. The idempotents e^K and E^K

Set

$$\Gamma=\Gamma[\mathfrak{o},4\mathfrak{o}]=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathfrak{o})\,\middle|\,c\in\mathfrak{p}^{2e}\right\}.$$

Recall that the genuine character $\varepsilon:\widetilde{\Gamma}\to\mathbb{C}^\times$ is defined by $\omega_\psi(g)\phi_0=\varepsilon(g)^{-1}\phi_0$, where ϕ_0 is the characteristic function of $\mathfrak{o}$.

Definition 3.1. The Hecke algebra $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(\widetilde{\Gamma}\backslash\widetilde{\mathrm{SL}_2}/\widetilde{\Gamma};\varepsilon)$ is the space of genuine functions $\tilde{\varphi} \in \widetilde{C}_0(\widetilde{\mathrm{SL}_2})$ such that $\tilde{\varphi}(\tilde{\gamma}_1\tilde{h}\tilde{\gamma}_2) = \varepsilon(\tilde{\gamma}_1)\varepsilon(\tilde{\gamma}_2)\tilde{\varphi}(\tilde{h})$. The multiplication of $\widetilde{\mathcal{H}}$ is given by the convolution product

$$(\tilde{\varphi}_1 * \tilde{\varphi}_2)(\tilde{g}) = \int_{\widetilde{\mathrm{SL}_2(F)}/\{\pm 1\}} \tilde{\varphi}_1(\tilde{g}\tilde{h}^{-1})\tilde{\varphi}_2(\tilde{h})\, d\tilde{h}.$$

Definition 3.2. We define a genuine function e^K on $\widetilde{\mathrm{SL}_2}$ by

$$e^K(g) = \begin{cases} q^e(\phi_0, \omega_\psi(g)\phi_0) & \text{if } g \in \widetilde{\mathrm{SL}_2(\mathfrak{o})}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\omega_\psi(g)\phi_0 = \varepsilon^{-1}(g)\phi_0$ for $g \in \widetilde{\Gamma}$, we have $e^K \in \widetilde{\mathcal{H}}$. Let $\mathcal{S}_0 = \mathcal{S}(2^{-1}\mathfrak{o}/\mathfrak{o})$ be the subspace of the Schwartz space $\mathcal{S}(F)$ which consists of all function $\phi \in \mathcal{S}(F)$ such that $\mathrm{Supp}(\phi) \subset 2^{-1}\mathfrak{o}$ and $\phi(x+y) = \phi(x)$ for any $x \in 2^{-1}\mathfrak{o}$ and $y \in \mathfrak{o}$. Then $\mathcal{S}_0$ is invariant under the action of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ by ω_ψ. We denote this representation by Ω_ψ.

For each $\lambda \in 2^{-1}\mathfrak{o}/\mathfrak{o}$, we denote by ϕ_λ the characteristic function of $\lambda + \mathfrak{o}$. Recall that $\mathbf{u}^\flat(z) = \left[\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\right] \in \widetilde{\mathrm{SL}_2(F)}$ for $z \in F$.

Lemma 3.1. *For $\lambda, \mu \in 2^{-1}\mathfrak{o}$, we have*

$$(\phi_\lambda, \omega_\psi(\mathbf{u}^\flat(z))\phi_\mu) = \alpha_\psi(z)|2z^{-1}|^{1/2}\boldsymbol{\omega}(z; \lambda - \mu).$$

Proof. We have

$$\omega_\psi(\mathbf{u}^\flat(z))\phi_\mu(x) = \overline{\alpha_\psi(z)}|2z^{-1}|^{1/2}\int_{y\in F} \phi_\mu(x+y)\psi(z^{-1}y^2)\, dy.$$

It follows that

$$\begin{aligned}(\phi_\lambda, \omega_\psi(\mathbf{u}^\flat(z))\phi_\mu) &= \alpha_\psi(z)|2z^{-1}|^{1/2}\int_{x\in F}\int_{y\in F} \phi_\lambda(x)\phi_\mu(x+y)\psi(-z^{-1}y^2)\, dy\, dx \\ &= \alpha_\psi(z)|2z^{-1}|^{1/2}\int_{y\in\mathfrak{o}} \psi(-z^{-1}(y+\lambda-\mu)^2)\, dy.\end{aligned}$$ □

Lemma 3.2. *Put $\mathrm{Sq}(\lambda) = \lambda^2$ for $\lambda \in F$. Then $\mathrm{Sq}: F \to F$ induces an injective map $\mathrm{Sq}: 2^{-1}\mathfrak{o}/\mathfrak{o} \to 4^{-1}\mathfrak{o}/\mathfrak{o}$.*

Proof. It is easy to see the map $\mathrm{Sq}: 2^{-1}\mathfrak{o}/\mathfrak{o} \to 4^{-1}\mathfrak{o}/\mathfrak{o}$ is well defined. It is enough to prove that the map $\mathrm{Sq}: \mathfrak{o}/2\mathfrak{o} \to \mathfrak{o}/4\mathfrak{o}$ is injective. Assume that $x^2 \equiv y^2 \bmod 4$ for $x, y \in \mathfrak{o}$. Then either $x - y \in 2\mathfrak{o}$ or $x + y \in 2\mathfrak{o}$. In either case, we have $x \equiv y \bmod 2\mathfrak{o}$, since $(x+y)-(x-y) = 2y \in 2\mathfrak{o}$. □

For each $0 \leqslant i \leqslant e$, put $\mathcal{S}^{(i)} = \mathcal{S}(\mathfrak{p}^{-e+i}/\mathfrak{o})$. Note that $\mathcal{S}_0 = \mathcal{S}^{(0)} \supset \cdots \supset \mathcal{S}^{(e)} = \mathbb{C}\phi_0$.

Proposition 3.3. *For each $0 \leqslant i \leqslant e$, $\mathcal{S}^{(i)}$ is an irreducible subspace of $\mathcal{S}_0$ under the action of $\widetilde{\Gamma[\mathfrak{o}, \mathfrak{p}^{2i}]}$.*

Proof. Put $N = \{\mathbf{u}^\sharp(b) | b \in \mathfrak{o}\}$. For each $\lambda \in 2^{-1}\mathfrak{o}$, we define a character ψ_{λ^2} of N by $\psi_{\lambda^2}(\mathbf{u}^\sharp(b)) = \psi(\lambda^2 b)$. Then we have $\Omega_\psi(\mathbf{u}^\sharp(b))\phi_\lambda = \psi_{\lambda^2}(b)\phi_\lambda$ for any $b \in \mathfrak{o}$. By Lemma 3.2, ψ_{λ^2} depends only on ($\lambda \bmod \mathfrak{o}$). Thus we have a multiplicity-free decomposition

$$\Omega_\psi|_N = \sum_{\lambda\in 2^{-1}\mathfrak{o}/\mathfrak{o}} \psi_{\lambda^2}.$$

It follows that any N-invariant subspace of $\mathcal{S}_0$ is of the form $\bigoplus_{\lambda\in X}\mathbb{C}\cdot\phi_\lambda$ for some subset $X\subset 2^{-1}\mathfrak{o}/\mathfrak{o}$. By Lemmas 3.1 and 2.10(1), we see that $\mathcal{S}^{(i)}$ is invariant under the action of $\widetilde{\Gamma[\mathfrak{o},\mathfrak{p}^{2i}]}$. Suppose that $\phi_\lambda,\phi_\mu\in\mathcal{S}^{(i)}$. By Lemmas 3.1 and 2.10(2), there exists $\mathbf{u}^\flat(z)\in\widetilde{\Gamma[\mathfrak{o},\mathfrak{p}^{2i}]}$ such that $(\phi_\lambda,\Omega_\psi(\mathbf{u}^\flat(z))\phi_\mu)\neq 0$. It follows that there is no non-trivial proper $\widetilde{\Gamma[\mathfrak{o},\mathfrak{p}^{2i}]}$-invariant subspace of $\mathcal{S}^{(i)}$. □

Proposition 3.4. *For each $0\leqslant i\leqslant e$, put*

$$e^{(i)}(g)=\begin{cases}\mathrm{Vol}(\Gamma[\mathfrak{o},\mathfrak{p}^{2i}])^{-1}q^{e-i}(\phi_0,\omega_\psi(g)\phi_0) & \text{if } g\in\widetilde{\Gamma[\mathfrak{o},\mathfrak{p}^{2i}]},\\ 0 & \text{otherwise.}\end{cases}$$

Then $e^{(i)}\in\widetilde{\mathcal{H}}$ is an idempotent. In particular, $e^{(0)}=e^K\in\widetilde{\mathcal{H}}$ is an idempotent.

Proof. This proposition follows from Proposition 3.3 and the Schur orthogonality. □

Lemma 3.5. (1) *We have*

$$e^{(e)}(g)=\begin{cases}\mathrm{Vol}(\Gamma)^{-1}\varepsilon(g) & \text{if } g\in\widetilde{\Gamma},\\ 0 & \text{otherwise.}\end{cases}$$

(2) *For $z\in\mathfrak{o}$, $z\neq 0$, we have*

$$\begin{aligned} e^K(\mathbf{u}^\flat(z)) &= \alpha(z)|2z|^{-1/2}\int_{y\in\mathfrak{o}}\overline{\psi(y^2/z)}\,dy\\ &= |2|^{-1}\int_{y\in\mathfrak{o}}\psi(zy^2/4)\,dy.\end{aligned}$$

(3) *For $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathfrak{o})$ with $c\in\mathfrak{o}^\times$, we have*

$$e^K([g])=\alpha_\psi(c)|2|^{-1/2}.$$

Proof. Assertion (1) is trivial. The first part of assertion (2) follows from Lemma 3.1. The last part of assertion (2) follows from Lemma 2.1(1). Now, assertion (3) follows from the first part of assertion (2). □

Lemma 3.6. *If $0<\mathrm{ord}(z)<2e$ and $2\nmid\mathrm{ord}(z)$, then $e^K(\mathbf{u}^\flat(z))=0$.*

Proof. We may assume the residual characteristic of F is 2. Assume $z=\varpi^{2r-1}u$, $u\in\mathfrak{o}^\times$, and $0<r\leqslant e$. Then

$$\begin{aligned}\int_{y\in\mathfrak{o}}\overline{\psi(y^2/z)}\,dy &= \int_{y\in\mathfrak{o}}\int_{x\in\mathfrak{o}}\overline{\psi((y+\varpi^{r-1}x)^2/z)}\,dx\,dy\\ &= \int_{y\in\mathfrak{o}}\int_{x\in\mathfrak{o}}\overline{\psi(y^2/z)\psi(2\varpi^{r-1}xy/z)\psi(x^2/\varpi u)}\,dx\,dy.\end{aligned}$$

Since $\mathrm{ord}(2\varpi^{r-1})>2r-1$, we have $\psi(2\varpi^{r-1}xy/z)=1$. Note that the map $\mathfrak{o}/\mathfrak{p}\to\mathfrak{o}/\mathfrak{p}$ given by $x\mapsto x^2$ is bijective. It follows that

$$\int_{x\in\mathfrak{o}}\overline{\psi(x^2/\varpi u)}\,dx=\int_{x\in\mathfrak{o}}\overline{\psi(x/\varpi u)}\,dx=0.$$ □

Recall that $\mathbf{w}_a=\left[\left(\begin{smallmatrix}0&-a^{-1}\\a&0\end{smallmatrix}\right)\right]\in\widetilde{\mathrm{SL}_2}$ for $a\in F^\times$.

Lemma 3.7. *We have* $\mathbf{w}_2^{-1}\widetilde{\Gamma}\mathbf{w}_2 = \widetilde{\Gamma}$ *and* $\varepsilon(\mathbf{w}_2^{-1}g\mathbf{w}_2) = \varepsilon(g)$ *for any* $g \in \widetilde{\Gamma}$.

Proof. The first part is easy. The second part follows from $\omega_\psi(\mathbf{w}_2)\phi_0 = \overline{\alpha(2)}\phi_0$. □

We set

$$E^K(g) = e^K(\mathbf{w}_2^{-1}g\mathbf{w}_2),$$
$$E^{(i)}(g) = e^{(i)}(\mathbf{w}_2^{-1}g\mathbf{w}_2) \quad (0 \leqslant i \leqslant e).$$

By Lemma 3.7, $E^K, E^{(i)} \in \widetilde{\mathcal{H}}$. It is clear that $\mathrm{Supp}(E^K) \subset \Gamma[\widetilde{4^{-1}\mathfrak{o}, 4}\mathfrak{o}]$. If $g \in \Gamma[\widetilde{4^{-1}\mathfrak{o}, 4}\mathfrak{o}]$, then

$$\begin{aligned} E^K(g) &= e^K(\mathbf{w}_2^{-1}g\mathbf{w}_2) \\ &= q^e(\phi_0, \omega_\psi(\mathbf{w}_2^{-1}g\mathbf{w}_2)\phi_0) \\ &= q^e(\omega_\psi(\mathbf{w}_2)\phi_0, \omega_\psi(g)\omega_\psi(\mathbf{w}_2)\phi_0). \end{aligned}$$

Since $\omega_\psi(\mathbf{w}_2)\phi_0 = \overline{\alpha_\psi(2)}\phi_0$, we have

$$E^K(g) = \begin{cases} q^e(\phi_0, \omega_\psi(g)\phi_0) & \text{if } g \in \Gamma[\widetilde{4^{-1}\mathfrak{o}, 4}\mathfrak{o}], \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have

$$E^{(i)}(g) = \begin{cases} \mathrm{Vol}(\Gamma[\mathfrak{o}, \mathfrak{p}^{2i}])^{-1}q^{e-i}(\phi_0, \omega_\psi(g)\phi_0) & \text{if } g \in \Gamma[\widetilde{4^{-1}\mathfrak{p}^{2i}, 4}\mathfrak{o}], \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $E^{(0)} = E^K$ and

$$E^{(e)}(g) = e^{(e)}(g) = \begin{cases} \mathrm{Vol}(\Gamma)^{-1}\varepsilon(g) & \text{if } g \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathbf{u}^\sharp(z) = \left[\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)\right] \in \widetilde{\mathrm{SL}_2(F)}$ for $z \in F$. Note that $\mathbf{w}_2^{-1}\mathbf{u}^\sharp(-z/4)\mathbf{w}_2 = \mathbf{u}^\flat(z)$.

Proposition 3.8. *We have*

$$\begin{aligned} E^K(\mathbf{w}_4\mathbf{u}^\sharp(-z/4)) &= q^{e/2}\alpha_\psi(1) \quad (z \in \mathfrak{o}), \\ E^K(\mathbf{u}^\sharp(-z/4)) &= q^e \int_{y\in\mathfrak{o}} \psi(zy^2/4)\, dy \quad (z \in \mathfrak{o}), \\ E^{(i)}(\mathbf{u}^\sharp(-z/4)) &= (1+q^{-1})q^{e+i} \int_{y\in\mathfrak{o}} \psi(zy^2/4)\, dy \quad (z \in \mathfrak{p}^{2i}, 1 \leqslant i \leqslant e). \end{aligned}$$

Moreover, we have $E^{(i)}(\mathbf{u}^\sharp(-z/4)) = 0$ *for* $0 < \mathrm{ord}(z) < 2e - 2i$, $2 \nmid \mathrm{ord}(z)$.

Proof. Since $\mathbf{w}_2^{-1} \cdot \mathbf{w}_4\mathbf{u}^\sharp(-z/4) \cdot \mathbf{w}_2 = \mathbf{u}^\sharp(-z)\mathbf{w}_1$, the first equation follows from Lemma 3.5(3). The second and the third equations follow immediately from $(\omega_\psi(\mathbf{u}^\sharp(z))\phi_0)(x) = \psi(zx^2)\phi_0(x)$. The last part follows from Lemma 3.6. □

As usual, λ denotes a left translation of a function, i.e., $\lambda(x)f(y) = f(x^{-1}y)$.

Proposition 3.9. *Put* $T^{(i)} = E^{(i)} - q^{-1}E^{(i+1)}$ *for* $i = 1, \dots, e-1$. *Then we have*

$$T^{(i)} = q^{e-i} \int_{x\in\mathfrak{o}^\times} \int_{y\in\mathfrak{o}} \psi(\varpi^{2i}xy^2/4)\lambda(\mathbf{u}^\sharp(-\varpi^{2i}x/4))E^{(e)}\, dy\, dx.$$

Proof. By Proposition 3.8, we have

$$\mathrm{Supp}(E^{(i)}) \subset \mathrm{Supp}(E^{(i+1)}) \coprod \bigg(\coprod_{x_j \in \mathfrak{o}^\times / U_{2e-2i}} \mathbf{u}^\sharp(-\varpi^{2i} x_j/4) \cdot \Gamma \bigg).$$

Here, x_j extends over a complete set of representatives for $\mathfrak{o}^\times / U_{2e-2i}$. Put

$$\varphi_j(g) = \begin{cases} E^{(i)}(g) & \text{if } g \in \mathbf{u}^\sharp(-\varpi^{2i} x_j/4) \cdot \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $T^{(i)} = \sum_{x_j} \varphi_j$. Since $\varphi_j * E^{(e)} = \varphi_j$, we have

$$\begin{aligned} \varphi_j &= E^{(e)}(\mathbf{1}_2)^{-1} E^{(i)}(\mathbf{u}^\sharp(-\varpi^{2i} x_j/4)) \lambda(\mathbf{u}^\sharp(-\varpi^{2i} x_j/4)) E^{(e)} \\ &= q^{-e+i} \int_{y \in \mathfrak{o}} \psi(\varpi^{2i} x_j y^2/4) \lambda(\mathbf{u}^\sharp(-\varpi^{2i} x_j/4)) E^{(e)}\, dy. \end{aligned}$$

Since this expression does not depend on the choice of the representative x_j,

$$\begin{aligned} \varphi_j &= \mathrm{Vol}(\mathfrak{p}^{2e-2i})^{-1} q^{-e+i} \int_{x \in x_j + \mathfrak{p}^{2e-2i}} \int_{y \in \mathfrak{o}} \psi(\varpi^{2i} x y^2) \lambda(\mathbf{u}^\sharp(-\varpi^{2i} x/4)) E^{(e)}\, dy\, dx \\ &= q^{e-i} \int_{x \in x_j + \mathfrak{p}^{2e-2i}} \int_{y \in \mathfrak{o}} \psi(\varpi^{2i} x y^2) \lambda(\mathbf{u}^\sharp(-\varpi^{2i} x/4)) E^{(e)}\, dy\, dx. \end{aligned}$$

It follows that

$$\begin{aligned} T^{(i)} &= \sum_{x_j} \varphi_j \\ &= q^{e-i} \sum_{x_j} \int_{x \in x_j + \mathfrak{p}^{2e-2i}} \int_{y \in \mathfrak{o}} \psi(\varpi^{2i} x y^2/4) \lambda(\mathbf{u}^\sharp(-\varpi^{2i} x/4)) E^{(e)}\, dy\, dx \\ &= q^{e-i} \int_{x \in \mathfrak{o}^\times} \int_{y \in \mathfrak{o}} \psi(\varpi^{2i} x y^2/4) \lambda(\mathbf{u}^\sharp(-\varpi^{2i} x/4)) E^{(e)}\, dy\, dx. \end{aligned}$$

Hence we have proved the proposition. □

By a similar argument, one can prove the following proposition.

PROPOSITION 3.10. *Assume that* $e > 0$. *Put* $T^K = (1 + q^{-1}) E^K - q^{-1} E^{(1)}$. *Then we have*

$$\begin{aligned} T^K &= \alpha_\psi(1) q^{e/2} \int_{x \in \mathfrak{o}} \lambda(\mathbf{w}_4 \mathbf{u}^\sharp(x/4)) E^{(e)}\, dx \\ &= \alpha_\psi(1) q^{e/2} \int_{x \in \mathfrak{p}} \lambda(\mathbf{w}_4 \mathbf{u}^\sharp(x/4)) E^{(e)}\, dx \\ &\quad + q^e \int_{x \in \mathfrak{o}^\times} \int_{y \in \mathfrak{o}} \psi(x y^2/4) \lambda(\mathbf{u}^\sharp(-x/4)) E^{(e)}\, dy\, dx. \end{aligned}$$

Proof. Note that

$$\begin{aligned} \mathrm{Supp}(T^K) &\subset \coprod_{x \in \mathfrak{o}/4\mathfrak{o}} \mathbf{w}_4 \cdot \mathbf{u}^\sharp(x/4) \cdot \Gamma \\ &= \bigg(\coprod_{x \in \mathfrak{p}/4\mathfrak{o}} \mathbf{w}_4 \cdot \mathbf{u}^\sharp(x/4) \cdot \Gamma \bigg) \coprod \bigg(\coprod_{x \in \mathfrak{o}^\times / U_{2e}} \mathbf{u}^\sharp(-x/4) \cdot \Gamma \bigg). \end{aligned}$$

By using this decompositions, one can prove the proposition as in the proof of Proposition 3.9. □

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4. Calculation of the Whittaker functions

In this section, we will calculate some Whittaker functions. The odd residual characteristic case has already been treated by Shimura [Shi85, Theorem 6.1] (see also Bump *et al.* [BFH91]). For $F = \mathbb{Q}_2$, the calculation was done by Ichino [Ich05].

In this section, we simply write $\alpha(x)$ for $\alpha_\psi(x)$ to simplify the notation. Let $B \subset \mathrm{SL}_2$ be the Borel subgroup consists of the upper triangular matrices. Let $\tilde{I}_\psi(s)$ be the induced representation of $\widetilde{\mathrm{SL}_2}$ induced from the genuine character of $\widetilde{B}$ such that

$$\mathbf{u}^\sharp(b)\mathbf{m}(a) \mapsto \frac{\alpha(1)}{\alpha(a)}|a|^s \quad (a \in F^\times, b \in F).$$

Thus $\tilde{I}_\psi(s)$ is the space of genuine functions f on $\widetilde{\mathrm{SL}_2}$ such that

$$f(\mathbf{u}^\sharp(b)\mathbf{m}(a)g) = \frac{\alpha(1)}{\alpha(a)}|a|^{s+1}f(g) \quad (a \in F^\times, b \in F).$$

The space $\tilde{I}_\psi(s)$ can be considered as a genuine representation of $\widetilde{\mathrm{SL}_2(F)}$ by right translation. It is well known that $\tilde{I}_\psi(s)$ is irreducible if $q^{-2s} \neq q^{\pm 1}$. We assume that $\tilde{I}_\psi(s)$ is irreducible.

For each integer t with $0 \leqslant t < e$, let $\mathcal{X}_t$ be the bi-Γ-invariant subset of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ given by

$$\mathcal{X}_t = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}) \,\middle|\, \mathrm{ord}(c) = 2t \right\}.$$

We set $\mathcal{X}_e = \Gamma$.

For $0 \leqslant t \leqslant e$, let $f_t \in \tilde{I}_\psi(s)$ be the element whose restriction to $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ is given by

$$f_t(k) = \begin{cases} q^{-e/2}\alpha(1)\overline{e^K(k)} & \text{if } k \in \widetilde{\mathcal{X}_t}, \\ 0 & \text{if } k \in \widetilde{\mathrm{SL}_2(\mathfrak{o})} \setminus \widetilde{\mathcal{X}_t}. \end{cases}$$

For each $f \in \tilde{I}(s)$ and $\xi \in F^\times$, we consider the integral $\mathcal{W}_{f,\xi}(g)$ defined by

$$\mathcal{W}_{f,\xi}(g) = \int_{x \in F} f(\mathbf{w}_1\mathbf{u}^\sharp(x)g)\overline{\psi(\xi x)}\, dx.$$

We will calculate the value $\mathcal{W}_{f_t,\xi}(1)$. Note that $\mathcal{W}_{f_t,\xi}(1) = 0$ for $\xi \notin \mathfrak{o}$ $(0 \leqslant t \leqslant e)$. Put $X = q^{-s}$.

Proposition 4.1. *Assume that $e = 0$ and $\xi \in \mathfrak{o}$. Then the value $\mathcal{W}_{f_0,\xi}(1)$ is equal to*

$$\mathcal{W}_{f_e,\xi}(1) = \begin{cases} 1 + (1 - q^{-1}) \displaystyle\sum_{n=1}^{\mathfrak{f}_\xi} X^{2n} + q^{-1/2}\chi_\xi X^{2\mathfrak{f}_\xi + 1} & \text{if } \chi_\xi \neq 0, \\ 1 + (1 - q^{-1}) \displaystyle\sum_{n=1}^{\mathfrak{f}_\xi} X^{2n} - q^{-1}X^{2\mathfrak{f}_\xi + 2} & \text{if } \chi_\xi = 0. \end{cases}$$

We omit a proof of this proposition, since it is easier than that of the following proposition. In fact, one can prove this proposition exactly as in the proof of assertion (3) of the following proposition.

PROPOSITION 4.2. *Assume that $e>0$ and $\xi\in\mathfrak{o}$. Then the following assertions, (1), (2) and (3), hold.*

(1) *We have $\mathcal{W}_{f_0,\xi}(1)=1$.*

(2) *If $0<t<e$, we have*

$$\mathcal{W}_{f_t,\xi}(1)=\begin{cases}(1-q^{-1})X^{2t} & \text{if } t\leqslant \mathfrak{f}_\xi+e,\\ -q^{-1}X^{2t} & \text{if } t=\mathfrak{f}_\xi+e+1,\\ 0 & \text{if } t>\mathfrak{f}_\xi+e+1.\end{cases}$$

(3) *The value $\mathcal{W}_{f_e,\xi}(1)$ is equal to*

$$\mathcal{W}_{f_e,\xi}(1)=\begin{cases}(1-q^{-1})\displaystyle\sum_{n=e}^{\mathfrak{f}_\xi+e}X^{2n}+q^{-1/2}\chi_\xi X^{2\mathfrak{f}_\xi+2e+1} & \text{if } \mathfrak{f}_\xi\geqslant 0,\ \chi_\xi\neq 0,\\ (1-q^{-1})\displaystyle\sum_{n=e}^{\mathfrak{f}_\xi+e}X^{2n}-q^{-1}X^{2\mathfrak{f}_\xi+2e+2} & \text{if } \mathfrak{f}_\xi\geqslant 0,\ \chi_\xi=0,\\ -q^{-1}X^{2e} & \text{if } \mathfrak{f}_\xi=-1,\\ 0 & \text{if } \mathfrak{f}_\xi<-1.\end{cases}$$

Proof. Assertion (1) is easy, since $f_0(\mathbf{w}_1\mathbf{u}^\sharp(x))$ is 1 for $x\in\mathfrak{o}$, and 0, otherwise. Next, we calculate $\mathcal{W}_{f_e,\xi}(1)$. Note that $f_e(\mathbf{w}_1\mathbf{u}^\sharp(x))=0$ unless $x^{-1}\in\mathfrak{p}^{2e}$. If $x^{-1}\in\mathfrak{p}^{2e}$, then we have $f_e(\mathbf{w}_1\mathbf{u}^\sharp(x))=q^{e/2}\alpha(x)|x|^{-s-1}$. It follows that

$$\begin{aligned}\mathcal{W}_{f_e}(1)&=q^{e/2}\sum_{n=2e}^{\infty}\int_{x\in\varpi^{-n}\mathfrak{o}^\times}\alpha(x)|x|^{-s-1}\overline{\psi(\xi x)}\,dx\\ &=\sum_{n=2e}^{\infty}q^{e/2}X^n\int_{x\in\mathfrak{o}^\times}\alpha(\varpi^{-n}x)\overline{\psi(\xi\varpi^{-n}x)}\,dx\\ &=\sum_{n=2e}^{\infty}q^{e/2}X^n\tilde{\mathcal{G}}(\varpi^{-n};\varpi^{-n}\xi)\\ &=\sum_{n=e}^{\infty}q^{e/2}X^{2n}\tilde{\mathcal{G}}(1;\varpi^{-2n}\xi)+\sum_{n=e}^{\infty}q^{e/2}X^{2n+1}\tilde{\mathcal{G}}(\varpi;\varpi^{-2n-1}\xi).\end{aligned}$$

By Proposition 2.9, we obtain assertion (3). Note that if $\mathfrak{f}_\xi<0$, then $\chi_\xi=0$, as we have observed after Lemma 2.5.

Next, we consider $\mathcal{W}_{f_t,\xi}(1)$ for $0<t<e$. Note that

$$f_t(\mathbf{w}_1\mathbf{u}^\sharp(x))=\begin{cases}|x|^{-s-(1/2)}\displaystyle\int_{y\in\mathfrak{o}}\psi(xy^2)\,dy & \text{if } x\in\varpi^{-2t}\mathfrak{o}^\times,\\ 0 & \text{otherwise}\end{cases}$$

by Lemma 3.5(2). It follows that

$$\begin{aligned}\mathcal{W}_{f_t,\xi}(1)&=\int_{x\in\varpi^{-2t}\mathfrak{o}^\times}|x|^{-s-(1/2)}\int_{y\in\mathfrak{o}}\psi(x(y^2-\xi))\,dy\,dx\\ &=q^tX^{2t}\int_{x\in\mathfrak{o}^\times}\int_{y\in\mathfrak{o}}\psi(\varpi^{-2t}x(y^2-\xi))\,dy\,dx\\ &=q^tX^{2t}T_{2t}(0,-\xi).\end{aligned}$$

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By Proposition 2.8, we obtain assertion (2). Note that if $t = \mathfrak{f}_\xi + e + 1$, then $\chi_\xi = 0$, since $\mathfrak{f}_\xi < 0$ implies $\chi_\xi = 0$. This completes the proof. □

Definition 4.1. For $\xi \in F^\times$, we define $\gamma(\xi, x) \in \mathbb{C}[x]$ and $\Psi(\xi, x) \in \mathbb{C}[x + x^{-1}]$ by

$$\gamma(\xi, x) = (1 - q^{-1}x^2)(1 - \chi_\xi q^{-1/2}x)^{-1},$$
$$\Psi(\xi, x) = \begin{cases} \dfrac{x^{\mathfrak{f}_\xi+1} - x^{-\mathfrak{f}_\xi-1}}{x - x^{-1}} - \chi_\xi q^{-1/2}\dfrac{x^{\mathfrak{f}_\xi} - x^{-\mathfrak{f}_\xi}}{x - x^{-1}} & \text{if } \mathfrak{f}_\xi \geqslant 0, \\ 0 & \text{if } \mathfrak{f}_\xi < 0. \end{cases}$$

Note that $\gamma(\xi, X) \neq 0$, as we have assumed that $\tilde{I}_\psi(s)$ is irreducible.

Proposition 4.3. *Put*

$$f_K^+ = \sum_{t=0}^{e} q^{-e+t}X^{-2t}f_t, \quad f_K^{[0]} = \sum_{t=0}^{e} f_t.$$

If $\xi \notin \mathfrak{o}$, *then* $\mathcal{W}_{f_K^+,\xi}(1) = \mathcal{W}_{f_K^{[0]},\xi}(1) = 0$. *For* $\xi \in \mathfrak{o}\backslash\{0\}$, *we have*

$$\mathcal{W}_{f_K^+,\xi}(1) = \gamma(\xi, X)X^{\mathfrak{f}_\xi}\Psi(\xi, X),$$
$$\mathcal{W}_{f_K^{[0]},\xi}(1) = \gamma(\xi, X)X^{\mathfrak{f}_\xi+e}\Psi(4\xi, X).$$

Proof. We may assume $\xi \in \mathfrak{o}\backslash\{0\}$. By Proposition 4.2(1) and (2), we have

$$\sum_{t=0}^{e-1} q^{-e+t}X^{-2t}\mathcal{W}_{f_t,\xi}(1) = \begin{cases} q^{-1} & \text{if } \mathfrak{f}_\xi \geqslant -1, \\ 0 & \text{if } \mathfrak{f}_\xi < -1. \end{cases}$$

In particular, if $\mathfrak{f}_\xi < 0$, then $\mathcal{W}_{f_K^+,\xi}(1) = 0$. Now we assume $\mathfrak{f}_\xi \geqslant 0$. By Propositions 4.1 and 4.2, we have

$$\mathcal{W}_{f_K^+,\xi}(1) = \begin{cases} q^{-1} + (1 - q^{-1})\displaystyle\sum_{n=0}^{\mathfrak{f}_\xi} X^{2n} + q^{-1/2}\chi_\xi X^{2\mathfrak{f}_\xi+1} & \text{if } \chi_\xi \neq 0, \\ q^{-1} + (1 - q^{-1})\displaystyle\sum_{n=0}^{\mathfrak{f}_\xi} X^{2n} - q^{-1}X^{2\mathfrak{f}_\xi+2} & \text{if } \chi_\xi = 0. \end{cases}$$

If $\chi_\xi \neq 0$, then

$$\begin{aligned} \mathcal{W}_{f_K^+,\xi}(1) &= q^{-1} + (1 - q^{-1})\sum_{n=0}^{\mathfrak{f}_\xi} X^{2n} + q^{-1/2}\chi_\xi X^{2\mathfrak{f}_\xi+1} \\ &= (X^2 - 1)^{-1}[q^{-1}(X^2 - 1) + (1 - q^{-1})(X^{2\mathfrak{f}_\xi+2} - 1) + q^{-1/2}\chi_\xi X^{2\mathfrak{f}_\xi+1}(X^2 - 1)] \\ &= (X^2 - 1)^{-1}(1 + q^{-1/2}\chi_\xi X)(-1 + q^{-1/2}\chi_\xi X - q^{-1/2}\chi_\xi X^{2\mathfrak{f}_\xi+1} + X^{2\mathfrak{f}_\xi+2}) \\ &= (1 + \chi_\xi q^{-1/2}X)X^{\mathfrak{f}_\xi}\Psi(\xi, X). \end{aligned}$$

If $\chi_\xi = 0$, one can show $\mathcal{W}_{f_K^+,\xi}(1)(1 - q^{-1}X^2)X^{\mathfrak{f}_\xi}\Psi(\xi, X)$ by a similar calculation. The case for $f_K^{[0]}$ is omitted since it is easier. □

Proposition 4.4. *We have*

$$f_K^+(g\mathbf{w}_2) = \overline{\alpha(2)}q^{-e/2}X^{-e}f_K^{[0]}(g),$$
$$f_K^{[0]}(g\mathbf{w}_2) = \overline{\alpha(2)}q^{e/2}X^{e}f_K^+(g).$$

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Proof. First note that these two equations are equivalent, since $\mathbf{w}_2^2 = [-\mathbf{1}_2, \langle -1, -1\rangle]$ and $\alpha(1)^2\overline{\alpha(2)}^2 = \langle 2, -1\rangle = 1$. It is enough to show

$$f_{e-t}(g\mathbf{w}_2) = \overline{\alpha(2)}q^{t-(e/2)}X^{e-2t}f_t(g)$$

for $0 \leqslant t \leqslant e$. For $t = 0, e$, this follows from $f_e(1) = q^{e/2}\alpha(1)$ and $f_0(\mathbf{w}_2) = \alpha(1)\overline{\alpha(2)}q^eX^{-e}$. Suppose $\mathbf{u}^\flat(z) \in \mathcal{X}_t$, $0 < t < e$. By Lemma 3.5,

$$f_t(\mathbf{u}^\flat(z)) = \alpha(1)q^{e/2}\int_{x\in\mathfrak{o}}\psi(-zx^2/4)\,dx.$$

Observe that

$$\mathbf{u}^\flat(z)\mathbf{w}_2 = [\mathbf{1}_2, \langle -2, -z\rangle]\cdot\mathbf{u}^\sharp(-z^{-1})\mathbf{m}(-2z^{-1})\mathbf{u}^\flat(-4z^{-1}).$$

It follows that

$$f_{e-t}(\mathbf{u}^\flat(z)\mathbf{w}_2) = \langle -2, -z\rangle\alpha(1)\alpha(2z)|2z^{-1}|^{s+1} \times\alpha(1)q^{-e/2}\alpha(z)|8z^{-1}|^{-1/2}\int_{x\in\mathfrak{o}}\psi(-zx^2/4)\,dx$$

by Lemma 3.5. Since $\langle -2, -z\rangle = \overline{\alpha(1)\alpha(2)\alpha(z)\alpha(2z)}$, we have

$$\begin{aligned} f_{e-t}(\mathbf{u}^\flat(z)\mathbf{w}_2) &= \alpha(1)\overline{\alpha(2)}q^tX^{e-2t}\int_{x\in\mathfrak{o}}\psi(-zx^2/4)\,dx \\ &= \overline{\alpha(2)}q^{t-(e/2)}X^{e-2t}f_t(\mathbf{u}^\flat(z)). \end{aligned}$$

Hence the proposition. □

We define a Whittaker function $W_\xi^+(g)$ by

$$W_\xi^+(g) = |\xi|^{1/2}\boldsymbol{\gamma}(\xi, X)^{-1}X^{-\mathfrak{f}_\xi}\mathcal{W}_{f_K^+,\xi}(g).$$

To determine the value of $W_\xi^+(g)$, it is enough to calculate the value on the set

$$\{\mathbf{m}(a) \mid a\in F^\times\}\cup\{\mathbf{m}(a)\mathbf{w}_1 \mid a\in F^\times\}\cup\{\mathbf{m}(a)\mathbf{u}^\flat(z) \mid a, z\in F^\times, 0<\mathrm{ord}(z)<2e\}.$$

PROPOSITION 4.5. (1) *For $a \in F^\times$, we have*

$$W_\xi^+(\mathbf{m}(a)) = \frac{\alpha(1)}{\alpha(a)}|\xi a^2|^{1/2}\Psi(\xi a^2, X),$$

$$W_\xi^+(\mathbf{m}(a)\mathbf{w}_1) = \begin{cases}\overline{\alpha(a)}|8\xi a^2|^{1/2}\Psi(16\xi a^2, X) & \text{if } 4\xi a^2\in\mathfrak{o},\\ 0 & \text{otherwise.}\end{cases}$$

(2) *Suppose that $\xi, a, z \in F^\times$, and that $0 < \mathrm{ord}(z) < 2e$. If $4\xi a^2z^{-2} \equiv \lambda^2 \bmod \mathfrak{o}$ for some $\lambda \in 2^{-1}\mathfrak{o}$, then we have*

$$W_\xi^+(\mathbf{m}(a)\mathbf{u}^\flat(z)) = \boldsymbol{\omega}(4z^{-1};\lambda)\psi(-\xi a^2z^{-1})\frac{\alpha(1)}{\alpha(a)}|4z^{-1}\xi a^2|^{1/2}\Psi(16\xi a^2z^{-2}, X).$$

If there is no such $\lambda\in 2^{-1}\mathfrak{o}$, then $W^+(\mathbf{m}(a)\mathbf{u}^\flat(z)) = 0$. Note that there is at most one $\lambda\in 2^{-1}\mathfrak{o}/\mathfrak{o}$ such that $4\xi a^2z^{-2}\equiv\lambda^2$ mod $\mathfrak{o}$.

Proof. The first part of assertion (1) follows from

$$\mathcal{W}_{f_K^+,\xi}(\mathbf{m}(a)) = \frac{\alpha(1)}{\alpha(a)}|a|^{-s+1}\mathcal{W}_{f_K^+,\xi a^2}(1).$$

The latter part of assertion (1) follows from

$$\begin{aligned}\mathcal{W}_{f_K^+,\xi}(\mathbf{m}(a)\mathbf{w}_1) &= \langle 2, a\rangle \mathcal{W}_{f_K^+,\xi}(\mathbf{m}(2a)\mathbf{w}_2)\\ &= \langle 2, a\rangle \overline{\alpha(2)} q^{-e/2} X^{-e} \mathcal{W}_{f_K^{[0]},\xi}(\mathbf{m}(2a))\\ &= \langle 2, a\rangle \overline{\alpha(2)} q^{-e/2} X^{-e} \frac{\alpha(1)}{\alpha(2a)} |a|^{-s+1} \mathcal{W}_{f_K^{[0]},4\xi a^2}(1)\\ &= \overline{\alpha(a)} q^{-e/2} |2a| \mathcal{W}_{f_K^{[0]},4\xi a^2}(1).\end{aligned}$$

We just explain how to prove assertion (2) briefly, since we do not use this assertion. For $\lambda \in 2^{-1}\mathfrak{o}$, let $f_K^{[\lambda]} \in \tilde{I}_\psi(s)$ be the function such that

$$f_K^{[\lambda]}(g) = \alpha(1)|2|^{-1/2}(\omega_\psi(g)\phi_\lambda, \phi_0)$$

for $g \in \widetilde{\mathrm{SL}_2(\mathfrak{o})}$. Here, $\phi_\lambda \in \mathcal{S}(2^{-1}\mathfrak{o}/\mathfrak{o})$ is the characteristic function of $\lambda + \mathfrak{o}$. Put

$$W_\xi^{[\lambda]}(g) = |\xi|^{1/2} \boldsymbol{\gamma}(\xi, X)^{-1} X^{-e-\mathfrak{f}_\xi} \mathcal{W}_{f_K^{[\lambda]},\xi}(g).$$

By Proposition 4.4, we have

$$W_\xi^+(g) = \alpha(2)|2|^{1/2} W_\xi^{[0]}(g\mathbf{w}_2).$$

Thus it is enough to calculate $W_\xi^{[0]}(\mathbf{m}(a)\mathbf{u}^\flat(z)\mathbf{w}_2)$. Using the decomposition

$$\mathbf{u}^\flat(z)\mathbf{w}_2 = [\mathbf{1}_2, \langle -2, -z\rangle] \cdot \mathbf{u}^\sharp(-z^{-1})\mathbf{m}(-2z^{-1})\mathbf{u}^\flat(-4z^{-1}),$$

it is reduced to a calculation of $W_\xi^{[0]}(\mathbf{m}(a')\mathbf{u}^\flat(z'))$, where $a' = -2az^{-1}$, $z' = -4z^{-1}$. We now replace a' and z' by a and z, respectively to simplify the notation. By Lemma 3.1, we have

$$\omega_\psi(\mathbf{u}^\flat(z))\phi_0 = \overline{\alpha(z)}|2z^{-1}|^{1/2} \sum_{\lambda \in 2^{-1}\mathfrak{o}/\mathfrak{o}} \overline{\boldsymbol{\omega}(z; \lambda)} \phi_\lambda.$$

It follows that

$$W_\xi^{[0]}(g\mathbf{u}^\flat(z)) = \overline{\alpha(z)}|2z^{-1}|^{1/2} \sum_{\lambda \in 2^{-1}\mathfrak{o}/\mathfrak{o}} \overline{\boldsymbol{\omega}(z; \lambda)} W_\xi^{[\lambda]}(g).$$

Now we calculate $W_\xi^{[\lambda]}(\mathbf{m}(a))$. By a similar calculation to that in the proof of assertion (1), we have

$$W_\xi^{[0]}(\mathbf{m}(a)\mathbf{w}_1) = \overline{\alpha(a)}|2\xi a^2|^{1/2}\Psi(4\xi a^2, X).$$

Using the equation $\omega_\psi(\mathbf{w}_1)\phi_0 = \overline{\alpha(1)}|2|^{1/2}\sum_\lambda \phi_\lambda$, we have

$$W_\xi^{[0]}(g\mathbf{w}_1) = \overline{\alpha(1)}|2|^{1/2} \sum_\lambda W_\xi^{[\lambda]}(g).$$

Since

$$W_\xi^{[\lambda]}(g\mathbf{u}^\sharp(x)) = \psi(\lambda^2 x) W_\xi^{[\lambda]}(g)$$

and

$$W_\xi^{[\lambda]}(\mathbf{m}(a)\mathbf{u}^\sharp(x)) = W_\xi^{[\lambda]}(\mathbf{u}^\sharp(a^2 x)\mathbf{m}(a)) = \psi(\xi a^2 x) W_\xi^{[\lambda]}(\mathbf{m}(a))$$

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for any $x \in \mathfrak{o}$, we conclude that $W_\xi^{[\lambda]}(\mathbf{m}(a)) \neq 0$ only when $\xi a^2 \equiv \lambda^2 \bmod \mathfrak{o}$. It follows that

$$W_\xi^{[\lambda]}(\mathbf{m}(a)) = \begin{cases} \dfrac{\alpha(1)}{\alpha(a)} |\xi a^2|^{1/2} \Psi(4\xi a^2, X) & \text{if } \xi a^2 \equiv \lambda^2 \bmod \mathfrak{o}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have

$$W_\xi^{[0]}(\mathbf{m}(a)\mathbf{u}^\flat(z)) = \overline{\boldsymbol{\omega}(z;\lambda)} \frac{\alpha(1)}{\alpha(a)\alpha(z)} |2\xi a^2 z^{-1}|^{1/2} \Psi(4\xi a^2, X)$$

for $\xi a^2 \equiv \lambda^2 \bmod \mathfrak{o}$. If there is no such λ, we have $W_\xi^{[\lambda]}(\mathbf{m}(a)) = 0$. Combining these results, we obtain assertion (2). □

The space $\tilde{I}_\psi(s)$ can be considered as a representation of $\widetilde{\mathrm{SL}_2}$ by right translation ρ. The Hecke algebra $\widetilde{\mathcal{H}}$ acts on $\tilde{I}_\psi(s)$ by

$$\rho(\varphi)f(g) = \int_{g_1 \in \widetilde{\mathrm{SL}_2}} f(gg_1)\varphi(g_1)\, dg_1 = f * \check{\varphi},$$

where $\check{\varphi}(g) = \varphi(g^{-1})$. Note that $\rho(e^K) f_K^{[0]} = f_K^{[0]} * \overline{e^K} = f_K^{[0]}$. Similarly, we have $\rho(E^K) f_K^+ = f_K^+$.

PROPOSITION 4.6. *Put* $\tilde{I}_\psi(s)^{e^K} = \{f \mid f \in \tilde{I}_\psi(s), \rho(e^K)f = f\}$ *and* $\tilde{I}_\psi(s)^{E^K} = \{f \mid f \in \tilde{I}_\psi(s), \rho(E^K)f = f\}$. *Then we have* $\tilde{I}_\psi(s)^{e^K} = \mathbb{C} \cdot f_K^{[0]}$ *and* $\tilde{I}_\psi(s)^{E^K} = \mathbb{C} \cdot f_K^+$.

Proof. The first identity is obvious, since $f \in \tilde{I}_\psi(s)$ is determined by its restriction to $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$. The second identity follows from

$$\tilde{I}_\psi(s)^{E^K} = \{\rho(\mathbf{w}_2)f \mid f \in \tilde{I}(s)^{e^K}\} = \mathbb{C} \cdot f_K^+.$$

This completes the proof. □

Assume that $s \in \sqrt{-1}\mathbb{R}$. Then $\tilde{I}_\psi(s)$ is a unitary representation with the inner product

$$(f, f') = \int_{x \in F} f(\mathbf{w}_1 \cdot \mathbf{u}^\sharp(x)) \overline{f'(\mathbf{w}_1 \cdot \mathbf{u}^\sharp(x))}\, dx \quad (f, f' \in \tilde{I}_\psi(s)).$$

PROPOSITION 4.7. *We have*

$$(f_K^+, f_K^+) = q^{-e}(1 + q^{-1}).$$

Proof. Recall that $f_K^+ = \sum_{t=0}^e q^{-e+t} X^{-2t} f_t$. Then we have $(f_K^+, f_K^+) = \sum_{t=0}^e q^{-2e+2t}(f_t, f_t)$. Obviously, we have $(f_0, f_0) = 1$ and

$$(f_e, f_e) = \sum_{r=2e}^{\infty} q^{e-2r} q^r (1 - q^{-1}) = q^{-e}.$$

For $0 < t < e$, we have

$$\begin{aligned}(f_t, f_t) &= q^{-2t} \int_{x \in \varpi^{-2t}\mathfrak{o}^\times} \int_{y \in \mathfrak{o}} \int_{z \in \mathfrak{o}} \psi(x(y^2 - z^2))\, dz\, dy\, dx \\ &= \int_{z \in \mathfrak{o}} T_{2t}(0, -z^2)\, dz \\ &= q^{-t}(1 - q^{-1}).\end{aligned}$$

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It follows that

$$(f_K^+, f_K^+) = q^{-2e} + \sum_{t=1}^{e-1} q^{-2e+t}(1-q^{-1}) + q^{-e} = q^{-e}(1+q^{-1}). \qquad \square$$

5. Representation theory of Jacobi groups

For the basic theory of Jacobi groups, we follow Berndt and Schmidt [BS98] and Schmidt [Sch98] (see also Ikeda [Ike94]). Let G^J be the subgroup of Sp_2 consisting of all elements of $\mathrm{Sp}_2(F)$ whose first column is equal to $\left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right)$. The algebraic group G^J is called a Jacobi group. The unipotent radical H of G^J is a Heisenberg group consisting of matrices of the form

$$(\lambda, \mu, \kappa) = \left(\begin{array}{cc|cc} 1 & \lambda & \kappa & \mu \\ 0 & 1 & \mu & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 \end{array}\right).$$

Then G^J is a semi-direct product $\mathrm{SL}_2 \ltimes H$. The center $Z^J(F)$ of $H(F)$ can be naturally identified with the additive group F by $(0,0,\kappa) \mapsto \kappa$. By the Stone–von-Neumann theorem, any irreducible representation of $H(F)$ on which $Z^J(F)$ acts by ψ is isomorphic to the so-called Schrödinger representation. It can be uniquely extended to the semi-direct product $\widetilde{\mathrm{SL}_2(F)} \ltimes H(F)$. The extended representation is called the Schrödinger–Weil representation, and is denoted by $\omega_\psi^{\mathrm{SW}}$. The Schrödinger–Weil representation $\omega_\psi^{\mathrm{SW}}$ can be realized on the Schwartz space $\mathcal{S}(F)$, and the action of $H(F)$ is given by

$$\omega_\psi^{\mathrm{SW}}((\lambda, \mu, \kappa))\phi(x) = \psi(\kappa + 2\mu x + \lambda\mu)\phi(x+\lambda).$$

The restriction of $\omega_\psi^{\mathrm{SW}}$ to $\widetilde{\mathrm{SL}_2(F)}$ is nothing but the Weil representation ω_ψ considered in § 1. It is well known that any irreducible representation of the Jacobi group $G^J(F)$ on which $Z^J(F)$ acts by ψ is isomorphic to $\omega_\psi^{\mathrm{SW}} \otimes \tilde{\pi}$, where $\tilde{\pi}$ is an irreducible genuine representation of $\widetilde{\mathrm{SL}_2(F)}$.

We assume the additive character ψ is of order 0. Put $K^J = G^J(F) \cap \mathrm{Sp}_2(\mathfrak{o})$. Let $\Pi = \omega_\psi^{\mathrm{SW}} \otimes \tilde{\pi}$ be an irreducible representation of $G^J(F)$. It is called spherical if it has a non-zero K^J-fixed vector. Schmidt [Sch98] classified the spherical representations when either residual characteristic of F is odd or $F = \mathbb{Q}_2$. In fact, his proof is applicable for any non-Archimedean local field of characteristic 0, as we explain as follows. Recall that the Weil representation ω_ψ of $\widetilde{\mathrm{SL}}_2$ has two irreducible component $\omega_\psi^\pm$, where ω_ψ^+ (respectively ω_ψ^-) consists of all even (respectively odd) functions in $\mathcal{S}(F)$.

Proposition 5.1. *Suppose that the irreducible representation $\Pi = \omega_\psi^{\mathrm{SW}} \otimes \tilde{\pi}$ of G^J is spherical. Then one of the following three cases occurs:*

(1) $\tilde{\pi} \simeq \tilde{I}_{\bar{\psi}}(s)$ *for* $s \in \mathbb{C}$ *such that* $q^{-2s} \neq q^{\pm 1}$;

(2) $\tilde{\pi} \simeq \omega_{\bar{\psi}}^+$;

(3) $\tilde{\pi} \simeq \omega_{\bar{\psi}_\eta}^+$ *with* $\eta \in \mathcal{U}_e \backslash \mathfrak{o}^{\times 2}$.

Proof. Let $\mathcal{H}(G^J, K^J)_1$ be the Hecke algebra consisting of smooth functions ϕ on $H(F)$ such that:

(i) $\phi((0,0,\kappa)h) = \psi(\kappa)^{-1}\phi(h)$ for any $\kappa \in F$ and $h \in H(F)$;

(ii) $\mathrm{Supp}(\phi)$ is compact modulo $Z^J(F)$;

(iii) $\phi(k_1 h k_2) = \phi(h)$ for any $h \in H(F)$ and $k_1, k_2 \in K^J$.

Schmidt [Sch98] has shown that the Hecke algebra $\mathcal{H}(G^J, K^J)_1$ is isomorphic to a polynomial ring generated by an element $T^J(\varpi)$. It follows that any irreducible spherical representation of G^J has a one-dimensional K^J-fixed vector space. Moreover, irreducible spherical representations of $G^J(F)$ are completely determined by the eigenvalue of $T^J(\varpi)$. Schmidt has shown that if $\tilde{\pi} = \tilde{I}_{\bar{\psi}}(s)$, then the space of K^J-fixed vectors is one-dimensional and the eigenvalue of $T^J(\varpi)$ is $q^{3/2}(q^{-s} + q^s)$. It is well known that $\tilde{I}_{\bar{\psi}}(s)$ is irreducible if and only if $q^{-2s} \neq q^{\pm 1}$. Hence if the $T^J(\varpi)$-eigenvalue does not equal to $\pm q(q+1)$, then case (1) occurs. If the $T^J(\varpi)$-eigenvalue is $q(q+1)$, then $q^{-s} = q^{\pm 1/2}$. It is well known that $\omega_{\bar{\psi}}^+$ is the unique irreducible subrepresentation of $I_{\bar{\psi}}(s)$ if $q^{-s} = q^{1/2}$. Clearly, $\omega_{\psi}^{\mathrm{SW}} \otimes \omega_{\bar{\psi}}^+$ is spherical. Therefore, if the $T^J(\varpi)$-eigenvalue is $q(q+1)$, then case (2) occurs. Similarly, one can easily show that if $\eta \in \mathcal{U}_e \backslash \mathfrak{o}^{\times 2}$, then $\omega_{\bar{\psi}_\eta}^+$ is the unique irreducible subrepresentation of $I_{\bar{\psi}}(s)$ if $q^{-s} = -q^{1/2}$. This follows from the fact that $a \mapsto \alpha_\psi(1)\overline{\alpha_\psi(\eta)\alpha_\psi(a)}\alpha_\psi(a\eta) = \langle \eta, a \rangle$ is the unique unramified character of $F^\times$ of order 2. Hence, if the $T^J(\varpi)$-eigenvalue is $-q(q+1)$, then case (3) occurs. □

Recall that the irreducible representation Ω_ψ of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ acts on the space $\mathcal{S}_0 = \mathcal{S}(2^{-1}\mathfrak{o}/\mathfrak{o})$.

LEMMA 5.2. *The representation* $\omega_{\psi}^{\mathrm{SW}} \otimes \tilde{\pi}$ *of* G^J *is spherical if and only if* $\mathrm{Hom}_{\widetilde{\mathrm{SL}_2(\mathfrak{o})}}(\Omega_{\bar{\psi}}, \tilde{\pi}) \neq \{0\}$. *Moreover,* $\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathrm{SL}_2(\mathfrak{o})}}(\Omega_{\bar{\psi}}, \tilde{\pi}) = 1$ *in this case.*

Proof. The space of $H(\mathfrak{o})$-fixed vectors of $\omega_{\psi}^{\mathrm{SW}}$ is exactly $\mathcal{S}_0$. By Proposition 3.3, the representation of $(\Omega_\psi, \mathcal{S}_0)$ of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ is irreducible. It follows that $\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathrm{SL}_2(\mathfrak{o})}}(\Omega_{\bar{\psi}}, \tilde{\pi})$ is equal to the dimension of K^J-fixed vectors of $\omega_{\psi}^{\mathrm{SW}} \otimes \tilde{\pi}$. Hence the lemma follows. □

For a genuine representation $(\tilde{\pi}, \mathcal{V})$ of $\widetilde{\mathrm{SL}_2(F)}$, we put $\mathcal{V}^{e^K} = \tilde{\pi}(e^K)\mathcal{V}$ and $\mathcal{V}^{E^K} = \tilde{\pi}(E^K)\mathcal{V}$.

PROPOSITION 5.3. *Let* $(\tilde{\pi}, \mathcal{V})$ *be a genuine irreducible representation of* $\widetilde{\mathrm{SL}_2(F)}$ *such that* $\mathcal{V}^{E^K} \neq \{0\}$. *Then one of the following three cases occurs:*

(1) $\tilde{\pi} \simeq \tilde{I}_\psi(s)$ *for* $s \in \mathbb{C}$ *such that* $q^{-2s} \neq q^{\pm 1}$;

(2) $\tilde{\pi} \simeq \omega_\psi^+$;

(3) $\tilde{\pi} \simeq \omega_{\psi_\eta}^+$ *with* $\eta \in \mathcal{U}_e \backslash \mathfrak{o}^{\times 2}$.

Moreover, we have $\dim_{\mathbb{C}} \mathcal{V}^{E^K} = 1$.

Proof. By Proposition 5.1 and Lemma 5.2, it is enough to show that

$$\dim_{\mathbb{C}} \mathcal{V}^{E^K} = \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathrm{SL}_2(\mathfrak{o})}}(\Omega_{\bar{\psi}}, \tilde{\pi}).$$

Since E^K and e^K are conjugate, $\mathcal{V}^{E^K} = \dim_{\mathbb{C}} \mathcal{V}^{e^K}$. Since e^K is a matrix coefficient of an irreducible representation Ω_ψ, we have $\dim_{\mathbb{C}} \mathcal{V}^{e^K} = \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{\mathrm{SL}_2(\mathfrak{o})}}(\Omega_{\bar{\psi}}, \tilde{\pi})$. □

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6. The case when $c_\psi \neq 0$

For global application, we need to consider the case when ψ is not necessarily of order 0. Let c_ψ be the maximal integer c such that $\psi(\varpi^{-c}\mathfrak{o}) = 1$. Put $\mathfrak{c} = \mathfrak{p}^{c_\psi}$. We fix an element $\boldsymbol{\delta} \in F^\times$ such that $\mathrm{ord}(\boldsymbol{\delta}) = c_\psi$. Put $\psi_0(x) = \psi(\boldsymbol{\delta}^{-1}x)$. Then ψ_0 is of order 0. We consider the following automorphism of $\widetilde{\mathrm{SL}}_2$.

Definition 6.1. The automorphism $\tilde{\boldsymbol{r}}_{\boldsymbol{\delta}}$ of $\widetilde{\mathrm{SL}_2(F)}$ is given by

$$\tilde{\boldsymbol{r}}_{\boldsymbol{\delta}}\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta\right]\right) = \begin{cases} \left[\begin{pmatrix} a & \boldsymbol{\delta}b \\ \boldsymbol{\delta}^{-1}c & d \end{pmatrix}, \zeta\right] & \text{if } c \neq 0, \\ \left[\begin{pmatrix} a & \boldsymbol{\delta}b \\ 0 & d \end{pmatrix}, \zeta\langle \boldsymbol{\delta}, d\rangle\right] & \text{if } c = 0. \end{cases}$$

Then we have $\omega_\psi = \omega_{\psi_0} \circ \tilde{\boldsymbol{r}}_{\boldsymbol{\delta}}$. By twisting by $\tilde{\boldsymbol{r}}_{\boldsymbol{\delta}}$, we can reduce the theory developed in previous sections to the case when ψ is of order 0. We just write down the results.

Put $\Gamma = \Gamma[\mathfrak{c}^{-1}, 4\mathfrak{c}]$. The genuine character of $\widetilde{\Gamma}$ is defined by $\omega_\psi(g)\phi_0 = \varepsilon^{-1}(g)\phi_0$, where $\phi_0 \in \mathcal{S}(F)$ is the characteristic function of $\mathfrak{o}$. The idempotents $e^K, E^K \in \widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(\widetilde{\Gamma}\backslash\widetilde{\mathrm{SL}_2(F)}/\widetilde{\Gamma}; \varepsilon)$ are defined by

$$e^K(g) = \begin{cases} q^e(\phi_0, \omega(g)\phi_0) & \text{if } g \in \widetilde{\Gamma[\mathfrak{c}^{-1}, \mathfrak{c}]}, \\ 0 & \text{otherwise}, \end{cases}$$

$$E^K(g) = \begin{cases} q^e(\phi_0, \omega(g)\phi_0) & \text{if } g \in \widetilde{\Gamma[(4\mathfrak{c})^{-1}, 4\mathfrak{c}]}, \\ 0 & \text{otherwise}. \end{cases}$$

For $i = 1, \ldots, e$, the idempotent $E^{(i)} \in \widetilde{\mathcal{H}}$ is defined by

$$E^{(i)}(g) = \begin{cases} (1 + q^{-1})q^{e+i}(\phi_0, \omega_\psi(g)\phi_0) & \text{if } g \in \widetilde{\Gamma[(4\mathfrak{c})^{-1}\mathfrak{p}^{2i}, 4\mathfrak{c}]}, \\ 0 & \text{otherwise}. \end{cases}$$

We set $E^{(0)} = E^K$. Then, we have $E^{(i)} * E^{(j)} = E^{(j)} * E^{(i)} = E^{(i)}$ for $0 \leqslant i \leqslant j \leqslant e$. When $e > 0$, we put $T^K = (1 + q^{-1})E^K - q^{-1}E^{(1)}$.

For the explicit value of E^K, we have

$$E^K\left(\mathbf{w}_{4\boldsymbol{\delta}} \cdot \mathbf{u}^\sharp\left(-\frac{z}{4\boldsymbol{\delta}}\right)\right) = \alpha_\psi(\boldsymbol{\delta})q^{e/2},$$

$$E^K\left(\mathbf{u}^\sharp\left(-\frac{z}{4\boldsymbol{\delta}}\right)\right) = q^e \int_{y\in\mathfrak{o}} \psi\left(\frac{zy^2}{4\boldsymbol{\delta}}\right) dy$$

for $z \in \mathfrak{o}$. Similarly, we have

$$E^{(i)}\left(\mathbf{u}^\sharp\left(-\frac{z}{4\boldsymbol{\delta}}\right)\right) = (1 + q^{-1})q^{e+i} \int_{y\in\mathfrak{o}} \psi\left(\frac{zy^2}{4\boldsymbol{\delta}}\right) dy$$

for $1 \leqslant i \leqslant e$, $z \in \mathfrak{p}^{2i}$.

Proposition 6.1. *For $i = 1, \ldots, e - 1$, we have*

$$E^{(i)} - q^{-1}E^{(i+1)} = q^{e-i} \int_{x\in\mathfrak{o}^\times} \int_{y\in\mathfrak{o}} \psi\left(\frac{\varpi^{2i}xy^2}{4\boldsymbol{\delta}}\right) \lambda\left(\mathbf{u}^\sharp\left(-\frac{\varpi^{2i}x}{4\boldsymbol{\delta}}\right)\right) E^{(e)}\, dy\, dx.$$

PROPOSITION 6.2. *Suppose that* $e > 0$. *Then we have*

$$\begin{aligned} T^K &= \alpha_\psi(\boldsymbol{\delta}) q^{e/2} \int_{x\in\mathfrak{o}} \lambda\bigg(\mathbf{w}_{4\boldsymbol{\delta}}\cdot \mathbf{u}^\sharp\bigg(\frac{x}{4\boldsymbol{\delta}}\bigg)\bigg) E^{(e)}\,dx \\ &= \alpha_\psi(\boldsymbol{\delta}) q^{e/2} \int_{x\in\mathfrak{p}} \lambda\bigg(\mathbf{w}_{4\boldsymbol{\delta}}\cdot \mathbf{u}^\sharp\bigg(\frac{x}{4\boldsymbol{\delta}}\bigg)\bigg) E^{(e)}\,dx \\ &\quad + q^e \int_{x\in\mathfrak{o}^\times}\int_{\in\mathfrak{o}} \psi\bigg(\frac{xy^2}{4\boldsymbol{\delta}}\bigg)\lambda\bigg(\mathbf{u}^\sharp\bigg(-\frac{x}{4\boldsymbol{\delta}}\bigg)\bigg) E^{(e)}\,dy\,dx. \end{aligned}$$

Let $\tilde{I}_\psi(s)$ be the space of genuine functions f on $\widetilde{\mathrm{SL}_2}$ such that

$$f(\mathbf{u}^\sharp(b)\mathbf{m}(a)g) = \frac{\alpha_\psi(1)}{\alpha_\psi(a)}|a|^{s+1} f(g) \quad (a\in F^\times, b\in F).$$

The space $\tilde{I}_\psi(s)$ can be considered as a representation of $\widetilde{\mathrm{SL}_2(F)}$ by the right translation. It is well known that $\tilde{I}_\psi(s)$ is irreducible if $q^{-s}\neq \pm q^{\pm 1/2}$. We assume $\tilde{I}_\psi(s)$ is irreducible. We define the element $f_K^{[0]}\in \tilde{I}_\psi(s)$ by the unique element such that

$$f_K^{[0]}(g) = \alpha_\psi(1) q^{c_\psi s-(e/2)}\,\overline{e^K(g)}$$

for $g\in \Gamma[\widetilde{\mathfrak{c}^{-1},\mathfrak{c}}]$. Put

$$f_K^+(g) = \alpha_\psi(2\boldsymbol{\delta}) q^{es-(e/2)} f_K^{[0]}(g\mathbf{w}_{2\boldsymbol{\delta}}).$$

Then we have

$$\tilde{I}_\psi(s)^{e^K} = \mathbb{C}\cdot f_K^{[0]}, \quad \tilde{I}_\psi(s)^{E^K} = \mathbb{C}\cdot f_K^+.$$

The Whittaker function $W_\xi^+(g)$ associated with $f_K^+\in\tilde{I}_\psi(s)$ is defined by

$$W_\xi^+(g) = |\xi|^{1/2}\boldsymbol{\gamma}(\xi, q^{-s})^{-1} q^{\mathfrak{f}_\xi s}\int_{x\in F} f_K^+(\mathbf{w}_1\mathbf{u}^\sharp(x)g)\overline{\psi(\xi x)}\,dx.$$

The value of W_ξ^+ is given by

$$W_\xi^+(\mathbf{m}(a)) = \frac{\alpha_\psi(1)}{\alpha_\psi(a)}|\xi a^2|^{1/2}\Psi(\xi a^2, q^{-s}),$$

$$W_\xi^+(\mathbf{m}(a)\mathbf{w}_{\boldsymbol{\delta}}) = \begin{cases} \dfrac{\alpha_\psi(1)}{\alpha_\psi(\boldsymbol{\delta})\alpha_\psi(a)}|8\xi a^2|^{1/2}\Psi(16\xi a^2, q^{-s}) & \text{if } 4\xi a^2\in\mathfrak{o}, \\ 0 & \text{otherwise.}\end{cases}$$

Moreover, for $\xi, a, z\in F^\times$, $0<\mathrm{ord}(z)<2e$,

$$W_\xi^+(\mathbf{m}(a)\mathbf{u}^\flat(\boldsymbol{\delta}z)) = \boldsymbol{\omega}(4\boldsymbol{\delta}^{-1}z^{-1};\lambda)\psi(-\xi a^2\boldsymbol{\delta}^{-1}z^{-1})\frac{\alpha_\psi(1)}{\alpha_\psi(a)}|4z^{-1}\xi a^2|^{1/2}\Psi(16\xi a^2 z^{-2}, q^{-s}),$$

if there exists $\lambda\in 2^{-1}\mathfrak{o}$ such that $4\xi a^2\boldsymbol{\delta}^2 z^{-2}\equiv\lambda^2 \bmod \mathfrak{o}$. Here,

$$\boldsymbol{\omega}(z;\lambda) = \int_{y\in\mathfrak{o}}\psi(-z^{-1}(y+\lambda)^2)\,dy.$$

If there is no such $\lambda\in 2^{-1}\mathfrak{o}$, then $W^+(\mathbf{m}(a)\mathbf{u}^\flat(z)) = 0$.

DEFINITION 6.2. Let $(\tilde{\pi},\mathcal{V})$ be an admissible representation of $\widetilde{\mathrm{SL}_2}$. A vector $\varphi\in\mathcal{V}$ is called ψ-pseudospherical if $\varphi\in\mathcal{V}^{E^K}$.

Proposition 6.3. *Let $(\tilde{\pi}, \mathcal{V})$ be a genuine irreducible representation of $\widetilde{\mathrm{SL}_2(F)}$ with non-zero ψ-pseudospherical vector. Then one of the following three cases occurs:*

(1) $\tilde{\pi} \simeq \tilde{I}_\psi(s)$ *for* $s \in \mathbb{C}$ *such that* $q^{-2s} \neq q^{\pm 1}$;

(2) $\tilde{\pi} \simeq \omega_\psi^+$;

(3) $\tilde{\pi} \simeq \omega_{\psi_\eta}^+$ *with* $\eta \in \mathcal{U}_e \backslash \mathfrak{o}^{\times 2}$.

Moreover, $\dim_{\mathbb{C}} \mathcal{V}^{E^K} = 1$.

7. The real metaplectic group $\widetilde{\mathbf{SL_2}(\mathbb{R})}$

In this section, we assume $F = \mathbb{R}$ and $\psi(x) = \mathbf{e}(x)$. The Weil constant $\alpha_\psi(a)$ is given by

$$\alpha_\psi(a) = \begin{cases} \exp(\pi\sqrt{-1}/4) & \text{if } a > 0, \\ \exp(-\pi\sqrt{-1}/4) & \text{if } a < 0. \end{cases}$$

The real metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ is the unique non-trivial topological double covering of $\mathrm{SL}_2(\mathbb{R})$. The maximal compact subgroup $\widetilde{\mathrm{SO}(2)}$ can be described as follows. Put

$$\tilde{\boldsymbol{k}}(\theta) = \begin{cases} [\boldsymbol{k}(\theta), 1] & \text{if } -\pi < \theta \leqslant \pi, \\ [\boldsymbol{k}(\theta), -1] & \text{if } \pi < \theta \leqslant 3\pi, \end{cases}$$

where

$$\boldsymbol{k}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Then the map $\theta \mapsto \tilde{\boldsymbol{k}}(\theta)$ can be extended to a homomorphism $\mathbb{R} \to \widetilde{\mathrm{SO}(2)}$, which induces an isomorphism $\mathbb{R}/4\pi\mathbb{Z} \simeq \widetilde{\mathrm{SO}(2)}$. The metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ acts on $\mathfrak{h}$ through $\mathrm{SL}_2(\mathbb{R})$. We define a factor of automorphy $\tilde{j} : \widetilde{\mathrm{SL}_2(\mathbb{R})} \times \mathfrak{h} \to \mathbb{C}$ by

$$\tilde{j}\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta\right], \tau\right) = \begin{cases} \zeta\sqrt{d} & \text{if } c = 0, d > 0, \\ -\zeta\sqrt{d} & \text{if } c = 0, d < 0, \\ \zeta(c\tau + d)^{1/2} & \text{if } c \neq 0. \end{cases}$$

Note that $\tilde{j}(\tilde{g}, \tau)$ is the unique factor of automorphy such that $\tilde{j}([g, \zeta], \tau)^2 = j(g, \tau)$, where $j(g, \tau)$ is the usual factor of automorphy on $\mathrm{SL}_2(\mathbb{R}) \times \mathfrak{h}$.

For non-negative integer κ, let $\mathcal{D}^+_{\kappa+(1/2)}$ be the lowest weight representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ with lowest weight $\kappa + (1/2)$. The representation $\mathcal{D}^+_{\kappa+(1/2)}$ has a unique Whittaker model for ψ. The Whittaker function with weight $\kappa + (1/2)$ is, up to constant, given by

$$W_\psi(\mathbf{m}(a)\tilde{\boldsymbol{k}}(\theta)) = \left(\frac{\alpha_\psi(1)}{\alpha_\psi(a)}\right)^{2\kappa+1} |a|^{\kappa+(1/2)} \mathbf{e}((\kappa + (1/2))\theta) e^{-2\pi a^2}.$$

8. Automorphic forms on $\widetilde{\mathbf{SL_2}(\mathbb{A})}$

In this section, we recall the theory of Hilbert modular forms of half-integral weight and the theory of automorphic forms on the metaplectic groups. For more detail, one can consult [Shi87, Shi93, Wal81].

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Let F be a totally real number field and ψ_1 be the non-trivial additive character of $\mathbb{A}/F$ such that the infinity component of ψ_1 is given by $x \mapsto \mathbf{e}(x)$ for any real place. Let $\mathfrak{S}$ be a set of bad places of F, which contains all places above 2 and ∞. We also assume that $\mathfrak{S}$ contains all non-Archimedean places v such that $c_{\psi_v} \neq 0$. Set

$$\mathrm{SL}_2(\mathbb{A})_{\mathfrak{S}} = \prod_{v\in\mathfrak{S}} \mathrm{SL}_2(F_v) \times \prod_{v\notin\mathfrak{S}} \mathrm{SL}_2(\mathfrak{o}_v).$$

The double covering of $\mathrm{SL}_2(\mathbb{A})_{\mathfrak{S}}$ defined by the 2-cocycle $\prod_{v\in\mathfrak{S}} \boldsymbol{c}_v(g_{1,v}, g_{2,v})$ is denoted by $\widetilde{\mathrm{SL}_2(\mathbb{A})}_{\mathfrak{S}}$, where $\boldsymbol{c}_v$ is the Kubota 2-cocycle for $\mathrm{SL}_2(F_v)$.

For $\mathfrak{S} \subset \mathfrak{S}'$, we can define an embedding

$$\iota_{\mathfrak{S}}^{\mathfrak{S}'} : \widetilde{\mathrm{SL}_2(\mathbb{A})}_{\mathfrak{S}} \to \widetilde{\mathrm{SL}_2(\mathbb{A})}_{\mathfrak{S}'}$$

by

$$[(g_v), \zeta] \to \left[(g_v), \zeta \prod_{v\in\mathfrak{S}'\setminus\mathfrak{S}} s_v(g_v)\right].$$

Here, $\boldsymbol{s}_v : \mathrm{SL}_2(\mathfrak{o}_v) \to \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} : g_v \mapsto [g_v, s_v(g_v)]$ is the unique splitting of the covering $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} \to \mathrm{SL}_2(\mathfrak{o}_v)$ for $v \notin \mathfrak{S}$. The adelic metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is the direct limit $\varinjlim \widetilde{\mathrm{SL}_2(\mathbb{A})}_{\mathfrak{S}}$. Then $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is a double covering of $\mathrm{SL}_2(\mathbb{A})$ and there exists a canonical embedding $\widetilde{\mathrm{SL}_2(F_v)} \hookrightarrow \widetilde{\mathrm{SL}_2(\mathbb{A})}$ for each place v of F. It is well known that $\mathrm{SL}_2(F)$ can be canonically embedded into $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. In fact, for each $\gamma \in \mathrm{SL}_2(F)$, the embedding is given by $\gamma \mapsto [\gamma, 1]$ for sufficiently large $\mathfrak{S}$. Let $\prod_v' \widetilde{\mathrm{SL}_2(F_v)}$ be the restricted direct product with respect to $\boldsymbol{s_v}(\mathrm{SL}_2(\mathfrak{o}_v))$. Then there is a canonical surjection $\prod_v' \widetilde{\mathrm{SL}_2(F_v)} \to \widetilde{\mathrm{SL}_2(\mathbb{A})}$. The image of $(g_v)_v \in \prod_v' \widetilde{\mathrm{SL}_2(F_v)}$ is also denoted by $(g_v)_v$. Note that this expression is not unique for an element of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. If $x = (x_v)_v \in \mathbb{A}$ is an adele, we define $\mathbf{u}^\sharp(x)$ and $\mathbf{u}^\flat(x)$ by $\mathbf{u}^\sharp(x) = (\mathbf{u}^\sharp(x_v))_v$ and $\mathbf{u}^\flat(x) = (\mathbf{u}^\flat(x_v))_v$, respectively. Similarly, if $a = (a_v)_v \in \mathbb{A}^\times$ is an idele, then we put $\mathbf{m}(a) = (\mathbf{m}(a_v))_v$.

Recall that a function f on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is a genuine function if $f(g[\mathbf{1}_2, \zeta]) = \zeta f(g)$ for any $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $\zeta \in \{\pm 1\}$. Suppose that a family of genuine functions f_v is given for each place v of F. We assume that there exists a set of bad primes $\mathfrak{S}_0$ such that $f_v(g_v) = 1$ for $g_v \in \boldsymbol{s}_v(\mathrm{SL}_2(\mathfrak{o}_v))$, $v \notin \mathfrak{S}_0$. Then one can define a genuine function $\prod_v f_v$ by $(\prod_v f_v)((g_v)_v) = \prod_v f_v(g_v)$.

Let $\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})$ be the finite part of $\mathrm{SL}_2(\mathbb{A})$ and Γ'_{f} a compact open subgroup of $\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})$. The inverse image of Γ'_{f} in $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is denoted by $\widetilde{\Gamma}'_{\mathrm{f}}$. A character $\varepsilon' : \widetilde{\Gamma}'_{\mathrm{f}} \to \mathbb{C}^\times$ is called a genuine character if $\varepsilon'([\mathbf{1}_2, -1]) = -1$.

Let $\{\infty_1, \ldots, \infty_n\}$ be the set of infinite places of F. The embedding $F \hookrightarrow \mathbb{R}$ corresponding to ∞_i is denoted by ι_i. Put $\Gamma' = \mathrm{SL}_2(F) \cap \Gamma'_{\mathrm{f}} \times \mathrm{SL}_2(\mathbb{R})^n$.

As usual, we embed $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{R})^n$ by $\gamma \mapsto (\iota_1(\gamma), \ldots, \iota_n(\gamma))$. Suppose that $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{Z}^n$ with $\kappa_1, \ldots, \kappa_n \geqslant 0$. We define a factor of automorphy $J^{\varepsilon', \kappa+(1/2)}(\gamma, z)$ for $\gamma \in \Gamma'$ and $z = (z_1, \ldots, z_n) \in \mathfrak{h}^n$ by

$$J^{\varepsilon', \kappa+(1/2)}(\gamma, z) = \prod_{v<\infty} \varepsilon'_v([\gamma, 1]) \prod_{i=1}^{n} \tilde{j}([\iota_i(\gamma), 1], z_i)^{2\kappa_i+1}.$$

Let $M_{\kappa+(1/2)}(\Gamma', \varepsilon')$ (respectively $S_{\kappa+(1/2)}(\Gamma', \varepsilon')$) be the space of Hilbert modular forms (respectively Hilbert cusp forms) on $\mathfrak{h}^n$ with respect to the automorphy factor $J^{\varepsilon', \kappa+(1/2)}(\gamma, z)$.

Thus each element $h(z) \in M_{\kappa+(1/2)}(\Gamma', \varepsilon')$ satisfies

$$h(\gamma(z)) = J^{\varepsilon', \kappa+(1/2)}(\gamma, z) h(z)$$

for any $\gamma \in \Gamma'$ and $z \in \mathfrak{h}^n$.

The element $h(z) \in M_{\kappa+(1/2)}(\Gamma', \varepsilon')$ can be considered as an automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$ as follows. For each $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$, there exist $\gamma \in \mathrm{SL}_2(F)$, $\tilde{g}_\infty \in \widetilde{\mathrm{SL}_2(\mathbb{R})}^n$, and $\tilde{g}_{\mathrm{f}} \in \widetilde{\Gamma}'_{\mathrm{f}}$ such that $g = \gamma \tilde{g}_\infty \tilde{g}_{\mathrm{f}}$ by the strong approximation theorem for $\mathrm{SL}_2(\mathbb{A})$. Then we set

$$\varphi_h(g) = h(\tilde{g}_\infty(\mathbf{i}))\varepsilon'(\tilde{g}_{\mathrm{f}})^{-1} \prod_{i=1}^{n} (\tilde{j}(\tilde{g}_{\infty_i}, \mathbf{i})^{2\kappa_i+1})^{-1}$$

Here, $\mathbf{i} = (\sqrt{-1}, \dots, \sqrt{-1}) \in \mathfrak{h}^n$. Then φ_h can be considered as a genuine automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$. We set

$$\mathcal{A}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \widetilde{\Gamma}'_{\mathrm{f}}, \varepsilon') = \{\varphi_h \mid h(z) \in M_{\kappa+(1/2)}(\Gamma', \varepsilon')\},$$
$$\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \widetilde{\Gamma}'_{\mathrm{f}}, \varepsilon') = \{\varphi_h \mid h(z) \in S_{\kappa+(1/2)}(\Gamma', \varepsilon')\}.$$

For each $\varphi_h \in \mathcal{A}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \widetilde{\Gamma}'_{\mathrm{f}}, \varepsilon')$, the element $h \in M_{\kappa+(1/2)}(\Gamma)$ is recovered as follows. For $z = (z_1, \dots, z_n) \in \mathfrak{h}^n$, there exists $g_\infty = (g_{\infty_1}, \dots, g_{\infty_n}) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}^n$ such that $z = g_\infty(\mathbf{i})$. Then we have

$$h(z) = \varphi_h(g_\infty) \prod_{i=1}^{n} \tilde{j}(g_{\infty_i}, \mathbf{i})^{2\kappa+1}.$$

We set

$$\mathcal{A}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}) = \bigcup_{(\widetilde{\Gamma}'_{\mathrm{f}}, \varepsilon')} \mathcal{A}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \widetilde{\Gamma}'_{\mathrm{f}}, \varepsilon'),$$
$$\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}) = \bigcup_{(\widetilde{\Gamma}'_{\mathrm{f}}, \varepsilon')} \mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \widetilde{\Gamma}'_{\mathrm{f}}, \varepsilon'),$$

where $(\widetilde{\Gamma}'_{\mathrm{f}}, \varepsilon')$ extends over all pairs of compact open subgroups $\widetilde{\Gamma}'_{\mathrm{f}} \subset \mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})$ and genuine characters $\varepsilon' : \widetilde{\Gamma}'_{\mathrm{f}} \to \mathbb{C}^\times$. Then $\widetilde{\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})}$ acts on $\mathcal{A}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})})$ and $\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})})$ by right translation ρ. The action of $\widetilde{\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})}$ on $\bigcup_{(\Gamma', \varepsilon')} M_{\kappa+(1/2)}(\Gamma', \varepsilon')$ is also denoted by ρ. Note that the right translation ρ induces the left action of the Hecke algebra $\widetilde{\mathcal{H}}(\widetilde{\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})})$ on $\mathcal{A}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})})$ by

$$\rho(\tilde{\phi})\varphi(g) = \int_{\widetilde{\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})}/\{\pm 1\}} \tilde{\phi}(g_1)\varphi(gg_1)\, dg_1 \quad (\tilde{\phi} \in \widetilde{\mathcal{H}}(\widetilde{\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})})).$$

Assume that

$$h(z) = \sum_{\xi \in F} c(\xi)\mathbf{e}(\xi z) \in M_{\kappa+(1/2)}(\Gamma', \varepsilon').$$

Then one can easily show that

$$\rho(\mathbf{u}^\sharp(x))h(z) = \sum_{\xi \in F} \psi_{1,v}(\xi x)c(\xi)\mathbf{e}(\xi z) \quad (x \in F_v)$$

if v is a non-Archimedean place of F. Similarly, suppose that $a \in F^\times$ is a totally positive element. Denote by a_{f} the finite part of the principal idele $a \in F^\times$. Then we have

$$\rho(\mathbf{m}(a_{\mathrm{f}}))h(z) = a^{-\kappa-(1/2)}h(a^{-2}z),$$

where $a^{-\kappa-(1/2)} = \prod_{i=1}^n \iota_i(a)^{-\kappa_i-(1/2)}$.

For irreducible cuspidal automorphic representation σ of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$, we denote by $\sigma[\kappa+(1/2)]$ the space of vectors of σ which has weight $\kappa_i+(1/2)$ at the real place ∞_i. Then we have

$$\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}) = \bigoplus_\sigma \sigma[\kappa+(1/2)].$$

Here, σ extends over all irreducible cuspidal representation whose ∞_i-component is a lowest weight representation with lowest weight $\kappa_i+(1/2)$.

For each pair of fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{ab} \subset \mathfrak{o}_F$, we define a congruence subgroup $\Gamma[\mathfrak{a},\mathfrak{b}] \subset \mathrm{SL}_2(F)$ by

$$\Gamma[\mathfrak{a},\mathfrak{b}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \,\middle|\, a, d \in \mathfrak{o}_F, b \in \mathfrak{a}, c \in \mathfrak{b} \right\}.$$

Similarly, if v is a non-Archimedean place, we define a compact open subgroup $\Gamma_v[\mathfrak{a}_v,\mathfrak{b}_v] \subset \mathrm{SL}_2(F_v)$ by

$$\Gamma_v[\mathfrak{a}_v,\mathfrak{b}_v] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F_v) \,\middle|\, a, d \in \mathfrak{o}_v, b \in \mathfrak{a}_v, c \in \mathfrak{b}_v \right\}.$$

Put $\Gamma = \Gamma[\mathfrak{d}_F^{-1}, 4\mathfrak{d}_F]$ and $\Gamma_v = \Gamma_v[\mathfrak{d}_v^{-1}, 4\mathfrak{d}_v]$, where $\mathfrak{d}_F$ is the different for $F/\mathbb{Q}$.

Suppose that $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{Z}^n$, $\kappa_1, \ldots, \kappa_n \geqslant 0$. Let $\eta \in \mathfrak{o}^\times$ be a unit such that $\mathrm{N}_{F/\mathbb{Q}}(\eta) = \prod_{i=1}^n (-1)^{\kappa_i}$. We fix such a unit η once and for all. Put $\psi(x) = \psi_1(\eta x)$. In this setting, we have $\mathfrak{c}_{\psi_v} = \mathfrak{d}_v = \mathfrak{d}\mathfrak{o}_v$ for any non-Archimedean place v. There exists a genuine character $\varepsilon_v : \Gamma_v \to \mathbb{C}^\times$ such that $\omega_{\psi_v}(g_v)\phi_{0,v} = \varepsilon_v(g_v)^{-1}\phi_{0,v}$ for each non-Archimedean place v by Lemma 1.1. Here, $\phi_{0,v} \in \mathcal{S}(F_v)$ is the characteristic function of $\mathfrak{o}_v$. We define a factor of automorphy $j^{\kappa+(1/2),\eta}(\gamma, z)$ for $\gamma \in \Gamma$ and $z = (z_1, \ldots, z_n) \in \mathfrak{h}^n$ by

$$j^{\kappa+(1/2),\eta}(\gamma, z) = \prod_{v<\infty} \varepsilon_v([\gamma, 1]) \prod_{i=1}^n \tilde{j}([\iota_i(\gamma), 1], z_i)^{2\kappa_i+1}.$$

When there is no fear of confusion, we simply write $j^{\kappa+(1/2)}(\gamma, z)$ for $j^{\kappa+(1/2),\eta}(\gamma, z)$.

DEFINITION 8.1. Let Γ, κ, and η be as above. We denote by $M_{\kappa+(1/2)}(\Gamma)$ the space of Hilbert modular forms for Γ with respect to the factor of automorphy $j^{\kappa+(1/2)}(\gamma, z)$. We also denote by $S_{\kappa+(1/2)}(\Gamma)$ the subspace of $M_{\kappa+(1/2)}(\Gamma)$ which consists of all cusp forms.

Remark 8.1. When $\kappa_1 = \cdots \kappa_n = 0$ and $\eta = 1$, the automorphy factor $j^{1/2}(\gamma, z)$ satisfies the formula

$$\theta_0(\gamma(z)) = j^{1/2}(\gamma, z)\theta_0(z), \tag{8.1}$$

where $\theta_0(z)$ is the basic theta function given by

$$\theta_0(z) = \sum_{\xi \in \mathfrak{o}_F} \mathbf{e}(\xi^2 z).$$

In particular, when $F = \mathbb{Q}$, our definition of $j^{\kappa+(1/2)}(\gamma, z)$ agrees with classical definition. For a proof of the formula (8.1), one can consult Shimura [Shi85], although our normalization of theta function is different from that given in [Shi85]. (Our $\theta_0(z)$ is $\theta(2z, 0; l_0)$ in Shimura's notation.)

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From now on, we set $G=\mathrm{SL}_2$ and denote $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$ by $G_F\backslash\widetilde{G_{\mathbb{A}}}$ to simplify the notation. Put $E^K=\prod_{v<\infty}E_v^K$. Then E^K can be considered as a genuine locally constant function on $\widetilde{\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})}$ with compact support. Put

$$\mathcal{A}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})^{E^K}=\{\varphi\in\mathcal{A}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})\mid\rho(E^K)\varphi=\varphi\},$$
$$\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})^{E^K}=\{\varphi\in\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})\mid\rho(E^K)\varphi=\varphi\}.$$

Clearly,

$$\mathcal{A}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})^{E^K}\subset\mathcal{A}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}};\Gamma_{\mathrm{f}},\varepsilon),$$
$$\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})^{E^K}\subset\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}};\Gamma_{\mathrm{f}},\varepsilon).$$

Definition 8.2. Let $M_{\kappa+(1/2)}(\Gamma)^{E^K}\subset M_{\kappa+(1/2)}(\Gamma)$ be the subspace corresponding to the subspace $\mathcal{A}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})^{E^K}\subset\mathcal{A}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})$. Similarly, let $S_{\kappa+(1/2)}(\Gamma)^{E^K}\subset S_{\kappa+(1/2)}(\Gamma)$ be the subspace corresponding to the subspace $\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})^{E^K}\subset\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})$. In other words,

$$M_{\kappa+(1/2)}(\Gamma)^{E^K}=\{h\in M_{\kappa+(1/2)}(\Gamma)\mid\rho(E^K)h=h\},$$
$$S_{\kappa+(1/2)}(\Gamma)^{E^K}=\{h\in S_{\kappa+(1/2)}(\Gamma)\mid\rho(E^K)h=h\}.$$

We identify $M_{\kappa+(1/2)}(\Gamma)^{E^K}$ and $\mathcal{A}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})^{E^K}$. Similarly, we identify $S_{\kappa+(1/2)}(\Gamma)^{E^K}$ and $\mathcal{A}^{\mathrm{cusp}}_{\kappa+(1/2)}(G_F\backslash\widetilde{G_{\mathbb{A}}})^{E^K}$. By Proposition 6.3, we have

$$S_{\kappa+(1/2)}(\Gamma)^{E^K}=\bigoplus_{\sigma}\sigma[\kappa+(1/2)]^{E_K},$$

where σ extends over all irreducible cuspidal automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ which satisfies the following conditions (i) and (ii).

(i) If v is a non-Archimedean place, then $\sigma_v\simeq\tilde{I}_\psi(s_v)$ for some $s_v\in\mathbb{C}$.

(ii) If ∞_i is a real place, then σ_{∞_i} is a lowest weight representation with lowest weight $\kappa_i+(1/2)$.

We also consider automorphic forms on $Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})$, where Z is the center of GL_2. We assume $\kappa_i\geqslant 1$ for $i=1,2,\ldots,n$. Let $\mathcal{K}$ be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathrm{f}})$. Choose a set of elements $\{\beta_1,\ldots,\beta_h\}$ such that $\{\det\beta_1,\ldots,\det\beta_h\}$ forms a complete set of representatives for $\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}(\det\mathcal{K})$. We may assume $\beta_t\in\mathrm{GL}_2(\mathbb{A}_{\mathrm{f}})$ for $t=1,2,\ldots,h$. Then we have $\mathrm{GL}_2(\mathbb{A})=\coprod_{t=1}^h Z(\mathbb{A})\mathrm{GL}_2(F)\beta_t\mathcal{K}\mathrm{GL}_2^+(\mathbb{R})^n$, and so

$$Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})/\mathcal{K}\cdot\mathrm{SO}(2)^n\simeq\coprod_{t=1}^h\Gamma_t\backslash\mathfrak{h}^n,$$

where $\Gamma_t=\mathrm{GL}_2(F)\cap Z(\mathbb{A})\beta_t\mathcal{K}\beta_t^{-1}\mathrm{GL}_2^+(\mathbb{R})^n$ for $t=1,\ldots,h$. Let $S_{2\kappa}(\Gamma_t)$ be the space of Hilbert cusp forms on $\mathfrak{h}^n$ with respect to Γ_t. Then $(f_1(z),\ldots,f_h(z))\in\oplus_{t=1}^n S_{2\kappa}(\Gamma_t)$ lifts to an automorphic form ϕ on $Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})$ by

$$\phi(g)=f_t(g_\infty(\mathbf{i}))\prod_{i=1}^n j(g_{\infty_i},z_i)^{-2\kappa_i},$$

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for $g = x\gamma\beta_t g_\infty k$, with $x \in Z(\mathbb{A})$, $\gamma \in \mathrm{GL}_2(F)$, $g_\infty = (g_{\infty_1}, \ldots, g_{\infty_n}) \in \mathrm{GL}_2^+(\mathbb{R})^n$, and $k \in \mathcal{K}$. Here, $j(g, z)$ is the usual automorphy factor for $\mathrm{GL}_2^+(\mathbb{R}) \times \mathfrak{h}$ defined by

$$j(g, z) = |\det g|^{-1/2}(cz + d), \quad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}), z \in \mathfrak{h}\right).$$

From now on, we set $H = \mathrm{PGL}_2$ and denote $\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})$ by $H_F\backslash H_\mathbb{A}$ to simplify the notation. We denote $\mathcal{A}_{2\kappa}^{\mathrm{cusp}}(H_F\backslash H_\mathbb{A}/\mathcal{K})$ the space of all ϕ obtained in this way, and put

$$\mathcal{A}_{2\kappa}^{\mathrm{cusp}}(H_F\backslash H_\mathbb{A}) = \bigcup_{\mathcal{K}} \mathcal{A}_{2\kappa}^{\mathrm{cusp}}(H_F\backslash H_\mathbb{A}/\mathcal{K}),$$

where $\mathcal{K}$ extends over all compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_\mathrm{f})$. For irreducible cuspidal automorphic representation τ of $\mathrm{PGL}_2(\mathbb{A})$, we denote by $\tau[2\kappa]$ the space of vectors of τ which has weight $2\kappa_i$ at the real place ∞_i. Then we have

$$\mathcal{A}_{2\kappa}^{\mathrm{cusp}}(H_F\backslash H_\mathbb{A}) = \bigoplus_{\tau} \tau[2\kappa].$$

Here, τ extends over all irreducible cuspidal representations whose ∞_i-components are discrete series representation with minimal weight $\pm 2\kappa_i$. Let $\mathcal{K}_0 = \prod_{v<\infty} \mathrm{GL}_2(\mathfrak{o}_v)$. Then we have

$$\mathcal{A}_{2\kappa}^{\mathrm{cusp}}(H_F\backslash H_\mathbb{A}/\mathcal{K}_0) = \bigoplus_{\tau} \tau[2\kappa]^{\mathcal{K}_0},$$

where τ extends over all irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies the following conditions (1) and (2).

(1) If v is a non-Archimedean place, then τ_v is an unramified principal series $I(s_v)$.

(2) If ∞_i is a real place, then τ_{∞_i} is a discrete series representation with minimal weight $\pm 2\kappa_i$.

Remark 8.2. Note that $\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}\prod_{v<\infty}\mathfrak{o}_v^\times \simeq \mathrm{Cl}^+(F)/\mathrm{Cl}^+(F)^2$, where $\mathrm{Cl}^+(F)$ is the narrow class group of F. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ be fractional ideals of F such that their images in $\mathrm{Cl}^+(F)$ form a set of representative of $\mathrm{Cl}^+(F)/\mathrm{Cl}^+(F)^2$. For each $t = 1, 2, \ldots, h$, choose an idele $\alpha_t \in \mathbb{A}_\mathrm{f}^\times$ such that $\alpha_t\mathfrak{o}_v = \mathfrak{a}_t\mathfrak{o}_v$. Put $\beta_t = \left(\begin{smallmatrix} \alpha_t^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{A}_\mathrm{f})$. Then we have $\mathrm{GL}_2(\mathbb{A}) = \coprod_{t=1}^h Z(\mathbb{A})\mathrm{GL}_2(F)\beta_t\mathcal{K}_0\mathrm{GL}_2^+(\mathbb{R})^n$, and so

$$Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})/\mathcal{K}_0 \cdot \mathrm{SO}(2)^n \simeq \coprod_{t=1}^h \Gamma_t\backslash\mathfrak{h}^n,$$

where

$$\Gamma_t = Z(F)\backslash\left(\mathrm{GL}_2(F) \cap \prod_{v<\infty} F_v^\times\hat{\Gamma}[\mathfrak{a}_{t,v}^{-1}, \mathfrak{a}_{t,v}] \cdot \mathrm{GL}_2^+(\mathbb{R})^n\right),$$

$$\hat{\Gamma}_v[\mathfrak{a}_v, \mathfrak{b}_v] = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_v) \,\middle|\, a, d \in \mathfrak{o}_v, b \in \mathfrak{a}_v, c \in \mathfrak{b}_v\right\}$$

is the Hurwitz–Maass extension, which is usually denoted by $\Gamma_m(\mathfrak{o}_F \oplus \mathfrak{a}_t)$ (cf. van der Geer [Gee88, ch. I]).

9. Application of Waldspurger's results

The correspondence between modular forms of integral weight and those of half-integral weight was first considered by Shimura [Shi73]. Waldspurger [Wal80, Wal91] treated the Shimura

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correspondence in terms of automorphic representations. In this section, we review Waldspurger's theory of the Shimura correspondence.

Let F be a number field and ψ a non-trivial additive character of $\mathbb{A}/F$. Let $\mathcal{A}_0$ be the space of genuine cusp forms of $G_F\backslash\widetilde{G_{\mathbb{A}}}$. The space of cusp forms orthogonal to the Weil representations associated to one-dimensional quadratic forms is denoted by $\mathcal{A}_{00}$. Then the multiplicity of an irreducible genuine cuspidal automorphic representation in $\mathcal{A}_{00}$ is one [Wal91, Theorem 3].

Let σ be an irreducible genuine cuspidal automorphic representation in $\mathcal{A}_{00}$. For $\xi \in F^\times$, we put $\psi_\xi(x) = \psi(\xi x)$. The additive character ψ_ξ of $\mathbb{A}/F$ is called a missing character of σ, if

$$\int_{F\backslash\mathbb{A}} f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\overline{\psi_\xi(x)}\,dx = 0$$

for any $f \in \sigma$ and $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$. We denote by $\theta(\sigma,\psi)$ the theta correspondence of σ for the dual pair $\mathrm{SL}_2 \times \mathrm{PGL}_2$. Then the theta correspondence $\theta(\sigma,\psi_\xi^{-1}) = 0$ if and only if ψ_ξ is a missing character [Wal80, Proposition 26]. Moreover, if ψ_ξ is not a missing character, then $\theta(\sigma,\psi_\xi^{-1}) \otimes \hat{\chi}_\xi$ does not depend on the choice of $\xi \in F^\times$ [Wal80, Proposition 28]. Here, $\hat{\chi}_\xi$ is the Hecke character of $\mathbb{A}^\times$ corresponding to $F(\sqrt{\xi})/F$. Put $\mathrm{Wald}(\sigma,\psi) = \theta(\sigma,\psi_\xi^{-1}) \otimes \hat{\chi}_\xi$. Then Waldspurger proved the following theorem.

Theorem 9.1. *Put $\tau = \mathrm{Wald}(\sigma,\psi)$. Let $L(s,\tau)$ be the L-function associated to τ. Then ψ is not a missing character if and only if the following two conditions are satisfied.*

(i) *There exists a non-zero ψ_v-Whittaker functional for σ_v for any place v.*

(ii) *We have $L(1/2,\tau) \neq 0$.*

Conversely, let τ be an automorphic representation of $H_{\mathbb{A}} = \mathrm{PGL}_2(\mathbb{A})$. Then the theta correspondence $\theta(\tau,\psi)$ is non-zero if and only if $L(1/2,\tau) \neq 0$. Waldspurger also proved the following theorem.

Theorem 9.2 (Waldspurger [Wal91, Theorem 4]). *Let τ be an irreducible cuspidal automorphic representation of $H_{\mathbb{A}}$ such that $\varepsilon(1/2,\tau) = 1$, where $\varepsilon(1/2,\tau)$ is the root number of τ. Let Σ be a finite set of places of F and $\delta > 0$ a positive number. Then there exists an element $\xi \in F^\times$ such that the following conditions (1) and (2) hold:*

(1) $|\xi - 1|_v < \delta$ *for each* $v \in \Sigma$;

(2) $L(1/2,\tau\otimes\hat{\chi}_\xi) \neq 0$.

For similar non-vanishing results, see [BFH90, FH95]. For a non-Archimedean place v of F and $s_v \in \mathbb{C}$, we define an unramified principal series $I(s_v)$ of $H_v = \mathrm{PGL}_2(F_v)$ as follows. Let B_v be the Borel subgroup of $\mathrm{GL}_2(F_v)$ which consists of upper triangular elements. Let $I(s_v)$ be the representation of $\mathrm{GL}_2(F_v)$ induced from the character B_v given by

$$\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} \mapsto |ad^{-1}|^{s_v}.$$

Then $I(s_v)$ can be considered as a representation of H_v.

Now let F be a totally real number field and ψ_1 the non-trivial additive character of $\mathbb{A}/F$ such that infinity component of ψ_1 is given by $x \mapsto \mathbf{e}(x)$ for any real place. For $\xi \in F^\times$, we put $\psi_\xi(x) = \psi_1(\xi x)$. We assume $\kappa_i \geqslant 1$ for $i = 1, 2, \ldots, n$. Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ which satisfies the following conditions (1) and (2).

(1) If v is a non-Archimedean place, then τ_v is an unramified principal series $I(s_v)$, $s_v \in \mathbb{C}$.

(2) If ∞_i is a real place, then τ_{∞_i} is a discrete series representation with minimal weight $\pm 2\kappa_i$.

Let $\eta \in \mathfrak{o}_F^\times$ be a unit such that $\mathrm{N}_{F/\mathbb{Q}}(\eta) = (-1)^{\sum_{i=1}^n \kappa_i}$. We shall show that $\varepsilon(1/2, \tau \otimes \hat{\chi}_\eta) = 1$. In fact, $\varepsilon(1/2, \tau_{\infty_i} \otimes \hat{\chi}_\eta) = (-1)^{\kappa_i}$ and $\varepsilon(1/2, \tau_v \otimes \hat{\chi}_\eta) = \langle -1, \eta \rangle_v$ for a non-Archimedean place v. It follows that

$$\varepsilon(1/2, \tau \otimes \hat{\chi}_\eta) = (-1)^{\sum_{i=1}^n \kappa_i} \prod_{v<\infty} \langle -1, \eta \rangle_v.$$

By the Hilbert product formula, $\prod_{v<\infty} \langle -1, \eta \rangle_v = \prod_{i=1}^n \langle -1, \eta \rangle_{\infty_i} = \mathrm{N}_{F/\mathbb{Q}}(\eta)$. It follows that $\varepsilon(1/2, \tau \otimes \hat{\chi}_\eta) = (-1)^{\sum_{i=1}^n \kappa_i} \cdot \mathrm{N}_{F/\mathbb{Q}}(\eta) = 1$.

By Theorem 9.2, there exists a totally positive element $\xi \in F^\times$ such that $L(1/2, \tau \otimes \hat{\chi}_{\xi\eta}) \neq 0$. Put $\sigma = \theta(\tau \otimes \hat{\chi}_{\xi\eta}, \psi_\xi)$. Then σ is an automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ which satisfies the following conditions (i), (ii) and (iii).

(i) If v is a non-Archimedean place, then σ_v is isomorphic to $\tilde{I}_{\psi_\eta}(s_v)$.

(ii) If ∞_i is a real place, then σ_{∞_i} is a lowest weight representation with minimal weight $\kappa_i + (1/2)$.

(iii) We have $\sigma \subset \mathcal{A}_{00}$.

Thus Waldspurger's theorems imply that there exists a one-to-one correspondence between the set of irreducible automorphic representation τ of $\mathrm{PGL}_2(\mathbb{A})$ satisfying conditions (1) and (2) and the set of irreducible genuine automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ satisfying conditions (i), (ii), and (iii).

LEMMA 9.3. *Let σ be an irreducible genuine automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ satisfying the condition* (ii). *If $\kappa_i > 1$ for some i, then $\sigma \subset \mathcal{A}_{00}$. If $\sigma \subset \mathcal{A}_{00}$, then σ_v is not isomorphic to a even Weil representation for any non-Archimedean place v.*

Proof. Since a holomorphic theta function associated to a one-dimensional orthogonal form has weight no greater than 3/2 (cf. Shimura [Shi87]), the first part follows. Since even Weil representations do not correspond to a generic representation of $\mathrm{PGL}_2(F_v)$ by the theta correspondence for $\widetilde{\mathrm{SL}_2} \times \mathrm{PGL}_2$, the latter part follows. □

Suppose that $\kappa_i > 1$ for some i. Then Waldspurger's theorem and Lemma 9.3 yield a one-to-one correspondence between the following two sets.

(a) The set of irreducible automorphic representation τ of $\mathrm{PGL}_2(\mathbb{A})$ with properties (1) and (2).

(b) The set of irreducible automorphic representation σ of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ with properties (i) and (ii).

Let $E^K = \prod_{v<\infty} E_v^K$ be an idempotent Hecke operator defined with respect to the additive character $\psi = \psi_\eta$.

It is well known that there exists a direct sum decomposition

$$\mathcal{A}_{2\kappa}^{\mathrm{cusp}}(H_F \backslash H_{\mathbb{A}} / \mathcal{K}_0) = \bigoplus_{i=1}^{d} \mathbb{C} \cdot f_i$$

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where f_i is a Hecke eigenform for any $1 \leqslant i \leqslant d$. Then the automorphic representation τ_i of $\mathrm{PGL}_2(\mathbb{A})$ generated by f_i is irreducible. Combining the results above and Proposition 6.3, we obtain the following theorem.

Theorem 9.4. *Suppose that $\kappa_i > 1$ for some $1 \leqslant i \leqslant n$. Then there exists a direct sum decomposition*

$$S_{\kappa+(1/2)}(\Gamma)^{E^K} = \bigoplus_{i=1}^{d} \mathbb{C} \cdot h_i$$

with the following properties (1) and (2).

(1) *The cusp form h_i is a Hecke eigenform with respect to the Hecke algebra $\widetilde{\mathcal{H}}_v$ for finite places $v \nmid 2$ for any $1 \leqslant i \leqslant d$.*

(2) *Let σ_i be the automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ generated by h_i. Then we have $\tau_i \simeq \mathrm{Wald}(\sigma_i, \psi)$ for any $1 \leqslant i \leqslant d$.*

Remark 9.1. In the case $\kappa_1 = \cdots = \kappa_n = 1$, the same conclusion holds if $S_{\kappa+(1/2)}(\Gamma)^{E^K}$ is replaced by $\mathcal{A}_{00} \cap S_{\kappa+(1/2)}(\Gamma)^{E^K}$.

Note that if $\sigma_{i,v} \simeq \tilde{I}_{\psi_v}(s_v)$ for a non-Archimedean place v, then $\tau_{i,v} \simeq I(s_v)$. The Satake parameter of f_i is $q_v^{s_v}$, by definition.

Theorem 9.4 implies that the strong multiplicity-one property holds for $S_{\kappa+(1/2)}(\Gamma)^{E^K}$. There exists a one-to-one correspondence between Hecke eigenforms of $S_{\kappa+(1/2)}(\Gamma)^{E^K}$ with respect to the Hecke algebras $\widetilde{\mathcal{H}}_v$ for $v \nmid 2$ and Hecke eigenforms of $\mathcal{A}_{2\kappa}^{\mathrm{cusp}}(H_F \backslash H_{\mathbb{A}} / \mathcal{K}_0)$ defined up to constant.

10. A Fourier coefficient formula

Assume that $\kappa_i > 1$ for some i. For each non-Archimedean place v of F and $\xi \in F_v$, let $\Psi_v(\xi, x)$ be the function defined in Definition 4.1.

Theorem 10.1. *Suppose that $h(z) = \sum_{\xi \in \mathfrak{o}_F} c(\xi) \mathbf{e}(\xi z) \in S_{\kappa+(1/2)}(\Gamma)^{E^K}$ is a Hecke eigenform. Let $f(z) \in \mathcal{A}_{2\kappa}^{\mathrm{cusp}}(H_F \backslash H_{\mathbb{A}} / \mathcal{K}_0)$ be a Hecke eigenform corresponding to h. Let α_v be the Satake parameter of f for non-Archimedean place v of F. Then for totally positive element $\xi \in F^\times$, the ξth Fourier coefficient is of the form*

$$c(\xi) = \beta_\xi \prod_{v < \infty} \Psi_v(\eta\xi, \alpha_v) \prod_{i=1}^{n} \iota_i(\xi)^{(\kappa_i/2)-(1/4)},$$

where the constant β_ξ satisfies

$$\beta_{\xi a^2} = \beta_\xi \prod_{i=1}^{n} \langle (-1)^{\kappa_i} \eta, a \rangle_{\infty_i} \quad (a \in F^\times).$$

Proof. Let $\varphi(g) \in \mathcal{A}_{\kappa+(1/2)}^{\mathrm{cusp}}(G_F \backslash \widetilde{G_{\mathbb{A}}})^{E^K}$ be the automorphic form on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ associated to $h \in S_{\kappa+(1/2)}(\Gamma)^{E^K}$. Consider the Whittaker function

$$\mathcal{W}_\xi(g) = \int_{x \in \mathbb{A}} \varphi(\mathbf{u}^\sharp(x) g) \overline{\psi_1(\xi x)} \, dx.$$

It is well known that the Fourier coefficient $c(\xi)$ is related to the Whittaker function by the formula $c(\xi)=\mathcal{W}_\xi(1)e^{2\pi \mathrm{tr}_{F/\mathbb{Q}}(\xi)}$. If $c(\xi a^2)=0$ for any $a\in F^\times$, then we put $\beta_\xi=0$. If $c(\xi a^2)\neq 0$ for some $a\in F^\times$, then replacing ξ by ξa^2, we may assume $c(\xi)\neq 0$. By the uniqueness of the Whittaker function, we have

$$\mathcal{W}_\xi(g)=\beta_\xi\prod_{v\leqslant\infty}W^+_{\xi,v}(g_v)$$

for some $\beta_\xi\in\mathbb{C}^\times$. In particular,

$$\begin{aligned}c(\xi)&=\beta_\xi\prod_{v<\infty}|\xi|_v^{1/2}\Psi_v(\eta\xi,\alpha_v)\prod_{i=1}^n\iota_i(\xi)^{(\kappa_i/2)+(1/4)}\\&=\beta_\xi\prod_{v<\infty}\Psi_v(\eta\xi,\alpha_v)\prod_{i=1}^n\iota_i(\xi)^{(\kappa_i/2)-(1/4)}.\end{aligned}$$

It is enough to prove that

$$c(\xi a^2)=\beta_\xi\prod_{v<\infty}\Psi_v(\eta\xi a^2,\alpha_v)\prod_{i=1}^n\langle(-1)^{\kappa_i}\eta,a\rangle\iota_i(\xi a^2)^{(\kappa_i/2)-(1/4)}$$

for any $a\in F^\times$. It is easy to see $\mathcal{W}_{\xi a^2}(1)=\mathcal{W}_\xi(\mathbf{m}(a))$. It follows that

$$\begin{aligned}c(\xi a^2)&=\mathcal{W}_\xi(\mathbf{m}(a))e^{2\pi\mathrm{tr}_{F/\mathbb{Q}}(\xi a^2)}\\&=\beta_\xi\prod_{v<\infty}\frac{\alpha_{\psi_v}(\eta)}{\alpha_{\psi_v}(\eta a)}|\xi a^2|_v^{1/2}\Psi_v(\eta\xi a^2,\alpha_v)\prod_{i=1}^n\left(\frac{\alpha_{\psi_{\infty,i}}(1)}{\alpha_{\psi_{\infty,i}}(a)}\right)^{2\kappa_i+1}\iota_i(\xi a^2)^{(\kappa_i/2)+(1/4)}.\end{aligned}$$

Note that

$$\begin{aligned}\prod_{v<\infty}\frac{\alpha_{\psi_v}(\eta)}{\alpha_{\psi_v}(\eta a)}&=\prod_{v<\infty}\frac{\alpha_{\psi_v}(1)}{\alpha_{\psi_v}(a)}\langle\eta,a\rangle_v,\\\prod_{i=1}^n\left(\frac{\alpha_{\psi_{\infty,i}}(1)}{\alpha_{\psi_{\infty,i}}(a)}\right)^{2\kappa_i+1}&=\prod_{i=1}^n\frac{\alpha_{\psi_{\infty,i}}(1)}{\alpha_{\psi_{\infty,i}}(a)}\langle(-1)^{\kappa_i},a\rangle_{\infty_i}.\end{aligned}$$

Hence the theorem follows by the product formulas. □

11. Review of the result of Baruch and Mao

In this section, we review the result of Baruch and Mao [BM07]. First we recall the definition of two local invariants $e(\varphi,\psi)$ and $e(\tilde{\varphi},\psi)$. Let F be a local field and ψ be a non-trivial additive character of F. Let $(\pi,\mathcal{V}_\pi)$ be an irreducible admissible unitary representation of $\mathrm{PGL}_2(F)$. A ψ-Whittaker functional L of π is a linear map $\mathcal{V}_\pi\longrightarrow\mathbb{C}$ such that

$$L(\pi(\mathbf{u}^\sharp(x))\varphi)=\psi(x)L(\varphi)$$

for any $x\in F$. The Whittaker function $W_\varphi(g)$ associated to L and φ is given by

$$W_\varphi(g)=L(\pi(g)\varphi)\quad(g\in\mathrm{PGL}_2(F)).$$

We assume $W_\varphi(\mathbf{1}_2)\neq 0$. Consider the inner product

$$(\varphi,\varphi')=\int_F W_\varphi\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)\overline{W_{\varphi'}\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)}\frac{da}{|a|}.\qquad(11.1)$$

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Here, da is the self-dual Haar measure of F with respect to ψ. Then Baruch and Mao defined the local invariant $e(\varphi, \psi)$ by

$$e(\varphi, \psi) = \frac{(\varphi, \varphi)}{|W_\varphi(\mathbf{1}_2)|^2}. \tag{11.2}$$

Note that $e(\varphi, \psi)$ does not depend on the choice of L.

Let $(\tilde{\pi}, \mathcal{V}_{\tilde{\pi}})$ be an irreducible admissible unitary representation of $\widetilde{\mathrm{SL}_2(F)}$. A ψ-Whittaker functional $\tilde{L}$ of $\tilde{\pi}$ is a linear map $\mathcal{V}_{\tilde{\pi}} \longrightarrow \mathbb{C}$ such that

$$\tilde{L}(\tilde{\pi}(\mathbf{u}^\sharp(x))\tilde{\varphi}) = \psi(x)\tilde{L}(\tilde{\varphi})$$

for any $x \in F$. The Whittaker function $\tilde{W}_{\tilde{\varphi}}(\tilde{g})$ associated to $\tilde{L}$ and $\tilde{\varphi}$ is given by

$$\tilde{W}_{\tilde{\varphi}}(\tilde{g}) = \tilde{L}(\tilde{\pi}(g)\tilde{\varphi}) \quad (\tilde{g} \in \widetilde{\mathrm{SL}_2(F)}).$$

We assume $\tilde{W}_{\tilde{\varphi}}([\mathbf{1}_2]) \neq 0$. Let $\{\delta_i\}$ be a set of representatives for $F/F^{\times 2}$ with $\delta_1 = 1$. Given a ψ-Whittaker functional $\tilde{L}$, Baruch and Mao proved that there exist ψ_{δ_i}-Whittaker functionals $\{\tilde{L}^{\delta_i}\}$ such that $\tilde{L}^{\delta_1} = \tilde{L}$ and such that the inner product

$$(\tilde{\varphi}, \tilde{\varphi}')_{BM} = \sum_{\delta_i} \frac{|2|}{2} \int_F \tilde{W}^{\delta_i}_{\tilde{\varphi}}(\mathbf{m}(a))\overline{\tilde{W}^{\delta_i}_{\tilde{\varphi}'}(\mathbf{m}(a))}\, \frac{da}{|a|}$$

is $\widetilde{\mathrm{SL}_2(F)}$-invariant. Here, $\tilde{W}^{\delta_i}_{\tilde{\varphi}}(g) = \tilde{L}^{\delta_i}(\tilde{\varphi})$. Then the local invariant $e(\tilde{\varphi}, \psi)$ is given by

$$e(\tilde{\varphi}, \psi) = \frac{(\tilde{\varphi}, \tilde{\varphi})_{BM}}{|\tilde{W}_{\tilde{\varphi}}([\mathbf{1}_2])|^2}. \tag{11.3}$$

Now, let F be a totally real number field and $\tilde{\pi} \subset \mathcal{A}_{00}$ be an irreducible cuspidal automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. Put $\pi = \mathrm{Wald}(\tilde{\pi}, \psi)$. We first assume that ψ is not a missing character of $\tilde{\pi}$. Note that in this case, $\tilde{\pi} = \theta(\pi, \psi)$ and $L(1/2, \pi) \neq 0$. Let S be a set of bad place of F containing all places of v where ψ_v is not of order 0, and all places where π_v is not unramified.

For $\varphi = \otimes_v \varphi_v \in \pi$ and $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v \in \tilde{\pi}$, Put

$$W^\psi_\varphi(g) = \int_{F\backslash\mathbb{A}} \varphi(\mathbf{u}^\sharp(x)g)\overline{\psi(x)}\, dx \quad (g \in \mathrm{PGL}_2(\mathbb{A})),$$
$$\widetilde{W}^\psi_{\tilde{\varphi}}(\tilde{g}) = \int_{F\backslash\mathbb{A}} \tilde{\varphi}(\mathbf{u}^\sharp(x)\tilde{g})\overline{\psi(x)}\, dx \quad (\tilde{g} \in \widetilde{\mathrm{SL}_2(\mathbb{A})}).$$

We assume $W^\psi_\varphi(\mathbf{1}_2) \neq 0$ and $\widetilde{W}^\psi_{\tilde{\varphi}}(\mathbf{1}_2) \neq 0$. Note that such φ and $\tilde{\varphi}$ exist by our assumption. We choose the Haar measure of dg on $\mathrm{SL}_2(\mathbb{A})$ and dh on $\mathrm{PGL}_2(\mathbb{A})$ as follows (cf. Lemma 9.1 of [BM07]). The Haar measure $dg = \prod_v dg_v$ and $dh = \prod_v dh_v$ are the product measures. If v is non-Archimedean, then dg_v and dh_v are normalized so that $\int_{\mathrm{SL}_2(\mathfrak{o}_v)} dg_v = \int_{\mathrm{PGL}_2(\mathfrak{o}_v)} dh_v = 1$. For $\mathrm{SL}_2(\mathbb{R})$, the measure is given by $dg_v = (2\pi)^{-1}|a|^{-2} da\, dx\, d\theta$, where $g_v = \mathbf{u}^\sharp(x)\mathbf{m}(a)\mathbf{k}(\theta)$. Similarly, for $\mathrm{PGL}_2^+(\mathbb{R})$, the measure is given by $dh_v = (2\pi)^{-1}|a|^{-2} da\, dx\, d\theta$, where h_v is the image of $\mathbf{u}^\sharp(x)\mathbf{m}(a)\mathbf{k}(\theta)$. Then [BM07, Equation (4.4)] says

$$\frac{|\widetilde{W}^\psi_{\tilde{\varphi}}(\mathbf{1}_2)|^2}{\langle\tilde{\varphi}, \tilde{\varphi}\rangle} = \frac{|W^\psi_\varphi(\mathbf{1}_2)|^2}{\langle\varphi, \varphi\rangle} L(1/2, \pi) \prod_{v \in S} \frac{e(\varphi_v, \psi_v)}{e(\tilde{\varphi}_v, \psi_v)L(1/2, \pi_v)}. \tag{11.4}$$

Here, the inner product $\langle\varphi,\varphi\rangle$ or $\langle\tilde\varphi,\tilde\varphi\rangle$ is taken with respect to the Haar measure chosen as above. It is well known that

$$\mathrm{Vol}(\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})) = 2\mathfrak{D}_F^{3/2}\xi_F(2),$$
$$\mathrm{Vol}(\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})) = \mathfrak{D}_F^{3/2}\xi_F(2),$$

where $\mathfrak{D}_F$ is the discriminant of F and $\xi_F(s)=\Gamma_{\mathbb{R}}(s)^n\zeta(s)$ is the complete Dedekind zeta function. Thus, in (11.4), we may and do replace the inner products $\langle\varphi,\varphi\rangle$ and $\langle\tilde\varphi,\tilde\varphi\rangle$ by the inner products with respect to the Tamagawa measures.

It can be shown that

$$\frac{|W_\varphi^\psi(\mathbf{1}_2)|^2}{\langle\varphi,\varphi\rangle} = \frac{\xi_F(2)}{2L(1,\pi,\mathrm{Ad})}\prod_{v\in S}\frac{|W_{\varphi_v}(1)|^2L(1,\pi_v,\mathrm{Ad})}{(\varphi_v,\varphi_v)L(2,\mathbf{1}_v)}.$$

Here, $L(s,\pi,\mathrm{Ad})$ is the adjoint L-function for π, and $L(s,\mathbf{1}_v)$ is the local Euler factor for $\xi_F(s)$. This can be proved by taking the residue of the integral representation (see Jacquet [Jac72]) of $L(s,\pi\times\pi)$. See also [FLO12, Appendix A]. It follows that

$$\frac{|\widetilde{W}_{\tilde\varphi}^\psi(\mathbf{1}_2)|^2}{\langle\tilde\varphi,\tilde\varphi\rangle} = \frac{\xi_F(2)L(1/2,\pi)}{2L(1,\pi,\mathrm{Ad})}\prod_{v\in S}\beta_v(\tilde\varphi_v,\psi_v), \tag{11.5}$$

where

$$\beta_v(\tilde\varphi_v,\psi_v) = \frac{|\widetilde{W}_{\tilde\varphi_v}^{\psi_v}(1)|^2L(1,\pi_v,\mathrm{Ad})}{(\tilde\varphi_v,\tilde\varphi_v)L(2,\mathbf{1}_v)L(1/2,\pi_v)}.$$

In the case when $\tilde\pi_v$ does not have a non-zero ψ_v-Whittaker functional, we set $\beta_v(\tilde\varphi_v,\psi_v)=0$. Then the formula (11.5) still holds when ψ is a missing character of $\tilde\pi$ by Theorem 9.1.

12. A generalization of the Kohnen–Zagier formula

In this section, we discuss a generalization of the Kohnen–Zagier formula [KZ81]. A similar result for Jacobi forms was also treated by [Koj12]. The following lemma follows from [BM07, Proposition 8.8].

Lemma 12.1. *Suppose that $F=\mathbb{R}$ and $\psi_\xi(x)=\mathbf{e}(\xi x)$, $\xi>0$. Assume that $(\tilde\pi,\mathcal{V}_{\tilde\pi})$ is the lowest weight representation of minimal weight $\kappa+(1/2)$ and $\tilde\varphi\in\mathcal{V}_{\tilde\pi}$ is a vector of weight $\kappa+(1/2)$. We define the Archimedean L-factor as in the work of Tate [Tat79]. Then we have*

$$\beta(\tilde\varphi,\psi_\xi) = e^{-4\pi\xi}2^{-1+3\kappa}\xi^{\kappa+(1/2)}.$$

Let F be a non-Archimedean local field. Let $(\tilde\pi,\mathcal{V}_{\tilde\pi})$ be an irreducible admissible unitary representation of $\widetilde{\mathrm{SL}_2}(F)$. Put $\tilde\pi_{\boldsymbol{\delta}}=\tilde\pi\circ\tilde{\boldsymbol{r}}_{\boldsymbol{\delta}}$, where $\tilde{\boldsymbol{r}}_{\boldsymbol{\delta}}$ is as in Definition 6.1. Thus there exists an isomorphism $\mathcal{I}_{\boldsymbol{\delta}}:\mathcal{V}_{\tilde\pi}\to\mathcal{V}_{\tilde\pi_\delta}$ such that $\mathcal{I}_{\boldsymbol{\delta}}\circ\tilde\pi=\tilde\pi_{\boldsymbol{\delta}}\circ\mathcal{I}_{\boldsymbol{\delta}}$. If $\tilde L:\mathcal{V}_{\tilde\pi}\to\mathbb{C}$ is a ψ-Whittaker functional of $\tilde\pi$, then $\tilde L\circ\mathcal{I}_{\boldsymbol{\delta}}^{-1}$ is a $\psi_{\boldsymbol{\delta}}$-Whittaker functional of $\tilde\pi_\delta$. Then it is easily seen that

$$e(\tilde\varphi,\psi) = |\boldsymbol{\delta}|^{1/2}e(\mathcal{I}_\delta(\tilde\varphi),\psi_{\boldsymbol{\delta}}) \quad (\tilde\varphi\in\mathcal{V}_{\tilde\pi}).$$

Lemma 12.2. *Suppose that F is a non-Archimedean local field. Suppose also that $\tilde\pi=\tilde I_\psi(s)$ is an irreducible unitary representation of $\widetilde{\mathrm{SL}_2}(F)$. Let $\tilde\varphi_0\in\tilde I_\psi(s)$ be a non-zero ψ-pseudospherical vector. Then we have*

$$\beta(\tilde\varphi_0,\psi_D) = q^{c_\psi/2}|2^{-1}D|\cdot|\Psi(D,X)|^2.$$

Here, $X=q^{-s}$ and $\Psi(D,X)$ are as in Definition 4.1.

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Proof. We may assume that $c_\psi = 0$. We follow the argument of Baruch and Mao [BM07]. We prove this lemma under the assumption that $\tilde{I}_\psi(s)$ is a unitary tempered principal series (i.e. $s \in \sqrt{-1}\mathbb{R}$), since we need only such cases. As for the complementary series case, the reader can refer to [BM07, § 8.2]. For $\tilde{\varphi}, \tilde{\varphi}' \in \tilde{I}_\psi(s)$, we define an inner product $(\tilde{\varphi}, \tilde{\varphi}')$ by

$$(\tilde{\varphi}, \tilde{\varphi}') = \int_{x \in F} \tilde{\varphi}(\mathbf{w}_1 \cdot \mathbf{u}^\sharp(x)) \overline{\tilde{\varphi}(\mathbf{w}_1 \cdot \mathbf{u}^\sharp(x))}\, dx$$

as in § 4. We define a $\psi_{D\delta_i}$-Whittaker functional $\tilde{L}^{D\delta_i}$ by

$$\tilde{L}^{D\delta_i}(\tilde{\varphi}) = |D\delta_i|^{1/2} \int_{x \in F} \tilde{\varphi}(\mathbf{w}_1 \cdot \mathbf{u}^\sharp(x)) \overline{\psi_{D\delta_i}(x)}\, dx \quad \tilde{\varphi} \in \tilde{I}_\psi(s).$$

Put $\tilde{W}^{D\delta_i}_{\tilde{\varphi}}(g) = \tilde{L}^{D\delta_i}(\tilde{\varphi})$. Baruch and Mao proved that

$$(\tilde{\varphi}, \tilde{\varphi}') = \sum_{\delta_i \in F^\times/F^{\times 2}} \frac{|2|}{2} \int_{a \in F^\times} \tilde{W}^{D\delta_i}_{\tilde{\varphi}}(\mathbf{m}(a)) \overline{\tilde{W}^{D\delta_i}_{\tilde{\varphi}}(\mathbf{m}(a))}\, \frac{da}{|a|}$$

for any $\tilde{\varphi}, \tilde{\varphi}' \in \tilde{I}_\psi(s)$ (cf. [BM07, § 8.1]). Thus we have $(\tilde{\varphi}, \tilde{\varphi}')_{BM} = (\tilde{\varphi}, \tilde{\varphi}')$. We may assume $\tilde{\varphi}_0 = f_K^+$, where f_K^+ is the ψ-pseudospherical vector defined in § 4. By Proposition 4.7, we have $(\tilde{\varphi}_0, \tilde{\varphi}_0) = |2|(1 + q^{-1})$. On the other hand,

$$|\tilde{W}^{D}_{\tilde{\varphi}_0}([\mathbf{1}_2])|^2 = |D| \cdot |\mathcal{W}_{f_K^+, D}(1)|^2 = |D| \cdot |\boldsymbol{\gamma}(D, X)\Psi(D, X)|^2.$$

Here, $\boldsymbol{\gamma}(D, X)$ is as in Definition 4.1. Let $\tau = \theta(\tilde{\pi}, \psi^{-1})$ be the unramified principal series of $\mathrm{PGL}_2(F)$ with Satake parameter $\{X, X^{-1}\}$. Then we have $\pi = \theta(\tilde{\pi}, \psi_D^{-1}) = \tau \otimes \hat{\chi}_D$ and

$$\begin{aligned}\frac{L(1, \pi, \mathrm{Ad})}{L(2, \mathbf{1}) L(1/2, \pi)} &= \frac{(1 - q^{-2})(1 - \chi_D q^{-1/2} X)(1 - \chi_D q^{-1/2} X^{-1})}{(1 - q^{-1} X^2)(1 - q^{-1})(1 - q^{-1} X^{-2})} \\ &= (1 + q^{-1})|\boldsymbol{\gamma}(D, X)|^2.\end{aligned}$$

Hence the lemma. □

Now we return to the situation of § 10. Let F be a totally real number field. Let $h(z) = \sum_{\xi \in \mathfrak{o}_F} c(\xi)\mathbf{e}(\xi z) \in S_{\kappa+(1/2)}(\Gamma)^{E^K}$ be a Hecke eigenform. We denote $\tilde{\varphi}_h \in \mathcal{A}_{\kappa+(1/2)}(G_F \backslash \widetilde{G_{\mathbb{A}}})^{E^K}$ by a corresponding automorphic form on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. Let $\psi_\xi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be the additive character given by $\psi_\xi(x) = \psi_1(\xi x)$. Then we have

$$\widetilde{W}^{\psi_\xi}_{\tilde{\varphi}_h}(\mathbf{1}_2) = c(\xi) e^{-2\pi \cdot \mathrm{tr}(\xi)}.$$

Let σ and τ be as in § 9. Thus σ be the automorphic representation of $\widetilde{G_{\mathbb{A}}}$ generated by $\tilde{\varphi}_h$ and $\tau = \mathrm{Wald}(\sigma, \psi_\eta)$. Note that τ_v is tempered for any v by Blasius [Bla06], and so the Satake parameter α_v has absolute value 1.

Now we apply (11.5) for $\tilde{\pi} = \sigma$, $\tilde{\varphi} = \tilde{\varphi}_h$, $\psi = \psi_\xi$. Note that $\pi = \mathrm{Wald}(\sigma, \psi_\xi) = \tau \otimes \hat{\chi}_{\eta\xi}$, where, $\hat{\chi}_{\eta\xi}$ is the Hecke character of $\mathbb{A}^\times/F^\times$ corresponding to $F(\sqrt{\eta\xi})/F$. Then we have

$$\frac{|\widetilde{W}^{\psi_\xi}_{\tilde{\varphi}_h}(\mathbf{1}_2)|^2}{\langle \tilde{\varphi}_h, \tilde{\varphi}_h \rangle} = \frac{\xi_F(2) L(1/2, \tau \otimes \hat{\chi}_{\eta\xi})}{2L(1, \tau, \mathrm{Ad})} \prod_{v \in S} \beta_v(\tilde{\varphi}_{h,v}, \psi_{\xi,v}).$$

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Here, the set S of bad places consists of all infinite places and finite places which divide $2\mathfrak{D}_F \mathrm{N}_{F/\mathbb{Q}}(\xi)$. Let S_∞ be the set of infinite places. By Lemma 12.1, we have

$$\prod_{v\in S_\infty} \beta_v(\tilde{\varphi}_{h,v}, \psi_{\xi,v}) = e^{-4\pi\cdot\mathrm{tr}(\xi)} 2^{-n+3|\kappa|} \prod_{i=1}^{n} \iota_i(\xi)^{\kappa_i+(1/2)}.$$

Here, $|\kappa| = \sum_{i=1}^{n} \kappa_i$. If v is a finite place, then $\tilde{\varphi}_{h,v}$ is a $\psi_{\eta,v}$-pseudospherical vector. By applying Lemma 12.2 for $\psi = \psi_{\eta,v}$ and $D = \eta^{-1}\xi$, we have

$$\prod_{v<\infty} \beta_v(\tilde{\varphi}_{h,v}, \psi_{\xi,v}) = 2^n \mathfrak{D}_F^{1/2} \mathrm{N}_{F/\mathbb{Q}}(\xi)^{-1} \prod_{v<\infty} |\Psi_v(\eta\xi, \alpha_v)|^2.$$

Putting this together, we obtain

$$\frac{|c(\xi)|^2}{\langle \tilde{\varphi}_h, \tilde{\varphi}_h \rangle} = \mathfrak{D}_F^{1/2} 2^{-1+3|\kappa|} \xi_F(2) \frac{L(1/2, \tau \otimes \hat{\chi}_{\eta\xi})}{L(1, \tau, \mathrm{Ad})} \prod_{v<\infty} |\Psi_v(\eta\xi, \alpha_v)|^2 \prod_{i=1}^{n} \iota_i(\xi)^{\kappa_i-(1/2)}. \quad (12.1)$$

The following theorem can be considered as a generalization of the Kohnen–Zagier formula [KZ81].

THEOREM 12.3. *Let $h = \sum_\xi c(\xi)\mathbf{e}(\xi z) \in S_{\kappa+(1/2)}(\Gamma)^{E^K} \cap \mathcal{A}_{00}$ and $\varphi_h \in \mathcal{A}_{\kappa+(1/2)}(G_F\backslash \widetilde{G_\mathbb{A}})^{E^K}$ be the automorphic form corresponding to h. The inner product $\langle \tilde{\varphi}_h, \tilde{\varphi}_h \rangle$ is defined by*

$$\langle \tilde{\varphi}_h, \tilde{\varphi}_h \rangle = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} |\tilde{\varphi}_h(g)|^2 \, dg,$$

where dg is the Tamagawa measure for $\mathrm{SL}_2(\mathbb{A})$. Put $\tau = \mathrm{Wald}(\sigma, \psi_\eta)$, where σ is the automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ generated by φ_h. For totally positive element $\xi \in F^\times$, define β_ξ to be as in Theorem 10.1. Then we have

$$\frac{|\beta_\xi|^2}{\langle \tilde{\varphi}_h, \tilde{\varphi}_h \rangle} = \mathfrak{D}_F^{1/2} 2^{-1+3|\kappa|} \xi_F(2) \frac{L(1/2, \tau \otimes \hat{\chi}_{\eta\xi})}{L(1, \tau, \mathrm{Ad})}.$$

Here, $L(s, \tau \otimes \hat{\chi}_{\eta\xi})$ and $L(s, \tau, \mathrm{Ad})$ are complete L-functions with gamma factors as in Tate [Tat79].

Theorem 12.3 follows from Theorem 10.1, (12.1), and the following lemma.

LEMMA 12.4. *Let notations be as above. Then there exists an element $a \in F^\times$ such that*

$$\prod_{v<\infty} \Psi_v(\eta\xi a^2, \alpha_v) \neq 0.$$

Proof. Replacing ξ by ξb^2 for some $b \in \mathfrak{o}_F$, we may assume $\mathfrak{f}_{\eta\xi,v} \geqslant 0$ for any finite place v. Put $\mathfrak{a} = \prod_{v<\infty} \mathfrak{p}_v^{\mathfrak{f}_{\eta\xi,v}}$. Choose a finite place v_0 such that $\mathfrak{a}^{-1}\mathfrak{p}_{v_0} = (a_1)$ is a principal ideal. We may assume that $v_0 \notin S$ and q_{v_0} is sufficiently large. If $\alpha_{v_0} + \alpha_{v_0}^{-1} - \chi_{\eta\xi,v_0} q_{v_0}^{-1/2} \neq 0$, then a_1 satisfies the condition of the lemma. Suppose that $\alpha_{v_0} + \alpha_{v_0}^{-1} - \chi_{\eta\xi,v_0} q_{v_0}^{-1/2} = 0$. Since q_{v_0} is sufficiently large and $\chi_{\eta\xi,v_0} = \pm 1$, α_{v_0} is not a root of unity. (If ζ is a primitive mth root of unity with $m > 12$, then $[\mathbb{Q}(\zeta + \zeta^{-1}) : \mathbb{Q}] > 2$.) It follows that the sequence $\{\alpha_{v_0}^{h_F}, \alpha_{v_0}^{2h_F}, \ldots\}$ is dense in the unit circle, where h_F is the class number of F. Therefore one can find a positive integer m such that

$$\alpha_{v_0}^{mh_F+2} - \alpha_{v_0}^{-mh_F-2} - \chi_{\eta\xi,v_0} q_{v_0}^{-1/2} (\alpha_{v_0}^{mh_F+1} - \alpha_{v_0}^{-mh_F-1}) \neq 0.$$

Then an element $a \in F^\times$ such that $(a) = \mathfrak{a}^{-1}\mathfrak{p}_{v_0}^{mh_F+1}$ satisfies the condition of the lemma. □

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13. The Kohnen plus space

In this section, we assume $\kappa_1, \ldots, \kappa_n \geqslant 0$.

Definition 13.1. Suppose that $\xi \in F$. Then $\xi \equiv \square \bmod 4$ if and only if there exists $y \in \mathfrak{o}_F$ such that $\xi \equiv y^2 \bmod 4\mathfrak{o}_F$.

Obviously, if $\xi \equiv \square \bmod 4$, then $\xi \in \mathfrak{o}_F$. For $\xi \in F$ and a non-Archimedean place v of F, let $\mathfrak{f}_{\xi,v}$ be the invariant defined in Definition 2.2 with respect to F_v.

Lemma 13.1. *Suppose $\xi \in F$. Then $\xi \equiv \square \bmod 4$ if and only if $\mathfrak{f}_{\xi,v} \geqslant 0$ for any place $v < \infty$ of F.*

Proof. Note that the condition is local, and so the lemma follows from Lemma 2.6. □

Definition 13.2. The Kohnen plus space $M^+_{\kappa+(1/2)}(\Gamma)$ is the space of $h(z) = \sum_{\xi \in \mathfrak{o}} c(\xi)\mathbf{e}(\xi z) \in M_{\kappa+(1/2)}(\Gamma)$ such that $c(\xi) = 0$ unless $\eta\xi \equiv \square \bmod 4$. We also set $S^+_{\kappa+(1/2)}(\Gamma) = M^+_{\kappa+(1/2)}(\Gamma) \cap S_{\kappa+(1/2)}(\Gamma)$, which is also called a Kohnen plus space.

Let $v|2$ be a place of F. The symbols q, ϖ, e, etc. defined with respect to F_v are denoted by q_v, ϖ_v, e_v, etc.

Lemma 13.2. *Suppose that $h \in M_{\kappa+(1/2)}(\Gamma)$. Then for $0 < i \leqslant e_v$, $\rho(E_v^{(i)})h = h$ if and only if h has a Fourier expansion*

$$h(z) = \sum_{\substack{\xi \in \mathfrak{o}_F \\ \mathfrak{f}_{\eta\xi,v} + i \geqslant 0}} c(\xi)\mathbf{e}(\xi z).$$

Proof. The case $i = e_v$ is trivial. We may assume that $h = \rho(E_v^{(i+1)})h$ and that $c(\xi) = 0$ unless $\mathfrak{f}_{\eta\xi,v} + i + 1 \geqslant 0$. By Proposition 6.1, we have

$$\begin{aligned} &\rho(E_v^{(i)})h(z) - q_v^{-1}h(z) \\ &\quad = q_v^{e_v - i} \int_{x \in \mathfrak{o}_v^\times} \int_{y \in \mathfrak{o}_v} \psi_{1,v}\left(\frac{\varpi_v^{2i}\eta x y^2}{4\boldsymbol{\delta}_v}\right) \rho\left(\mathbf{u}^\sharp\left(-\frac{\varpi_v^{2i}x}{4\boldsymbol{\delta}_v}\right)\right) h(z)\, dy\, dx. \end{aligned}$$

Here,

$$\begin{aligned} &\int_{x \in \mathfrak{o}_v^\times} \int_{y \in \mathfrak{o}_v} \psi_{1,v}\left(\frac{\varpi_v^{2i}\eta x y^2}{4\boldsymbol{\delta}_v}\right) \rho\left(\mathbf{u}^\sharp\left(-\frac{\varpi_v^{2i}x}{4\boldsymbol{\delta}_v}\right)\right) h(z)\, dy\, dx \\ &\quad = \sum_{\substack{\xi \in \mathfrak{o}_F \\ \mathfrak{f}_{\eta\xi,v} + i + 1 \geqslant 0}} c(\xi)\mathbf{e}(\xi z) \int_{x \in \mathfrak{o}_v^\times} \int_{y \in \mathfrak{o}_v} \psi_{1,v}\left(\frac{\varpi_v^{2i}x(\eta y^2 - \xi)}{4\boldsymbol{\delta}_v}\right) dy\, dx \\ &\quad = \sum_{\substack{\xi \in \mathfrak{o}_F \\ \mathfrak{f}_{\eta\xi,v} + i + 1 \geqslant 0}} c(\xi) T_{2e_v - 2i}(0, -\eta\xi)_v \mathbf{e}(\xi z). \end{aligned}$$

By Proposition 2.8,

$$T_{2e_v - 2i}(0, -\eta\xi)_v = \begin{cases} q_v^{-e_v + i}(1 - q_v^{-1}) & \text{if } \mathfrak{f}_{\eta\xi,v} + i \geqslant 0, \\ -q_v^{-e_v + i - 1} & \text{if } \mathfrak{f}_{\eta\xi,v} + i + 1 = 0,\ \chi_{\eta\xi,v} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathfrak{f}_{\eta\xi,v} + i + 1 = 0$ implies $\chi_{\eta\xi,v} = 0$. Hence the lemma follows. □

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PROPOSITION 13.3. *Suppose that $h \in M_{\kappa+(1/2)}(\Gamma)$. Assume that $h(z)$ has a Fourier expansion*

$$h(z) = \sum_{\substack{\xi \in \mathfrak{o}_F \\ \mathfrak{f}_{\eta\xi,v} \geqslant 0}} c(\xi)\mathbf{e}(\xi z).$$

Then we have $\rho(E_v^K)h = h$.

Proof. By Lemma 13.2, we have $\rho(E_v^{(1)})h = h$. Put $T_v^K = (1+q_v^{-1})E^K - q_v^{-1}E_v^{(1)}$. Then we have

$$\rho(E_v^K)h(z) = (q_v+1)^{-1}h(z) + (1+q_v^{-1})^{-1}\rho(T_v^K)h(z).$$

By Proposition 6.2, we have

$$\begin{aligned}\rho(T_v^K)h(z) &= \alpha_{\psi_v}(\boldsymbol{\delta}_v)q_v^{e_v/2}\rho(\mathbf{w}_{4\boldsymbol{\delta}_v})\int_{x\in\mathfrak{o}_v}\rho\left(\mathbf{u}^\sharp\left(\frac{x}{4\boldsymbol{\delta}_v}\right)\right)h(z)\,dx \\ &= \alpha_{\psi_v}(\boldsymbol{\delta}_v)q_v^{e_v/2}\rho(\mathbf{w}_{4\boldsymbol{\delta}_v})\int_{x\in\mathfrak{p}_v}\rho\left(\mathbf{u}^\sharp\left(\frac{x}{4\boldsymbol{\delta}_v}\right)\right)h(z)\,dx \\ &\quad + q_v^{e_v}\int_{x\in\mathfrak{o}_v^\times}\int_{y\in\mathfrak{o}_v}\psi_{1,v}\left(\frac{\eta x y^2}{4\boldsymbol{\delta}_v}\right)\rho\left(\mathbf{u}^\sharp\left(-\frac{x}{4\boldsymbol{\delta}_v}\right)\right)h(z)\,dy\,dx.\end{aligned}$$

As in the proof of Lemma 13.2, we have

$$\begin{aligned}&q_v^{e_v}\int_{x\in\mathfrak{o}_v^\times}\int_{y\in\mathfrak{o}_v}\psi_{1,v}\left(\frac{\eta x y^2}{4\boldsymbol{\delta}_v}\right)\rho\left(\mathbf{u}^\sharp\left(-\frac{x}{4\boldsymbol{\delta}_v}\right)\right)h(z)\,dy\,dx \\ &= q_v^{e_v}\sum_{\substack{\xi \in \mathfrak{o}_F \\ \mathfrak{f}_{\eta\xi,v} \geqslant 0}} c(\xi)\mathbf{e}(\xi z)\int_{x\in\mathfrak{o}_v^\times}\int_{y\in\mathfrak{o}_v}\psi_{1,v}\left(\frac{x(\eta y^2-\xi)}{4\boldsymbol{\delta}_v}\right)dy\,dx \\ &= q_v^{e_v}\sum_{\substack{\xi \in \mathfrak{o}_F \\ \mathfrak{f}_{\eta\xi,v} \geqslant 0}} c(\xi)T_{2e_v}(0,-\eta\xi)_v\mathbf{e}(\xi z) \\ &= (1-q_v^{-1})h(z).\end{aligned}$$

On the other hand, observe that

$$\int_{x\in\mathfrak{p}_v}\rho\left(\mathbf{u}^\sharp\left(\frac{x}{4\boldsymbol{\delta}_v}\right)\right)h(z)\,dx = q_v^{-1}\int_{x\in\mathfrak{o}_v}\rho\left(\mathbf{u}^\sharp\left(\frac{x}{4\boldsymbol{\delta}_v}\right)\right)h(z)\,dx,$$

since $\mathfrak{f}_{\eta\xi,v} \geqslant 0$ and $\mathrm{ord}_v(\xi) \geqslant 2e_v - 1$ imply $\mathrm{ord}_v(\xi) \geqslant 2e_v$. It follows that $\rho(T_v^K)h(z) = q_v^{-1}\rho(T_v^K)h(z) + (1-q_v^{-1})h(z)$, and so we have $\rho(T^K)h(z) = h(z)$. Therefore we have

$$\rho(E_v^K)h(z) = (q_v+1)^{-1}h(z) + (1+q_v^{-1})^{-1}h(z) = h(z).$$

Thus we have proved the proposition. □

Proposition 13.3 implies $M^+_{\kappa+(1/2)}(\Gamma) \subset M_{\kappa+(1/2)}(\Gamma)^{E^K}$. We shall show the converse. For a finite idele $a = (a_v)_v \in \mathbb{A}_{\mathbf{f}}^\times$, we define an element $\mathbf{w}(a)_{\mathrm{f}} \in \widetilde{\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})} \subset \widetilde{\mathrm{SL}_2(\mathbb{A})}$ by $\mathbf{w}(a)_{\mathrm{f}} = (g_v)_v$, where $g_v = \mathbf{w}_{a_v}$ for $v < \infty$ and $g_v = 1$ for $v|\infty$. Let $\boldsymbol{\delta}$ be the finite idele whose vth component is $\boldsymbol{\delta}_v$.

PROPOSITION 13.4. *We have $M_{\kappa+(1/2)}(\Gamma)^{E^K} \subset M^+_{\kappa+(1/2)}(\Gamma)$.*

Proof. Put $K = \prod_{v<\infty}\Gamma[\mathfrak{d}_v^{-1}, \mathfrak{d}_v]$. Then K is a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})$. Put $\mathcal{S}_0 = \mathcal{S}(2^{-1}\hat{\mathfrak{o}}/\hat{\mathfrak{o}})$, where $\hat{\mathfrak{o}} = \prod_{v<\infty}\mathfrak{o}_v$. Note that $2^{-1}\hat{\mathfrak{o}}/\hat{\mathfrak{o}}$ can be canonically identified with $2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$.

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Then as in Proposition 3.3, $\widetilde{K}$ acts on $\mathcal{S}_0$ by the finite Weil representation Ω_ψ. By Proposition 3.3, Ω_ψ is an irreducible representation of $\widetilde{K}$. For $\lambda \in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$, we denote by ϕ_λ the characteristic function of $\lambda + 2^{-1}\mathfrak{o}_F$. Then $\{\phi_\lambda \mid \lambda \in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F\}$ is a complete orthonormal basis of $\mathcal{S}_0$. The basis $\{\phi_\lambda\}$ satisfies the following conditions:

$$\begin{aligned}\Omega_\psi(e^K)\phi_0 &= \phi_0,\\ \Omega_\psi(\mathbf{u}^\sharp(x))\phi_\lambda &= \psi_{\mathrm{f}}(\lambda^2 x)\phi_\lambda \quad (x \in \hat{\mathfrak{d}}^{-1}),\\ \Omega_\psi(\mathbf{w}(\boldsymbol{\delta})_{\mathrm{f}})\phi_0 &= 2^{-n/2}\prod_{v<\infty}\overline{\alpha_{\psi_v}(\boldsymbol{\delta}_v)}\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}\phi_\lambda.\end{aligned}$$

Here, $\hat{\mathfrak{d}}^{-1} = \mathfrak{d}_F^{-1}\hat{\mathfrak{o}}$ and $\psi_{\mathrm{f}} = \prod_{v<\infty}\psi_v$. Suppose that $h(z) \in M_{\kappa+(1/2)}(\Gamma)^{E^K}$. Put

$$h^{[0]} = 2^{\sum_{i=1}^n \kappa_i}\prod_{v<\infty}\overline{\alpha_{\psi_v}(\boldsymbol{\delta}_v)}\langle 2, \boldsymbol{\delta}_v\rangle_v \cdot \rho(\mathbf{w}(2\boldsymbol{\delta})_{\mathrm{f}}^{-1})h(z).$$

Then we have $h^{[0]} \in M_{\kappa+(1/2)}(\Gamma)$ and $\rho(e^K)h^{[0]} = h^{[0]}$. Let $\mathcal{V}$ be the vector space over $\mathbb{C}$ generated by $\{\rho(k)h^{[0]} \mid k \in \widetilde{K}\}$. Since e^K is a matrix coefficient of the irreducible representation Ω_ψ, $\mathcal{V}$ is isomorphic to $\mathcal{S}_0$ as a representation of $\widetilde{K}$. It follows that there exists an isomorphism $i : \mathcal{S}_0 \simeq \mathcal{V}$. such that $i(\phi_0) = h^{[0]}$. Put $h^{[\lambda]} = i(\phi_\lambda)$ for each $\lambda \in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$. Then we have

$$\rho(e^K)h^{[0]} = h^{[0]}, \tag{13.1}$$

$$\rho(\mathbf{u}^\sharp(x))h^{[\lambda]} = \psi_{\mathrm{f}}(\lambda^2 x)h^{[\lambda]} \quad (x \in \hat{\mathfrak{d}}^{-1}), \tag{13.2}$$

$$\rho(\mathbf{w}(\boldsymbol{\delta})_{\mathrm{f}})h^{[0]} = 2^{-n/2}\prod_{v<\infty}\overline{\alpha_{\psi_v}(\boldsymbol{\delta}_v)}\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F} h^{[\lambda]}. \tag{13.3}$$

Note that

$$\begin{aligned}\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F} h^{[\lambda]}(z) &= 2^{n/2}\prod_{v<\infty}\alpha_{\psi_v}(\boldsymbol{\delta}_v)\cdot\rho(\mathbf{w}(\boldsymbol{\delta})_{\mathrm{f}})h^{[0]}(z)\\ &= 2^{\sum_{i=1}^n \kappa_i + (n/2)}\prod_{v<\infty}\langle 2, \boldsymbol{\delta}_v\rangle_v\cdot\rho(\mathbf{w}(\boldsymbol{\delta})_{\mathrm{f}}\mathbf{w}(2\boldsymbol{\delta})_{\mathrm{f}}^{-1})h(z)\\ &= 2^{\sum_{i=1}^n \kappa_i + (n/2)}\cdot\rho(\mathbf{m}(2_{\mathrm{f}}))h(z)\\ &= h(z/4).\end{aligned}$$

Put

$$h^{[\lambda]}(z) = \sum_{\xi\in F} c_\lambda(\xi)\mathbf{e}(\xi z/4).$$

For $x \in \hat{\mathfrak{d}}^{-1}$, we have

$$\begin{aligned}\rho(\mathbf{u}^\sharp(x))h^{[\lambda]}(z) &= \psi_{\mathrm{f}}(\lambda^2 x)\sum_{\xi\in F} c_\lambda(\xi)\mathbf{e}(\xi z/4)\\ &= \psi_{1,\mathrm{f}}(\eta\lambda^2 x)\sum_{\xi\in F} c_\lambda(\xi)\mathbf{e}(\xi z/4)\end{aligned}$$

by (13.2). On the other hand, we have

$$\rho(\mathbf{u}^\sharp(x))h^{[\lambda]}(z) = \sum_{\xi\in F} c_\lambda(\xi)\psi_{1,\mathrm{f}}(\xi x/4)\mathbf{e}(\xi z/4).$$

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It follows that $c_\lambda(\xi)=0$ unless $\xi\equiv 4\eta\lambda^2 \bmod 4\mathfrak{o}_F$. Thus, we have

$$h(z/4)=\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F} h^{[\lambda]}(z)=\sum_{\eta\xi\equiv\square(4)} c(\xi)\mathbf{e}(\xi z/4),$$

where

$$c(\xi)=\begin{cases} c_\lambda(\xi) & \text{if } \xi\equiv 4\eta\lambda^2 \bmod 4\mathfrak{o}_F \text{ for some } \lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F,\\ 0 & \text{if } \eta\xi\not\equiv\square \bmod 4.\end{cases}$$

Note that for each $\xi\in F$, there exists at most one $\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$ such that $\xi\equiv 4\eta\lambda^2 \bmod 4\mathfrak{o}_F$. Hence the proposition follows. □

By Propositions 13.3 and 13.4, we have the following theorem.

Theorem 13.5. *We have*

$$M^+_{\kappa+(1/2)}(\Gamma)=M_{\kappa+(1/2)}(\Gamma)^{E^K},\quad S^+_{\kappa+(1/2)}(\Gamma)=S_{\kappa+(1/2)}(\Gamma)^{E^K}.$$

Proposition 13.6. *Suppose that $h(z)=\sum_{\xi\in\mathfrak{o}_F} c(\xi)\mathbf{e}(\xi z)\in M^+_{\kappa+(1/2)}(\Gamma)$. Then $h^{[\lambda]}(z)=\sum_{\eta\xi\equiv\lambda^2 \bmod \mathfrak{o}_F} c(4\xi)\mathbf{e}(\xi z)$ satisfies the conditions (13.1)–(13.3) for any $\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$. Conversely, if $\{h^{[\lambda]}(z)\mid \lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F\}$ satisfies the conditions (13.1)–(13.3), then*

$$h(z)=\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F} h^{[\lambda]}(4z)\in M^+_{\kappa+(1/2)}(\Gamma).$$

Proof. The first part is already proved in the proof of Proposition 13.4. Suppose that $\{h^{[\lambda]}(z)\mid \lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F\}$ satisfies the conditions (13.1)–(13.3). Then we have

$$\begin{aligned} h(z)&=2^{-\sum_{i=1}^n\kappa_i-(n/2)}\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}\rho(\mathbf{m}(2_{\mathrm{f}}))h^{[\lambda]}(z)\\ &=2^{-\sum_{i=1}^n\kappa_i}\prod_{v<\infty}\alpha_{\psi_v}(\boldsymbol{\delta}_v)\cdot\rho(\mathbf{m}(2_{\mathrm{f}})\mathbf{w}(\boldsymbol{\delta})_{\mathrm{f}})h^{[0]}(z)\\ &=2^{-\sum_{i=1}^n\kappa_i}\prod_{v<\infty}\alpha_{\psi_v}(\boldsymbol{\delta}_v)\langle 2,\boldsymbol{\delta}_v\rangle_v\cdot\rho(\mathbf{w}(2\boldsymbol{\delta})_{\mathrm{f}})h^{[0]}(z).\end{aligned}$$

By (13.1), we have $\rho(E^K)h=h$. □

14. Relation to Jacobi forms

In this section, we assume that $\sum_{i=1}^n\kappa_i\equiv n \bmod 2$. We also assume that $\eta=-1$, and so $\psi(x)=\psi(\eta x)=\psi_1(-x)=\overline{\psi_1(x)}$. Since the relation between Kohnen plus space and the space of Jacobi forms is already established for $F=\mathbb{Q}$ (see [EZ85]), we assume $F\neq\mathbb{Q}$. Let G^J be the subgroup of $\mathrm{Sp}_2(F)$ consisting of all elements of $\mathrm{Sp}_2(F)$ whose first column is equal to ${}^t(1\,0\,0\,0)$. We define the Jacobi group $\Gamma^J(1)$ by

$$\Gamma^J=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix}\in G^J\,\middle|\, A,D\in M_2(\mathfrak{o}_F),\,B\in M_2(\mathfrak{d}_F^{-1}),\,C\in M_2(\mathfrak{d}_F)\right\}.$$

For a holomorphic function $\phi(z,w)$ on $(z,w)\in\mathfrak{h}^n\times\mathbb{C}^n$, we set

$$\tilde{\phi}\left(\begin{pmatrix}\omega&w\\w&z\end{pmatrix}\right)=\mathbf{e}(\omega)\phi(z,w)\quad(\omega\in\mathfrak{h}^n).$$

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Then $\phi(z,w)$ is a Jacobi form on weight $\kappa+1$ and index 1 if $\tilde{\phi}|_{\kappa+1}\gamma=\tilde{\phi}$ for any $\gamma\in\Gamma^J$. Note that we do not need the cusp conditions by the Köcher principle. A Jacobi cusp form is a Jacobi form with some cuspidality condition at the cusps. The space of Jacobi forms and the space of Jacobi cusp forms of weight $\kappa+1$ and index 1 is denoted by $J_{\kappa+1,1}(\Gamma^J)$ and $J^{\mathrm{cusp}}_{\kappa+1,1}(\Gamma^J)$, respectively. A Jacobi form can be considered as an automorphic form on $G^J(\mathbb{A})$.

For $\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$, we define

$$\theta_\lambda(z,w)=\sum_{\xi\in\mathfrak{o}_F}\mathbf{e}((\xi+\lambda)^2z)\mathbf{e}(2(\xi+\lambda)w).$$

The element $\lambda\in 2^{-1}\mathfrak{O}_F/\mathfrak{O}_F$ is called the theta characteristic of $\theta_\lambda(z,w)$. Since $\theta_\lambda(z,w)$ is the theta function associated to $\phi_\lambda\in\mathcal{S}_0$, we have

$$\begin{aligned}\rho(e^K_{\psi_1})\theta_0&=\theta_0,\\ \rho(\mathbf{u}^\sharp(x))\theta_\lambda&=\psi_{1,\mathrm{f}}(\lambda^2x)\theta_\lambda\quad(x\in\hat{\mathfrak{d}}^{-1}),\\ \rho(\mathbf{w}(\boldsymbol{\delta})_{\mathrm{f}})\theta_0&=2^{-n/2}\prod_{v<\infty}\overline{\alpha_{\psi_{1,v}}(\boldsymbol{\delta}_v)}\cdot\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}\theta_\lambda.\end{aligned}$$

Here, $e^K_{\psi_1}$ is the idempotent defined with respect to ψ_1. Note that $e^K_{\psi_1}=\overline{e^K}$, where e^K is defined with respect to ψ.

It is well known that a Jacobi form $\phi(z,w)$ has a theta expansion

$$\phi(z,w)=\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}\theta_\lambda(z,w)k_\lambda(z),$$

where $k_\lambda(z)$ is a modular form of weight $\kappa+(1/2)$ with respect to some congruence subgroup. Note that $\phi(z,w)$ is a Jacobi cusp form if and only if $k_\lambda(z)$ is a cusp form for any $\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$.

Conversely, suppose $k_\lambda(z)$ is a Hilbert modular form of weight $\kappa+(1/2)$ for some congruence subgroup for each $\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$. Then $\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}\theta_\lambda(z,w)k_\lambda(z)\in J_{\kappa+1,1}(\Gamma^J)$ if and only if

$$\begin{aligned}\rho(e^K)k_0&=k_0,\\ \rho(\mathbf{u}^\sharp(x))k_\lambda&=\psi_{\mathrm{f}}(\lambda^2x)k_\lambda\quad(x\in\hat{\mathfrak{d}}^{-1}),\\ \rho(\mathbf{w}(\boldsymbol{\delta})_{\mathrm{f}})k_0&=2^{-n/2}\prod_{v<\infty}\overline{\alpha_{\psi_v}(\boldsymbol{\delta}_v)}\cdot\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}k_\lambda.\end{aligned}$$

For $\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$ and $h(z)=\sum_{-\xi\equiv\square\ (4)}c(\xi)\mathbf{e}(\xi z)\in M^+_{\kappa+(1/2)}(\Gamma)$, we set

$$h^{[\lambda]}(z)=\sum_{\xi\equiv-\lambda^2\bmod\mathfrak{o}_F}c(4\xi)\mathbf{e}(\xi z).$$

By Proposition 13.6, we obtain the following theorem.

Theorem 14.1. *The map $M^+_{\kappa+(1/2)}(\Gamma)\to J_{\kappa+1,1}(\Gamma^J)$ given by*

$$h(z)\mapsto\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}\theta_\lambda(z,w)h^{[\lambda]}(z)$$

is an isomorphism. The inverse map is given by

$$\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}\theta_\lambda(z,w)k_\lambda(z)\mapsto\sum_{\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F}k_\lambda(4z).$$

This isomorphism induces an isomorphism $S^+_{\kappa+(1/2)}(\Gamma)\simeq J^{\mathrm{cusp}}_{\kappa+1,1}(\Gamma^J)$.

Note that the isomorphism $M^+_{\kappa+(1/2)}(\Gamma) \simeq J_{\kappa+1,1}(\Gamma^J)$ is also given by

$$\sum_{-\xi \equiv \square(4)} c(\xi)\mathbf{e}(\xi z) \mapsto \sum_{m,r\in\mathfrak{o}} c(4r-m^2)\mathbf{e}(mz+rw).$$

15. Examples

In this section, we consider only parallel weights, so the weight $(\kappa,\ldots,\kappa)$ is just denoted by κ.

Lemma 15.1. *Assume that the class number of F is one. Then $h(z)=\sum_{\xi\in\mathfrak{o}_F} c(\xi)\mathbf{e}(\xi z)\in M^+_{\kappa+(1/2)}(\Gamma)$ is a cusp form if and only if $c(0)=0$. In particular, $\dim M^+_{\kappa+(1/2)}(\Gamma)\leqslant \dim S^+_{\kappa+(1/2)}(\Gamma)+1$.*

Proof. Let $h^{[\lambda]}(z)$ $(\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F)$ be as in § 13. Let $\mathcal{V}$ be the vector space generated by $h^{[\lambda]}(z)$ over $\mathbb{C}$. By Proposition 13.6, $\rho([(\gamma)_{\mathrm{f}},\pm 1])h^{[0]}\in\mathcal{V}$ for any $\gamma\in\Gamma[\mathfrak{d}_F^{-1},\mathfrak{d}_F]$. Here, $(\gamma)_{\mathrm{f}}\in \mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})$ is a finite part of $\gamma\in\widetilde{\mathrm{SL}_2(\mathbb{A})}$. Since the class number of F is one, we have $\mathrm{SL}_2(F)=B(F)\cdot\Gamma[\mathfrak{d}_F^{-1},\mathfrak{d}_F]$, where B is the Borel subgroup of SL_2 consisting of upper triangular matrices. It follows that the constant term of $\rho([(\gamma)_{\mathrm{f}},\pm 1])h(z)$ is zero for any $\gamma\in\mathrm{SL}_2(F)$. □

For $\lambda\in 2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$, we define the theta function $\theta_\lambda(z)$ by

$$\theta_\lambda(z)=\sum_{\xi\in\mathfrak{o}_F}\mathbf{e}((\xi+\lambda)^2 z).$$

If κ is even, we define the Eisenstein series $E_\kappa(z)$ of weight κ by

$$E_\kappa(z)=2^{-n}\zeta_F(1-\kappa)+\sum_{\substack{\xi\in\mathfrak{o}_F\\ \xi\gg 0}}\sum_{\mathfrak{a}|(\xi)}\mathrm{N}(\mathfrak{a})^{\kappa-1}\mathbf{e}(\xi z)\in M_\kappa(\mathrm{SL}_2(\mathfrak{o}_F)).$$

Here, $\mathrm{N}(\mathfrak{a})$ is the norm of $\mathfrak{a}$.

Example 1. Assume that $F=\mathbb{Q}(\sqrt{13})$. The narrow class number of F is 1, and so the Hurwitz–Maass extension $\Gamma_m(\mathfrak{o}_F\oplus\mathfrak{o}_F)$ is equal to $\mathrm{SL}_2(\mathfrak{o}_F)/\{\pm 1\}$. It is known that $\dim_{\mathbb{C}} S_4(\mathrm{SL}_2(\mathfrak{o}_F))=1$. In fact, it is spanned by the Doi–Naganuma lift from $S_4(\Gamma_0(13),\hat{\chi}_{13})$, where $\hat{\chi}_{13}$ is the non-trivial real Dirichlet character of conductor 13. Theorem 9.4 implies $\dim_{\mathbb{C}} S^+_{5/2}(\Gamma)=1$. Put $g(z)=\theta_0(z)E_2(4z)\in M^+_{5/2}(\Gamma)$. Decomposing $g(z)$ into eigenvectors of some Hecke operator, we obtain two elements

$$h_1(z)=q-4q^{(7+\sqrt{13})/2}-4q^{(7-\sqrt{13})/2}+3q^4+\cdots,$$
$$h_2(z)=29+10q+250q^{(7+\sqrt{13})/2}+250q^{(7-\sqrt{13})/2}+610q^4+\cdots.$$

Here, $q^\xi=\mathbf{e}(\xi z)$. By Lemma 15.1, we have $h_1(z)\in S^+_{5/2}(\Gamma)$ and so $S^+_{5/2}(\Gamma)=\mathbb{C}\cdot h_1(z)$. We give a table of Fourier coefficients of $h_1(z)$ and $h_2(z)$ below.

2ξ	0	2	$7\pm\sqrt{13}$	8	10	$15\pm\sqrt{13}$	16	18
h_1	0	1	-4	3	13	-26	39	16
h_2	29	10	250	610	768	2640	3000	6250

2ξ	$19\pm 3\sqrt{13}$	$20\pm 4\sqrt{13}$	$22\pm 4\sqrt{13}$	$23\pm\sqrt{13}$
h_1	-26	13	26	-65
h_2	2640	2160	4320	10080

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Example 2. Assume that $F=\mathbb{Q}(\sqrt{5})$. The narrow class number of F is 1, and so the Hurwitz–Maass extension $\Gamma_m(\mathfrak{o}_F\oplus\mathfrak{o}_F)$ is equal to $\mathrm{SL}_2(\mathfrak{o}_F)/\{\pm 1\}$. It is known that $\dim_{\mathbb{C}} S_6(\mathrm{SL}_2(\mathfrak{o}_F))=1$. In fact, it is spanned by the Doi–Naganuma lift from $S_6(\Gamma_0(5),\hat{\chi}_5)$, where $\hat{\chi}_5$ is the non-trivial real Dirichlet character of conductor 5. Theorem 9.4 implies $\dim_{\mathbb{C}} S^+_{7/2}(\Gamma)=1$.

We define operators $\mathbf{U},\mathbf{W}: M_{\kappa+(1/2)}(\Gamma)\to M_{\kappa+(1/2)}(\Gamma)$ by

$$\mathbf{U}: h(z)\mapsto \sum_{\nu\in\mathfrak{o}_F/4\mathfrak{o}_F} h\left(\frac{z+\nu}{4}\right),$$

$$\mathbf{W}: h(z)\mapsto (-2\sqrt{5}z_1z_2)^{-\kappa-(1/2)} h\left(-\frac{1}{20z}\right).$$

By the result of § 13, $h(z)\in M^+_{\kappa+(1/2)}(\Gamma)$ if and only if $\mathbf{WU}h(z)=4h(z)$. Let $\lambda_1,\lambda_2,\lambda_3$ be the distinct non-zero elements of $2^{-1}\mathfrak{o}_F/\mathfrak{o}_F$. By explicit calculation, we have

$$\begin{aligned}\mathbf{U}(\theta_0^3)&=\theta_0^3+6\theta_{\lambda_1}\theta_{\lambda_2}\theta_{\lambda_3},\\ \mathbf{W}(\theta_0^3)&=\theta_0^3,\\ \mathbf{U}(\theta_{\lambda_1}\theta_{\lambda_2}\theta_{\lambda_3})&=4\theta_{\lambda_1}\theta_{\lambda_2}\theta_{\lambda_3},\\ \mathbf{W}(\theta_{\lambda_1}\theta_{\lambda_2}\theta_{\lambda_3})&=\theta_0^3-\theta_{\lambda_1}\theta_{\lambda_2}\theta_{\lambda_3}.\end{aligned}$$

It follows that $f_0(z)=4\theta_0^3-3\theta_{\lambda_1}\theta_{\lambda_2}\theta_{\lambda_3}\in M^+_{3/2}(\Gamma)$. As in Example 1, we decompose $f_0(z)E_2(4z)$ by some Hecke operator, and obtain two elements

$$\begin{aligned}h_1(z)&=q^{(5+\sqrt{5})/2}+q^{(5-\sqrt{5})/2}-3q^3+2q^4+\cdots,\\ h_2(z)&=67+504q^{(5+\sqrt{5})/2}+504q^{(5-\sqrt{5})/2}+2240q^3+9450q^4+\cdots.\end{aligned}$$

By Lemma 15.1, we have $h_1(z)\in S^+_{7/2}(\Gamma)$ and so $S^+_{7/2}(\Gamma)=\mathbb{C}\cdot h_1(z)$. We give a table of Fourier coefficients of $h_1(z)$ and $h_2(z)$ below.

2ξ	0	$5\pm\sqrt{5}$	6	8	$13\pm\sqrt{5}$	14	16	$17\pm 3\sqrt{5}$
h_1	0	1	−3	2	−1	31	−44	−11
h_2	67	504	2240	9450	100800	155520	298620	264600

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