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A note on theorem of Sah

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In this note we show that if H is any subgroup of the finite group G and if D is a normal subgroup of H such that H/D is soluble and the order of H/D is relatively prime to the index of H in G then the existence of a normal subgroup N of G such that $NH = G$ and $N \cap H$ is contained in D is equivalent to the condition that every irreducible character of H/D can be extended to one of G. This is a generalization of a result due to Sah for the case when D is the identity subgroup.

In his paper on complements in finite groups [2], Sah proved the following theorem:

"If H is a soluble Hall subgroup of the group G then the following two conditions are equivalent:

1. *H has a normal complement in G ;*

2. *every irreducible character of H can be extended to one of G".*

A similar theorem for normal supplements is proved in this note. The proof follows closely that of Sah.

By the word "group", we shall mean "finite group". Suppose that H is a subgroup of the group G and that D is a normal subgroup of H (that is, $D \trianglelefteq H \leq G$) . If there is a normal subgroup N of G which satisfies

$$NH = G \quad \text{and} \quad N \cap H \leq D$$

N is called a *normal supplement over* D to H in G . If $D = \{1\}$,

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the group consisting of the identity alone, N is called a *normal complement* of H in G .

LEMMA. *Suppose* $D \trianglelefteq H \leq G$. *If each irreducible character of* H/D *can be extended to one of* G *then for every* K *,* $D \trianglelefteq K \trianglelefteq H$ *, there is an* $N \trianglelefteq G$ *such that* $N \cap H = K$ *and every irreducible character of* HN/N *can be extended to one of* G/N .

Proof. For each irreducible character of H/D , trivial on K/D , select an extension to G . The kernel of this extension must contain K . Let N be the intersection of all such kernels. Then $N \trianglelefteq G$ and $N \cap H$ contains K . From the orthogonality relations [1; Theorem 16.6.9, p. 274], the irreducible characters of H/K separate the H/K-classes of H/K . Therefore there is at least one character non-trivial on any H/K-class and it follows that $N \cap H = K$.

Each irreducible character of HN/N gives rise to an irreducible character of H/K since HN/N is isomorphic to H/K . By the choice of N , the characters of H/K can be extended to those of G trivial on N , that is to those of G/N .

Let $|G : H|$ denote the index of H in G , and (a, b) denote the greatest common factor of a and b . If $\left(|G : H|, |H : \{1\}|\right) = 1$, is called a *Hall* subgroup of G .

THEOREM. *If* $D \trianglelefteq H \leq G$ *,* $(|G : H|, |H : D|) = 1$ *and* H/D *is soluble then the following conditions are equivalent:*

1. H *has a normal supplement over* D *in* G *;*

2. *every irreducible character of* H/D *can be extended to one of* G .

Proof. Obviously 1. implies 2., so assume every irreducible character of H/D can be extended to one of G .

From the lemma, there is $N \trianglelefteq G$ such that $N \cap H = D$ and every irreducible character of HN/N may be extended to one of G/N . H/D is isomorphic to HN/N so HN/N is soluble. Since $|G/N : HN/N|$ equals $|G : HN|$ which divides $|G : H|$ and $(|G : H|, |H : D|) = 1$, HN/N is a Hall subgroup of G/N . Therefore we may apply Sah's theorem to find $L/N \trianglelefteq G/N$ such that

$$(L/N)(HN/N) = G/N \text{ and } L/N \cap HN/N = \{1\} .$$

Hence $LH = G$ and $L \cap HN \leq N$.

But $HN \geq H$ so $L \cap H \leq L \cap HN \leq N$. Therefore $(L \cap H) \cap H \leq N \cap H = D$ which implies $L \cap H \leq D$ and the theorem is proved.

References

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[3] Michio Suzuki, "On the existence of a normal Hall subgroup", *J. Math. Soc. Japan* 15 (1963), 387-391.

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