LETTER TO THE EDITOR

Dear Editor,

On conditional passage-time structure of birth-death processes

Sumita (1984) has given an analytic proof of the interesting fact that, for any continuous-time Markov birth and death process $\{X(t), t \geqq 0\}$ on $[0, 1, \cdots)$ with strictly positive birth and death rates, any $1 \leqq m < n < \infty$, the conditional first-passage times ${}^{(m-1)}T_{m,n+1}$ and ${}^{(n+1)}T_{n,m-1}$ have the same distribution. (Here ${}^{(k)}T_{ij}$ denotes the time it takes to reach j from i given that j is reached before k.) He also states that this identity has 'no clear probabilistic interpretation'. He appears to have overlooked the following elementary argument.

Let ${}^{(k)}N_{ij}$ denote the analogous conditional first-passage times for the corresponding jump-chain $\{X_r, r \geqq 0\}$, which is a Markov chain on $[0, 1, \cdots)$ with $p_{i,i+1} = p_i$, $p_{i,i-1} = q_i$, $0 < p_i < 1$, $p_i + q_i = 1$, $i = 1, 2, \cdots$, $p_{01} = 1$. (Thus $(X_r, r \geqq 0)$ is a discrete birth and death process, or a generalized random walk.) Take fixed $1 \leqq m < n < \infty$ and set $N = {}^{(m-1)}N_{m,n+1}$, $\tilde{N} = {}^{(n+1)}N_{n,m-1}$ and let $\bar{p}_{ij}^{(r)}$ denote the r-step transition probabilities with taboo set $\{m-1, n+1\}$. Then clearly

$$P\{N = r\} = \alpha_r \Big/ \sum_1^\infty \alpha_r, \quad P\{\tilde{N} = r\} = \tilde{\alpha}_r \Big/ \sum_1^\infty \tilde{\alpha}_r,$$

where

$$\alpha_r = p_n \cdot \bar{p}_{m,n}^{(r-1)}, \quad \tilde{\alpha}_r = q_m \cdot \bar{p}_{n,m}^{(r-1)}.$$

By associating with each path of length j starting at m, finishing at n and not visiting $m-1$ or $n+1$ the reversed path starting at n and finishing at m, it is easily seen that for all j such that $\bar{p}_{m,n}^{(j)} > 0$,

$$\bar{p}_{m,n}^{(j)} = \{p_m p_{m+1} \cdots p_{n-1} / q_n q_{n-1} \cdots q_{m+1}\} \cdot \bar{p}_{n,m}^{(j)}.$$

Hence $\alpha_r = c\tilde{\alpha}_r$ for all r, where $c = \Pi_m^n (p_i / q_i)$, and N and $\tilde{N}$ have the same distribution.

To extend this to $\{X(t), t \geqq 0\}$, notice that the above argument assigns to each path of $\{X_r, r \geqq 0\}$ from m to $n+1$ which has positive probability, conditional on not hitting $m-1$, a path from n to $m-1$ which has the same probability, conditional on not hitting $n+1$. Furthermore the correspondence is such that both paths visit each state l the same number of times, for each $m \leqq l \leqq n$. Thus the times spent by $\{X(t), t \geqq 0\}$ in traversing the two paths will have the same distribution, and it follows easily that ${}^{(m-1)}T_{m,n+1}$ and ${}^{(n+1)}T_{n,m-1}$ have the same distribution.

In the case that the birth rates and death rates are constant, Sumita also established the stronger result that for each $0 \leqq k \leqq n - m$, ${}^{(m-1)}T_{m+k,n+1}$ and ${}^{(n+1)}T_{n-k,m-1}$ have the same distribution. This is also susceptible to a similar elementary proof, based on a consideration of the reflected process $\tilde{X}_n = m + n - X_n$.

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Yours sincerely,
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Reference

SUMITA, U. (1984) On conditional passage-time structure of birth–death processes. *J. Appl. Prob.* **21**, 10–21.