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The Chen–Ruan Cohomology of Weighted Projective Spaces

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Abstract. In this paper we study the Chen–Ruan cohomology ring of weighted projective spaces. Given a weighted projective space $\mathbf{P}^n_{q_0,\dots,q_n}$, we determine all of its twisted sectors and the corresponding degree shifting numbers. The main result of this paper is that the obstruction bundle over any 3-multi-sector is a direct sum of line bundles which we use to compute the orbifold cup product. Finally we compute the Chen–Ruan cohomology ring of weighted projective space $\mathbf{P}^5_{1,2,2,3,3,3}$.

1 Introduction

The notion of Chen–Ruan orbifold cohomology has appeared in physics as a result of studying the string theory on global quotient orbifold, (see [6, 7]). In addition to the usual cohomology of the global quotient, this space included the cohomology of so-called twisted sectors. Zaslow [18] gave many examples of global quotients and computed their orbifold cohomology spaces. But the real mathematical definition of orbifold cohomology was given by Chen and Ruan [5] for arbitrary orbifolds. The most interesting feature of this new cohomology theory, besides the generalization of non global quotients, is the existence of a ring structure which was previously missing. This ring structure is obtained from Chen–Ruan's orbifold quantum cohomology construction by restricting to the class called ghost maps, the same as the ordinary cup product may be obtained by quantum cup product. Since the Chen–Ruan cohomology appeared, the problem of how to calculate the orbifold cohomology ring has been considered by several authors.[1] Chen and Ruan gave several simple examples. Park and Poddar [16] considered the Chen–Ruan cohomology ring of the mirror quintic. In this paper we calculate the Chen–Ruan cohomology ring of weighted projective spaces.

To achieve this goal, we take the weighted projective space as a simplicial toric variety with local isotropy groups the finite cyclic groups. Using the properties of toric varieties induced from the fans, we calculate the Chen–Ruan cohomology group of any weighted projective space. To compute the Chen–Ruan cohomology ring of the weighted projective space, we first prove that the obstruction bundle is the direct sum of some line bundles; then we introduce the localization techniques which should work for toric varieties to compute the 3-point function which is the key in the orbifold cup product [5]. In particular, we give a concrete example.

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[1] Recently Borisov, Chen and Smith used the algebraic method to solve the orbifold Chow ring of all simplicial toric varieties, see [3].

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On the other hand, a very interesting aspect of calculating the Chen–Ruan cohomology ring of weighted projective spaces lies in a conjecture of Ruan. In string theory, physicists suggest that the orbifold string theory of an orbifold should be equivalent to the ordinary string theory of its crepant resolution. For the orbifold cohomology, Ruan's cohomology hyperkahler resolution conjecture (see [17]) states that the Chen–Ruan cohomology ring of an orbifold should be isomorphic to the ordinary cohomology ring of its hyperkahler resolution. I hope that my calculation of the Chen–Ruan cohomology ring of the weighted projective space may contribute to this interesting problem.

The paper is organized as follows. Section 2 is a review of some facts concerning Chen–Ruan cohomology. In Section 3 we introduce the basic concept of the weighted projective space. In Section 4 we discuss the Chen–Ruan cohomology group of any weighted projective space. And in the Section 5 we compute the ring structure of the Chen–Ruan cohomology of the weighted projective space.

2 Preliminaries on Chen–Ruan Cohomology

2.1 Orbifold and Orbifold Vector Bundle

Definition 2.1.1 An orbifold structure on a Hausdorff, separable topological space X is given by an open cover $\mathcal{U}$ of X satisfying the following conditions.

(i) Each element U in $\mathcal{U}$ is uniformized, say by (V, G, π). Namely, V is a smooth manifold and G is a finite group acting smoothly on V such that $U = V/G$ with π as the quotient map. Let $\mathrm{Ker}(G)$ be the subgroup of G acting trivially on V.

(ii) For $U' \subset U$, there is a collection of injections $(V', G', \pi') \to (V, G, \pi)$. Namely, the inclusion $i\colon U' \subset U$ can be lifted to maps $\widetilde{i}\colon V' \to V$ and an injective homomorphism $i_*\colon G' \to G$ such that i_* is an isomorphism from $\mathrm{Ker}(G')$ to $\mathrm{Ker}(G)$ and $\widetilde{i}$ is i_*-equivariant.

(iii) For any point $x \in U_1 \cap U_2$, $U_1, U_2 \in \mathcal{U}$, there is a $U_3 \in \mathcal{U}$ such that $x \in U_3 \subset U_1 \cap U_2$.

For any point $x \in X$, suppose that (V, G, π) is a uniformizing neighborhood and $\overline{x} \in \pi^{-1}(x)$. Let G_x be the stabilizer of G at $\overline{x}$. Up to conjugation, it is independent of the choice of $\overline{x}$ and is called the *local group* of x. Then there exists a sufficiently small neighborhood V_x of $\overline{x}$ such that (V_x, G_x, π_x) uniformizes a small neighborhood of x, where π_x is the restriction $\pi|V_x$, and (V_x, G_x, π_x) is called a *local chart* at x. The orbifold structure is called *reduced* if the action of G_x is effective for every x.

Let $pr\colon E \to X$ be a rank k complex *orbifold bundle* over an orbifold X [5]. Then a uniformizing system for $E \mid U = pr^{-1}(U)$ over a uniformized subset U of X consists of the following data:

- A uniformizing system (V, G, π) of U.
- A uniformizing system $(V \times \mathbf{C}^{\mathbf{k}}, G, \widetilde{\pi})$ for $E \mid U$. The action of G on $V \times \mathbf{C}^k$ is an extension of the action of G on V given by $g \cdot (x, v) = (g \cdot x, \rho(x, g)v)$ where $\rho : V \times G \to \mathrm{Aut}(\mathbf{C}^k)$ is a smooth map satisfying:

$$\rho(g \cdot x, h) \circ \rho(x, g) = \rho(x, hg), g, h \in G, x \in V.$$

- The natural projection map $\widetilde{pr}\colon V \times \mathbf{C}^k \to V$ satisfies $\pi \circ \widetilde{pr} = pr \circ \widetilde{\pi}$.

By an orbifold connection $\triangle$ on E, we mean an equivariant connection that satisfies $\triangle = g^{-1} \triangle g$ for every uniformizing system of E. Such a connection can always be obtained by averaging an equivariant partition of unity.

2.2 Twisted Sectors and Chen–Ruan Cohomology

The most physical idea is twisted sectors. Let X be an orbifold. Consider the set of pairs:

$$\widetilde{X}_k = \{(p, (\mathbf{g})_{G_p}) \mid p \in X, \mathbf{g} = (g_1, \dots, g_k), g_i \in G_p\},$$

where $(\mathbf{g})_{G_p}$ is the conjugacy class of k-tuple $\mathbf{g} = (g_1, \dots, g_k)$ in G_p. We use G^k to denote the set of k-tuples. If there is no confusion, we will omit the subscript G_p to simplify the notation. Suppose that X has an orbifold structure $\mathcal{U}$ with uniformizing systems $(\widetilde{U}, G_U, \pi_U)$. From Chen and Ruan [5], see also [12], we have $\widetilde{X}_k$ is naturally an orbifold with the generalized orbifold structure at $(p, (\mathbf{g})_{G_p})$ given by $(V_p^{\mathbf{g}}, C(\mathbf{g}), \pi\colon V_p^{\mathbf{g}} \to V_p^{\mathbf{g}}/C(\mathbf{g}))$, where $V_p^{\mathbf{g}} = V_p^{g_1} \cap \dots \cap V_p^{g_k}$ and $C(\mathbf{g}) = C(g_1) \cap \dots \cap C(g_k)$. Here $\mathbf{g} = (g_1, \dots, g_k)$ and V_p^g stands for the fixed point set of g in V_p. When X is almost complex, $\widetilde{X}_k$ inherits an almost complex structure from X, and when X is closed, $\widetilde{X}_k$ is finite disjoint union of closed orbifolds.

Now we describe the connected components of $\widetilde{X}_k$. Recall that every point p has a local chart (V_p, G_p, π_p) which gives a local uniformized neighborhood $U_p = \pi_p(V_p)$. If $q \in U_p$, up to conjugation there is a unique injective homomorphism $i_*\colon G_q \to G_p$. For $\mathbf{g} \in (G_q)^k$, the conjugation class $i_*(\mathbf{g})_q$ is well defined. We define an equivalence relation $i_*(\mathbf{g})_q \cong (\mathbf{g})_q$. Let T_k denote the set of equivalence classes. To abuse the notation, we use $(\mathbf{g})$ to denote the equivalence class which $(\mathbf{g})_q$ belongs to. We will usually denote an element of T_1 by (g). It is clear that $\widetilde{X}_k$ can be decomposed as a disjoint union of connected components

$$\widetilde{X}_k = \bigsqcup_{(\mathbf{g}) \in T_k} X_{(\mathbf{g})},$$

where $X_{(\mathbf{g})} = \{(p, (\mathbf{g}')_p) \mid \mathbf{g}' \in (G_p)^k, (\mathbf{g}')_p \in (\mathbf{g})\}$. Note that for $\mathbf{g} = (1, \dots, 1)$, we have $X_{(\mathbf{g})} = X$. A component $X_{(\mathbf{g})}$ is called a k-multisector, if $\mathbf{g}$ is not the identity. A component of $X_{(g)}$ is simply called a *twisted sector*. If X has an almost complex, complex or kahler structure, then $X_{(\mathbf{g})}$ has the analogous structure induced from X. We define

$$T_3^0 = \{(\mathbf{g}) = (g_1, g_2, g_3) \in T_3 \mid g_1 g_2 g_3 = 1\}.$$

Note that there is a one-to-one correspondence between T_2 and T_3^0 given by $(g_1, g_2) \mapsto (g_1, g_2, (g_1 g_2)^{-1})$.

Now we define the Chen–Ruan cohomology. Assume that X is an n-dimensional compact almost complex orbifold with almost structure J. Then for a point p with nontrivial group G_p, J gives rise to an effective representation $\rho_p\colon G_p \to \mathrm{GL}(n, \mathbf{C})$. For any $g \in G_p$, we write $\rho_p(g)$, up to conjugation, as a diagonal matrix

$$\operatorname{diag}\bigl(e^{2\pi i \frac{m_{1,g}}{m_g}}, \dots, e^{2\pi i \frac{m_{n,g}}{m_g}} \operatorname{diag} r\bigr).$$

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where m_g is the order of g in G_p, and $0 \leq m_{i,g} < m_g$. Define a function $\iota\colon \widetilde{X}_1 \to \mathbf{Q}$ by

$$\iota(p,(g)_p) = \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}.$$

We see that the function $\iota\colon \widetilde{X}_1 \to \mathbf{Q}$ is locally constant and $\iota = 0$ if $g = 1$. Denote its value on $X_{(g)}$ by $\iota_{(g)}$. We call $\iota_{(g)}$ the degree shifting number of $X_{(g)}$. It has the following properties:

- $\iota_{(g)}$ is an integer if and only if $\rho_p(g) \in SL(n,\mathbf{C})$;
- $\iota_{(g)} + \iota_{(g^{-1})} = \operatorname{rank}(\rho_p(g) - \mathrm{Id}) = n - \dim_{\mathbf{C}} X_{(g)}$.

Definition 2.2.1 ([5]) Let X be a closed complex orbifold, we define the orbifold cohomology group of X by

$$H^d_{\mathrm{orb}}(X,\mathbf{Q}) := \bigoplus_{(g)\in T_1} H^{d-2\iota_{(g)}}(X_{(g)},\mathbf{Q}).$$

2.3 The Obstruction Bundle

Choose $(\mathbf{g}) = (g_1,g_2,g_3) \in T_3^0$. Let $(p,(\mathbf{g})_p)$ be a generic point in $X_{(\mathbf{g})}$. Let $K(\mathbf{g})$ be the subgroup of G_p generated by g_1 and g_2. Consider an orbifold Riemann sphere with three orbifold points $(S^2,(p_1,p_2,p_3),(k_1,k_2,k_3))$. When there is no confusion, we will simply denote it by S^2. The orbifold fundamental group is

$$\pi_1^{\mathrm{orb}}(S^2) = \{\lambda_1,\lambda_2,\lambda_3 \mid \lambda_i^{k_i} = 1, \lambda_1\lambda_2\lambda_3 = 1\},$$

where λ_i is represented by a loop around the marked p_i. There is a surjective homomorphism $\rho\colon \pi_1^{\mathrm{orb}}(S^2) \to K(\mathbf{g})$ specified by mapping $\lambda_i \mapsto g_i$. $\mathrm{Ker}(\rho)$ is a finite-index subgroup of $\pi_1^{orb}(S^2)$. Let $\widetilde{C}$ be the orbifold universal cover of S^2. Let $C = \widetilde{C}/\mathrm{Ker}(\rho)$. Then C is smooth and compact, and $C/K(\mathbf{g}) = S^2$. The genus of C can be computed using the Riemann–Hurwitz formula for Euler characteristics of a branched covering, and turns out to be

$$g(C) = \frac{1}{2}\Big(2 + |K(\mathbf{g})| - \sum_{i=1}^{3} \frac{|K(\mathbf{g})|}{k_i}\Big). \tag{2.1}$$

Since $K(\mathbf{g})$ acts holomorphically on C, $K(\mathbf{g})$ acts on $H^1(C,\mathcal{O}_C)$. The "obstruction bundle" $E_{(\mathbf{g})}$ over $X_{(\mathbf{g})}$ is constructed as follows. On the local chart $(V_p^{\mathbf{g}}, C(\mathbf{g}), \pi)$ of $X_{(\mathbf{g})}$, $E_{(\mathbf{g})}$ is given by $(TV_p \otimes H^1(C,\mathcal{O}_C))^{K(\mathbf{g})} \times V_p^{\mathbf{g}} \to V_p^{\mathbf{g}}$, where $(TV_p \otimes H^1(C,\mathcal{O}_C))^{K(\mathbf{g})}$ is the $K(\mathbf{g})$-invariant subspace. We define an action of $C(\mathbf{g})$ on $TV_p \otimes H^1(C,\mathcal{O}_C)$, which is the usual one on TV_p and trivial on $H^1(C,\mathcal{O}_C)$. The actions of $C(\mathbf{g})$ and $K(\mathbf{g})$ commute, and $(TV_p \otimes H^1(C,\mathcal{O}_C))^{K(\mathbf{g})}$ is invariant under $C(\mathbf{g})$. Thus we have obtained an action of $C(\mathbf{g})$ on $(TV_p \otimes H^1(C,\mathcal{O}_C))^{K(\mathbf{g})} \times V_p^{\mathbf{g}} \to V_p^{\mathbf{g}}$, extending the usual one on $V_p^{\mathbf{g}}$. These trivializations fit together to define the bundle $E_{(\mathbf{g})}$ over $X_{(\mathbf{g})}$. If we set $e\colon X_{(\mathbf{g})} \to X$ to be the map given by $(p,(\mathbf{g})_p) \mapsto p$, one may think of $E_{(\mathbf{g})}$ as $(e^*TX \otimes H^1(C,\mathcal{O}_C))^{K(\mathbf{g})}$. The rank of $E_{(\mathbf{g})}$ is given by the formula (see [5])

$$\operatorname{rank}_{\mathbf{C}}(E_{(\mathbf{g})}) = \dim_{\mathbf{C}}(X_{(\mathbf{g})}) - \dim_{\mathbf{C}}(X) + \Sigma_{j=1}^{3}\iota_{(g_j)}. \tag{2.2}$$

2.4 Orbifold Cup Product

First, there is a natural map $I\colon X_{(g)} \to X_{(g^{-1})}$ defined by $(p,(g)_p) \mapsto (p,(g^{-1})_p)$.

Definition 2.4.1 Let $n = \dim_{\mathbf{C}}(X)$. For any integer $0 \le n \le 2n$, the pairing

$$\langle\ ,\ \rangle_{\mathrm{orb}}\colon H^d_{\mathrm{orb}}(X) \times H^{2n-d}_{\mathrm{orb}}(X) \to \mathbf{Q}$$

is defined by taking the direct sum of

$$\langle\ ,\ \rangle^{(g)}_{\mathrm{orb}}\colon H^{d-2\iota_{(g)}}(X_{(g)};\mathbf{Q}) \times \mathbf{H^{2n-d-2\iota_{(g^{-1})}}(X_{(g^{-1})};Q)} \to \mathbf{Q},$$

where

$$\langle \alpha,\beta\rangle^{(g)}_{\mathrm{orb}} = \int^{\mathrm{orb}}_{X_{(g)}} \alpha \wedge I^*(\beta)$$

for $\alpha \in H^{d-2\iota_{(g)}}(X_{(g)};\mathbf{Q})$, and $\beta \in H^{2n-d-2\iota_{(g^{-1})}}(X_{(g^{-1})};\mathbf{Q})$.

Choose an orbifold connection A on $E_{(\mathbf{g})}$. Let $e_A(E_{(\mathbf{g})})$ be the Euler form computed from the connection A by Chen–Weil theory. Let $\eta_j \in H^{d_j}(X_{(g_j)};\mathbf{Q})$, for $j = 1,2,3$. Define maps $e_j\colon X_{(\mathbf{g})} \to X_{(g_j)}$ by $(p,(\mathbf{g})_p) \mapsto (p,(g_j)_p)$.

Definition 2.4.2 Define the 3-point function to be

$$\langle \eta_1,\eta_2,\eta_3\rangle_{\mathrm{orb}} := \int^{\mathrm{orb}}_{X_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge e_A(E_{(\mathbf{g})}). \tag{2.3}$$

Note that the above integral does not depend on the choice of A. As in Definition 2.4.1, we extend the 3-point function to $H^*_{\mathrm{orb}}(X)$ by linearity. We define the orbifold cup product by the relation

$$\langle \eta_1 \cup_{\mathrm{orb}} \eta_2,\eta_3\rangle_{\mathrm{orb}} := \langle \eta_1,\eta_2,\eta_3\rangle_{\mathrm{orb}} \tag{2.4}$$

Again we extend $\cup_{\mathrm{orb}}$ to $H^*_{\mathrm{orb}}(X)$ via linearity. Note that if $(\mathbf{g}) = (1,1,1)$, then $\eta_1 \cup_{\mathrm{orb}} \eta_2$ is just the ordinary cup product $\eta_1 \cup \eta_2$ in $H^*(X)$.

3 The Weighted Projective Spaces

3.1 The Orbifold Structure of the Weighted Projective Space

Definition 3.1.1 ([10]) Let $Q = (q_0,\dots,q_n)$ be an $(n+1)$-tuple of positive integers. The weighted projective space of type Q, $\mathbf{P}(Q) = \mathbf{P}^n_{q_0,\dots,q_n}$ is defined by

$$\mathbf{P}(Q) = \left\{ z \in (\mathbf{C}^{n+1})^* \mid z \sim \lambda(\mathbf{q})\cdot z, \lambda \in \mathbf{C}^* \right\}$$

where $\lambda(\mathbf{q}) = \mathrm{diag}(\lambda^{q_0},\dots,\lambda^{q_n})$.

Remark 3.1.2 The above $\mathbf{C}^*$-action is free if and only if $q_i = 1$ for $i = 0, \dots, n$. If $\gcd(q_0, \dots, q_n) = d \neq 1$, then $\mathbf{P}^n_{q_0,\dots,q_n}$ is homeomorphic to $\mathbf{P}^n_{q_0/d,\dots,q_n/d}$(by identification of λ^d with λ).

Weighted projective spaces are, in general, orbifolds where the singularities have cyclic structure groups acting diagonally. Moreover, if all the q_i's are mutually prime, all these orbifold singularities are isolated. In fact, as is usually done for complex projective spaces, we can consider the sets

$$U_i = \{[z]_Q \in \mathbf{P}(Q) : z_i \neq 0\} \subset \mathbf{P}(Q)$$

and the bijective maps ϕ_i from U_i to $\mathbf{C}^n/\mu_{q_i}(Q_i)$ given by

$$\phi_i([z]_Q) = \Big(\frac{z_0}{(z_i)^{q_0/q_i}}, \dots, \frac{\hat{z}_i}{z_i}, \dots, \frac{z_n}{(z_i)^{q_n/q_i}} \Big)_{q_i},$$

where $(z_i)^{1/q_i}$ is a q_i-root of z_i and $(\,\cdot\,)_{q_i}$ is a μ_{q_i}-orbit in $\mathbf{C}^n/\mu_{q_i}(Q_i)$ with μ_{q_i} acting on $\mathbf{C}^n$ by $\xi \cdot z = \xi(Q_i)z$, $\xi \in \mu_{q_i}$. Here $Q_i = (q_0, \dots, \hat{q}_i, \dots, q_n)$ and $\xi(Q_i) = \operatorname{diag}(\xi^{q_0}, \dots, \xi^{q_n})$.

3.2 Toric Structure of the Weighted Projective Spaces

Given $Q = (q_0, \dots, q_n) \in \mathbf{Z}^{n+1}$, define a grading of $\mathbf{C}[X_0, \dots, X_n]$ by $\deg X_i = q_i$. We denote this ring by $S(Q)$. Then $\mathbf{P}(Q) := \mathbf{P}^n_{q_0,\dots,q_n} = \operatorname{proj}(S(Q))$ is the weighted projective space of type Q, and $\mathbf{P}(Q)$ is covered by the affine open sets $D_+(X_i) := \operatorname{spec} S(Q)_{X_i}$, $(i = 0, \dots, n)$. The monic monomials of $S(Q)_{X_i}$ are of type $X_i^{-l}\Pi_{j\neq i}X_j^{\lambda_j}$, where $lq_i = \Sigma_{j\neq i}\lambda_j q_j$ and l, λ_j are non-negative integers. So each such monomial is uniquely determined by the n-tuple $(\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n)$ of its non-negative exponents. The exponents occurring are just the points lying in the intersection of the cone $e := \operatorname{pos}\{e_1, \dots, e_n\}$ and the lattice $N_{Q,q_i} \subseteq \mathbf{Z}^n$ that is defined as follows.

Consider $Q_i = (q_0, \dots, \hat{q}_i, \dots, q_n)$ as an element of $\operatorname{Hpm}_{\mathbf{Z}}(\mathbf{Z}^n, \mathbf{Z})$ by setting

$$Q_i(a_1, \dots, a_n) := q_0 a_1 + \cdots + q_n a_n,$$

where $\mathbf{Z}^n$ is equipped with its canonical basis. Let $\pi_i \colon \mathbf{Z} \to \mathbf{Z}_{q_i}$ denote the canonical projection. Then $N_{Q,q_i} := \operatorname{Ker}(\pi_i \circ Q_i)$ is a sublattice of $\mathbf{Z}^n$. Denote by M_{Q,q_i} the dual lattice. We have an isomorphism of semigroup rings

$$S(Q)_{(X_i)} \cong \mathbf{C}[e \cap N_{Q,q_i}],$$

revealing $D_+(X_i)$ to be the affine toric variety associated with $\check{e}$ with respect to M_{Q,q_i}.

Proposition 3.2.1 ([4]) *Let $C_i = (c^i_1, \dots, c^i_n)$ be a basis of N_{Q,q_i}, and denote by $r^i_1, \dots, r^i_n$ the row vectors of C_i. Let $\sigma_i := \operatorname{pos}\{r^i_1, \dots, r^i_n\}$. Then there is an isomorphism of semigroups*

$$\check{\sigma}_i \cap \mathbf{Z}^n \simeq e \cap N_{Q,q_i}.$$

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From the above proposition we see that the weighted projective space $\mathbf{P}(Q)$ is a toric variety. From Fulton [9], we construct the fan Σ of $\mathbf{P}(Q)$ as follows. Let the fan Σ be generated by vectors $\{v_0, \ldots, v_n\}$ so that $q_0 v_0 + q_1 v_1 + \cdots + q_n v_n = 0$. Then the toric variety X_Σ is the weighted projective space $\mathbf{P}(Q)$.

Remark 3.2.2

(i) Conrad [4] gives a method to compute the lattice vectors $\{v_0, \ldots, v_n\}$ such that they generate a fan Σ for the weighted projective space $\mathbf{P}(Q)$;

(ii) If $\gcd(q_0, \ldots, q_n) = d \neq 1$, we see that $\mathbf{P}^n_{q_0,\ldots,q_n}$ and $\mathbf{P}^n_{q_0/d,\ldots,q_n/d}$ have the same fans, so they are homeomorphic.

4 The Chen–Ruan Cohomology Groups of Weighted Projective Spaces

4.1 The Ordinary Cohomology Groups of Weighted Projective Spaces

Let $Q = (q_0, \ldots, q_n)$, $\gcd(q_0, \ldots, q_n) = 1$ and $\mathbf{P}(Q)$ be the weighted projective space of type Q. The ordinary cohomology group of $\mathbf{P}(Q)$ has already been studied by several authors, see [1, 11]. Here we only give the results.

Theorem 4.1.1 *Let $Q = (q_0, \ldots, q_n)$ and $\mathbf{P}(Q)$ be the weighted projective space of type Q. Then the cohomology group of $\mathbf{P}(Q)$ with rational coefficient is*

$$H^i(\mathbf{P}(Q), \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{if } i = 2r, 0 \leq r \leq n, \\ 0 & \text{if } i \text{ is odd or } i > 2n. \end{cases}$$

4.2 Orbiford Structure from Toric Varieties

Let $\mathbf{P}(Q)$ be the weighted projective space of type Q. And let $\{v_0, \ldots, v_n\}$ be the one dimensional generators of the fan Σ of $\mathbf{P}(Q)$. We have the following proposition.

Proposition 4.2.1 $\mathbf{P}(Q)$ *has the orbifold structure given by the following atlas.*

$$\{(U_{\sigma_k'}, G_{\sigma_k}, \pi_{\sigma_k}) : \sigma_k = (v_0, \ldots, \hat{v}_k, \ldots, v_n), k = 0, \ldots, n\}$$

In particular, $G_{\sigma_k} = \mathbf{Z}_{q_k}$ is the cyclic group of order q_k.

Proof Because the fan Σ of toric variety $\mathbf{P}(Q)$ is generated by $\{v_0, \ldots, v_n\}$, we have all $n+1$ of the n-dimensional cones $\sigma_0, \ldots, \sigma_n$. From Poddar [15], we see that

$$\{(U_{\sigma_k'}, G_{\sigma_k}, \pi_{\sigma_k}) : \sigma_k = (v_0, \ldots, \hat{v}_k, \ldots, v_n), k = 0, \ldots, n\}$$

forms the orbifold structure of $\mathbf{P}(Q)$. Now we prove that for any σ_k, $G_{\sigma_k} = \mathbf{Z}_{q_k}$. Since we have $G_{\sigma_k} = N/N_{\sigma_k}$, let $\pi_k \colon \mathbf{Z} \to \mathbf{Z}_{q_k}$ be the standard projection. Define the map $Q_k \colon \mathbf{Z}^n \to \mathbf{Z}$ such that $Q_k(a_1, \ldots, a_n) = q_0 a_1 + \cdots + q_n a_n$, $Q_k = (q_0, \ldots, \hat{q}_k, \ldots, q_n)$. Then we let $N_{Q,q_k} := \operatorname{Ker}(\pi_k \circ Q_k)$. From Proposition 3.2.1, $N_{Q,q_k} \cong N_{\check{\sigma}_k}$. Because Σ is simplicial, from the construction [9] of the toric variety we have $N_{\sigma_k} = N_{\check{\sigma}_k}$, so $N_{Q,k} = N_{\sigma_k}$. Thus we have: $G_{\sigma_k} = N/N_{\sigma_k} = \mathbf{Z}^n/N_{Q,q_k} \cong \mathbf{Z}_{q_k}$. ∎

Remark 4.2.2 From the proposition above, $U_{\sigma_k'} = \mathbf{C}^n$, so $U_{\sigma_k} = U_{\sigma_k'}/G_{\sigma_k} = \mathbf{C}^n/\mathbf{Z}_{q_k}$, and the action of $\mathbf{Z}_{q_k}$ is the diagonal action. Its matrix representation can be computed from Poddar's method [15] which we will use in the following sections.

4.3 Twisted Sectors of Weighted Projective Spaces and Degree Shifting Numbers

The toric variety $\mathbf{P}(Q)$ has orbit decomposition $\mathbf{P}(Q) = \bigsqcup_{\tau\in\Sigma} O_\tau$. From Proposition 4.2.1, if $\sigma \in \Sigma$ is an n-dimensional cone, suppose $\sigma = (v_0, \dots, v_{n-1})$, then $G_\sigma = \mathbf{Z}_{q_n}$. We also know that $G_\sigma = \{k_a = \Sigma_{i=0}^{n-1} a_i v_i : k_a \in N, a_i \in [0,1)\}$. If $\tau = (v_0, \dots, v_{i-1})$ is a face of σ, then $G_\tau = \{g_a \in G_\sigma : a_i = 0 \text{ if } j+1 \le i \le n\}$, *i.e.*, $G_\tau = \{k_a = \Sigma_{j=1}^{i-1} a_j v_j : k_a \in N, a_j \in [0,1)\}$. Furthermore, G_τ can be taken as the local group of the points in O_τ.

Proposition 4.3.1 *Let $\tau = (v_0, \dots, v_{i-1})$ be a cone of Σ. Then we have $G_\tau \cong \mathbf{Z}_d$, where $d = \gcd(q_i, \dots, q_n)$. In particular, $(q_i, \dots, q_n)$ is the maximal subset of $(q_0, \dots, q_n)$ whose* gcd *is d if and only if the dimension of the fixed point set of $\mathbf{Z}_d$ is $n-i$.*

Proof Since G_τ is a subgroup of G_{σ_k} for each $i \le k \le n$, $|G_\tau|$ divides d. It remains to show that G_τ has an element of order d. Let $u = \sum_{j=0}^{i-1} (q_i/d \bmod 1) v_j$; then $u \in G_\tau$, since $\sum_{j=0}^{n} q_j v_j = 0$. Now suppose u has order m. Then d divides $m(q_j \bmod d)$ for each $0 \le j \le i-1$. Hence d divides $m \cdot \gcd\{q_0, \dots, q_{i-1}\}$. Since $\gcd\{q_0, \dots, q_n\} = 1$, d and $\gcd\{q_0, \dots, q_{i-1}\}$ have no common factor. Thus d divides m. But $m \le d$. So $m = d$ and $u \in G_\tau$ is an element of order d. [2]

Suppose $(q_i, \dots, q_n)$ is a maximal subset of $(q_0, \dots, q_n)$ that satisfies the condition $\gcd(q_i, \dots, q_n) = d$, while the dimension of the fixed point set of $\mathbf{Z}_d$ is not $n-i$. Let $g_a = a_0 v_0 + \dots + a_{i-1} v_{i-1}$ be a generator of $G_\tau = \mathbf{Z}_d$. Because the dimension of O_τ is $n-i$, we must have $a_s = 0$ for some $s \le i-1$. If we let $\rho = (v_0, \dots, \hat{v}_s, \dots, v_{i-1})$, then $G_\rho = \mathbf{Z}_d$, and from the first part of the proposition, $d = \gcd(q_s, q_i, \dots, q_n)$, contradicting the maximality of $(q_i, \dots, q_n)$.

Conversely, suppose the dimension of the fixed point set of $\mathbf{Z}_d$ is $n-i$, *i.e.*, the dimension of the orbit O_τ. If we have a subset strictly bigger than $(q_i, \dots, q_n)$ with gcd d, say $(q_s, q_i, \dots, q_n)$ without loss of generality, then let

$$\delta = (v_0, \dots, v_{s-1}, v_{s+1}, \dots, v_{i-1}).$$

From the first part of the theorem we have

$$G_\delta = \mathbf{Z}_d = \{k_a = a_0 v_0 + \dots + a_{s-1} v_{s-1} + a_{s+1} v_{s+1} + \dots + a_{i-1} v_{i-1} : a_i \in [0,1)\}.$$

We see that the dimension of the fixed point set of $\mathbf{Z}_d$ exceeds $n-i+1$, a contradiction. ∎

Now we discuss the twisted sectors of the weighted projective space $\mathbf{P}(Q)$. From the theorem of Poddar[15] for the twisted sectors of general toric varieties, we have the following.

[2] This proof was suggested by a referee.

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Theorem 4.3.2 ([15]) *A twisted sector of a weighted projective space is isomorphic to a subvariety $\overline{O}_\tau$ of $X_\Sigma = \mathbf{P}(Q)$ for some $\tau \in \Sigma$. There is a one-to-one correspondence between the set of twisted sectors of the type $\overline{O}_\tau$ and the set of integral vectors in the interior of τ which are linear combinations of the 1-dimensional generators of τ with coefficients in $(0,1)$.*

Proposition 4.3.3 *Given a weighted projective space $\mathbf{P}(Q)$ of type $Q = (q_0, \ldots, q_n)$. Let $\Sigma = \{v_0, \ldots, v_n\}$ be the fan of $\mathbf{P}(Q)$. If $\tau = (v_0, \ldots, v_{i-1})$ is a cone in Σ, then $\overline{O}_\tau$ is the weighted projective space $\mathbf{P}(Q_\tau)$, where $Q_\tau = (q_i, \ldots, q_n)$.*

Proof From Fulton[9], $\overline{O}_\tau$ is a toric variety and the fan $\mathrm{Star}(\tau)$ can be described as follows. Let N_τ be the sublattice of N generated by $\tau \in N$, and let $N(\tau) = N/N_\tau$, $M(\tau) = \tau^\perp \cap M$ be the quotient lattice and the dual. The star of a cone τ can be defined abstractly as the set of cones σ in Σ that contain τ as a face. Such cones σ are determined by their images in $N(\tau)$, *i.e.*, by

$$\overline{\sigma} = (\sigma + (N_\tau)_{\mathbf{R}})/(N_\tau)_{\mathbf{R}} \subset N_{\mathbf{R}}/(N_\tau)_{\mathbf{R}} = N(\tau)_{\mathbf{R}}.$$

These cones $\{\overline{\sigma} : \tau < \sigma\}$ form a fan in $N(\tau)$, and we denote this fan by $\mathrm{Star}(\tau)$. The corresponding toric variety is $n-k$-dimensional. For the toric variety $\mathbf{P}(Q)$, let $\overline{v}_i, \ldots, \overline{v}_n$ be the images of $v_0, \ldots, v_n$ in the quotient lattice $N(\tau)$. Since $q_0 v_0 + \cdots + q_n v_n = 0$ in N, we have $q_i \overline{v}_i + \cdots + q_n \overline{v}_n = 0$ in the quotient lattice $N(\tau)$. So from the definition of weighted projective space, we conclude that the toric variety corresponding to the fan $\mathrm{Star}(\tau)$ in the quotient lattice $N(\tau)$ is the weighted projective space $\mathbf{P}(Q_\tau)$, where $Q_\tau = (q_i, \ldots, q_n)$. ∎

Remark 4.3.4 The gcd of $(q_i, \ldots, q_n)$ need not necessarily be 1. In general, if $d = \gcd(q_i, \ldots, q_n)$, then $\mathbf{P}(Q_\tau)$ is a nonreduced orbifold, and it has a corresponding reduced orbifold $\mathbf{P}(Q'_\tau)$, where $Q'_\tau = (q_i/d, \ldots, q_n/d)$.

Theorem 4.3.5 *If $\tau = (v_0, \ldots, v_{i-1})$ is a cone in Σ, then $\overline{O}_\tau = \mathbf{P}(Q_\tau)$, where $Q_\tau = (q_i, \ldots, q_n)$. Let $\gcd(q_i, \ldots, q_n) = d$. Then $\overline{O}_\tau = \mathbf{P}(Q_\tau)$ is a twisted sector if and only if $(q_i, \ldots, q_n)$ is the maximal subset of $(q_0, \ldots, q_n)$ that satisfies the condition $\gcd(q_i, \ldots, q_n) = d$.*

Proof If $\overline{O}_\tau$ is a twisted sector, then suppose $\overline{O}_\tau = X_{(g_a)}$, where $g_a = \sum_{j=0}^{i-1} a_j v_j$ is a generator of $G_\tau = \mathbf{Z}_d$. If we have $q_k \notin (q_i, \ldots, q_n)$ with $\gcd(q_k, q_i, \ldots, q_n) = d$, let $\delta = (v_0, \ldots, \hat{v}_k, \ldots, v_{i-1})$. Then by Proposition 4.3.1, $G_\delta = \mathbf{Z}_d$. Therefore $G_\tau = G_\delta$ and $g_a \in G_\delta$. However from Poddar [15], $G_\delta = \{\Sigma_{j=0}^{i-1} a_j v_j : j \neq k, a_j \in [0,1)\}$. Hence the coefficient a_k in g_a must be zero. Then by Theorem 4.3.2, O_τ cannot be a twisted sector, which is a contradiction.

Suppose $(q_i, \ldots, q_n)$ is the maximal subset of $(q_0, \ldots, q_n)$ whose gcd is d. Let $\tau = (v_0, \ldots, v_{i-1})$. Then by Proposition 4.3.1, the fixed point set of G_τ has dimension $n - i$. Therefore, all coefficients a_j in a generator $g_a = \sum_{j=0}^{i-1} a_j v_j$ of G_τ must be positive. Hence by Theorem 4.3.2, $\overline{O}_\tau$ is a twisted sector. ∎

Remark 4.3.6 From the above analysis, if $\tau = (v_0, \dots, v_{i-1})$ is a cone in Σ, we describe the orbifold structure of twisted sector $X_{(g_a)} = \overline{O}_\tau$ as follows. In the points of O_τ, the local group is $\mathbf{Z}_d$, $d = \gcd(q_i, \dots, q_n)$, and $C(g_a) = \mathbf{Z}_d$ acts trivially. If $\delta > \tau$ is a cone, then $O_\delta \subset \overline{O}_\tau$ and the local group at a point $y \in O_\delta$ is the cyclic group $\mathbf{Z}_t$, where $\delta = (v_0, \dots, v_{i-1}, v_{t_1}, \dots, v_{t_s})$ and $t = \gcd(q_i, \dots, \widehat{q}_{t_1}, \dots, \widehat{q}_{t_s}, \dots, q_n)$.

The degree shifting numbers can be computed easily. For instance, let $X_{(g_a)} = \overline{O}_\tau$ be a twisted sector, and $\tau = (v_0, \dots, v_{i-1})$. We can write g_a as

$$g_a = \sum_{j=0}^{i-1} a_j v_j, a_j \in (0,1).$$

The degree shifting number $\iota_{(g_a)}$ of $X_{(g_a)}$ is

$$\iota_{(g_a)} = \sum_{j=0}^{i-1} a_j.$$

4.4 The Chen–Ruan Cohomology Groups of Weighted Projective Spaces

Based on Theorem 4.3.5, we can write the Chen–Ruan cohomology group of $\mathbf{P}(Q)$ as follows.

Theorem 4.4.1 *We write the orbifold cohomology group of* $\mathbf{P}(Q)$ *as*

$$H^p_{orb}(\mathbf{P}(Q); \mathbf{Q}) \cong \bigoplus_{\sigma \in \Sigma, l \in \mathbf{Q}} H^{p-2l}(\overline{O}_\sigma) \otimes \bigoplus_{t \in \sigma_l} \mathbf{Q}t,$$

where $\sigma_l = \{\sum_{v_i \subset \sigma} a_i v_i \in N : a_i \in (0,1), \sum_{v_i \subset \sigma} a_i = l\}$, *and* p *is a rational number in* $[0, n]$.

4.5 Example

For $Q = (2,3,4)$, $\mathbf{P}(Q) = \mathbf{P}^2_{2,3,4}$, and we have $q_0 = 2, q_1 = 3, q_2 = 4$. From Conrads [4], we compute the generators of the fan Σ: $v_0 = (-3,-2), v_1 = (2,0), v_2 = (0,1)$. For $\sigma_2 = (v_0, v_1) = ((-3,-2),(2,0))$, we have $G_{\sigma_2} = N/N_{\sigma_2} = \mathbf{Z}_4$. We use 3-tuples $g_a = (a_0, a_1, a_2)$ to represent the action of the element $g_a \in G_{\sigma_2}$. For example, $g_{\sigma_2} = (\frac{1}{2}, \frac{1}{4}, 0)$ represents the matrix

$$\begin{pmatrix} e^{2\pi i \cdot \frac{1}{2}} & 0 \\ 0 & e^{2\pi i \cdot \frac{1}{4}} \end{pmatrix}.$$

So $\mathbf{Z}_4$ is generated by g_{σ_2} and the action on $U_{\sigma_2'} = \mathbf{C}^2$ is through the corresponding matrix. For the later examples, we always use the n-tuple to represent the group elements.

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For $\sigma_1 = (v_0, v_2) = ((-3,-2),(0,1))$, $G_{\sigma_1} = \mathbf{Z}_3$. We have $g_{\sigma_1} = (\frac{1}{3}, 0, \frac{2}{3})$, $g_{\sigma_1}^2 = (\frac{2}{3}, 0, \frac{1}{3})$, $g_{\sigma_1}^3 = (0,0,0) = 1$.

For $\sigma_0 = (v_1, v_2) = ((2,0),(0,1))$, $G_{\sigma_0} = \mathbf{Z}_2$, we have the generator $g_{\sigma_0} = (0, \frac{1}{2}, 0)$.

Then we have the twisted sectors: $p_1 = [0,1,0]$, $p_2 = [0,0,1]$, $X_{(g_{\sigma_2})} = X_{(g_{\sigma_2}^3)} = p_2$, $\iota_{(g_{\sigma_2})} = \frac{3}{4}$, $\iota_{(g_{\sigma_2}^3)} = \frac{5}{4}$; $X_{(g_{\sigma_2}^2)}$ and $X_{(g_{\sigma_0})}$ are the same twisted sector $\mathbf{P}(2,0,4)$, and $\iota_{(g_{\sigma_2}^2)} = \iota_{(g_{\sigma_0})} = \frac{1}{2}$; $X_{(g_{\sigma_1})} = X_{(g_{\sigma_1}^2)} = p_1$, $\iota_{(g_{\sigma_1})} = \iota_{(g_{\sigma_1}^2)} = 1$. So

$$\begin{aligned} H^p_{orb}(\mathbf{P}(Q);\mathbf{Q}) = H^p(\mathbf{P}(Q);\mathbf{Q}) \oplus H^{p-2\iota_{(g_{\sigma_2})}}(\{p_2\};\mathbf{Q}) \oplus H^{p-2\iota_{(g_{\sigma_2}^3)}}(\{p_2\};\mathbf{Q}) \\ \oplus H^{p-2\iota_{(g_{\sigma_2}^2)}}(\mathbf{P}(2,0,4);\mathbf{Q}) \oplus 2H^{p-2\iota_{(g_{\sigma_1})}}(\{p_1\};\mathbf{Q}) \end{aligned}$$

We compute the orbifold cohomology group of $\mathbf{P}(Q)$ as

$$H^0_{\mathrm{orb}}(\mathbf{P}(Q);\mathbf{Q}) = \mathbf{Q}, \quad H^1_{\mathrm{orb}}(\mathbf{P}(Q);\mathbf{Q}) = \mathbf{Q},$$
$$H^{3/2}_{\mathrm{orb}}(\mathbf{P}(Q);\mathbf{Q}) = \mathbf{Q}, \quad H^2_{\mathrm{orb}}(\mathbf{P}(Q);\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q}, \quad H^{5/2}_{\mathrm{orb}}(\mathbf{P}(Q);\mathbf{Q}) = \mathbf{Q},$$
$$H^3_{\mathrm{orb}}(\mathbf{P}(Q);\mathbf{Q}) = \mathbf{Q}, \quad H^4_{\mathrm{orb}}(\mathbf{P}(Q);\mathbf{Q}) = \mathbf{Q}.$$

All the other dimensions of the Chen–Ruan cohomology groups are zero.

5 The Chen–Ruan Cohomology Ring of Weighted Projective Spaces

5.1 The Ordinary Cohomology Ring of Weighted Projective Spaces

In this section we recall the ordinary cohomology ring of the weighted projective space. The readers may refer to [1]. Let $Q = (q_0, \dots, q_n)$ and $\mathbf{P}(Q) = \mathbf{P}^n_{q_0,\dots,q_n}$ be the weighted projective space of type Q. Let $\mathbf{P}^n$ be the n-dimensional complex projective space. As in [1], let $\varphi\colon \mathbf{P}^n \to \mathbf{P}(Q)$ be the map taking $[x_0, \dots, x_n]$ to $[x_0^{q_0}, \dots, x_n^{q_n}]$. Take $k \in \{0, \dots, n\}$, and consider $I = \{i_0, \dots, i_k\}$ with $0 \le i_0 < \dots < i_k \le n$. Put $l_I = l_I(q_{i_0}, \dots, q_{i_k}) = q_{i_0} \dots q_{i_k} / \gcd(q_{i_0}, \dots, q_{i_k})$, and let

$$l_k = l_k(q_0, \dots, q_n) = \mathrm{lcm}\{l_I | I \subset \{0, \dots, n\}, |I| = k+1\}.$$

Theorem 5.1.1 ([1]) *For each $k, 0 \le k \le n$, there exists a unique $\beta_k \in H^{2k}(\mathbf{P}(Q);\mathbf{Q})$ such that $\varphi^*(\beta_k) = l_k \beta^k$, and $\{1, \beta, \dots, \beta^n\}$ is a $\mathbf{Q}$-basis of the free abelian group $H^{2k}(\mathbf{P}(Q);\mathbf{Q})$. In other words there are commutative diagrams:*

$$\begin{tikzcd} H^{2k}(\mathbf{P}(Q);\mathbf{Q}) \arrow[r, "\varphi^*"] \arrow[d, equal] & H^{2k}(\mathbf{P}^n;\mathbf{Q}) \arrow[d, equal] \\ \mathbf{Q} \arrow[r, "\cdot l_k"'] & \mathbf{Q} \end{tikzcd}$$

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So we can make precise the multiplicative structure of the cohomology $H^{2k}(\mathbf{P}(Q);\mathbf{Q})$. Since $\varphi^*\colon H^*(\mathbf{P}(Q);\mathbf{Q}) \to H^*(\mathbf{P}^n;\mathbf{Q})$ is a ring homomorphism,

$$\xi_i\xi_j = \begin{cases} e_{ij}\xi_{i+j} & \text{if } i+j \leq n,\\ 0 & \text{if not,}\end{cases}$$

where $e_{ij} = l_il_j/l_{i+j}$, $1 \leq i, j \leq n$.

5.2 Three-Multisectors

In this section for a 3-multisector $X_{\mathbf{g}}$, we suppose that $\mathbf{g} = (g_1, g_2, g_3) \in T_3^0$.

Theorem 5.2.1 ([16]) *If $\tau_1[1] \cup \tau_2[1]$ generates an element of Σ, then for every pair $g_{a_1} \in G(\tau_1) \cap \mathrm{Int}(\tau_1)$, $g_{a_2} \in G(\tau_2) \cap \mathrm{Int}(\tau_2)$, we have a unique 3-multisector $X_{(\mathbf{g})}$, which is analytically isomorphic to $\overline{O}_{\tau_1} \cap \overline{O}_{\tau_2}$. As we vary over τ_1, τ_2, we obtain all the 3-multisectors.*

Since the fan $\Sigma = \{v_0, \dots, v_n\}$ of $\mathbf{P}(Q)$ for $Q = (q_0, \dots, q_n)$ has $n+1$ primitive 1-dimensional generators. If τ_1 and τ_2 are two cones of Σ and the cardinality of $\tau_1[1] \cup \tau_2[1]$ is less than $n+1$, then we have that $\tau_1[1] \cup \tau_2[1]$ forms an element $\tau = \tau_1[1] \cup \tau_2[1]$ of Σ, so $\overline{O}_{\tau_1} \cap \overline{O}_{\tau_2}$ is a 3-multisector of $\mathbf{P}(Q)$. Moreover, we can prove the 3-multisectors of $\mathbf{P}(Q)$ are actually twisted sectors.

Theorem 5.2.2 *Let $X = \mathbf{P}(Q)$. Suppose $X_{(g_1)}$ and $X_{(g_2)}$ are two twisted sectors of X corresponding to the cones τ_1 and τ_2, respectively,* i.e., *$X_{(g_1)} = \overline{O}_{\tau_1}, X_{(g_2)} = \overline{O}_{\tau_2}$. Then $X_{(\mathbf{g})} = X_{(g_1,g_2,(g_1g_2)^{-1})} = \overline{O}_{\tau_1} \cap \overline{O}_{\tau_2}$ is still a twisted sector.*

Proof First, if $\tau_1 \subset \tau_2$, then from Theorem 5.2.1, $\tau_1[1] \cup \tau_2[1]$ generates τ_2, so $X_{(\mathbf{g})} = \overline{O}_{\tau_2} = X_{(g_2)}$.

If $\tau_1[1] \cap \tau_2[1] = \varnothing$, let $\tau = (\tau_1[1] \cup \tau_2[1])$. Then $X_{(\mathbf{g})} = X_{(g_1,g_2,(g_1g_2)^{-1})} = \overline{O}_\tau$. Since $G_\tau = \{\sum_{v_i \subset \tau} a_iv_i \mid a_i \in [0,1)\}$, we can always find an element $g \in G_\tau$ such that $g = \sum_{v_i\subset\tau} a_iv_i$ for all $a_i \neq 0$. If we take $g_1 = \sum_{v_i\subset\tau_1} a_iv_i (a_i \neq 0)$ and $g_2 = \sum_{v_i\subset\tau_2} a_iv_i (a_i \neq 0)$, and let $g = g_1g_2$, then from Theorem 4.3.2, $X_{(\mathbf{g})} = \overline{O}_\tau = X_{(g)}$.

If τ_1 and τ_2 do not satisfy the above two types of conditions, without loss of generality, we suppose

$$\tau_1 = (v_0, \dots, v_s), \tau_2 = (v_0, \dots, v_j, v_{s+1}, \dots, v_t), \; j < s < t < n.$$

Then let $\tau = \tau_1[1] \cup \tau_2[1] = (v_0, \dots, v_j, \dots, v_s, v_{s+1}, \dots, v_t)$. We know from Proposition 4.3.3 that $\overline{O}_\tau = \mathbf{P}(Q_\tau)$, where $Q_\tau = (0, \dots, 0, q_{t+1}, \dots, q_n)$. While $\overline{O}_{\tau_1} = \mathbf{P}(Q_{\tau_1})$, where $Q_{\tau_1} = (0, \dots, 0, q_{s+1}, \dots, q_t, q_{t+1}, \dots, q_n)$, we have $\overline{O}_{\tau_2} = \mathbf{P}(Q_{\tau_2})$, where $Q_{\tau_2} = (0, \dots, 0, q_{j+1}, \dots, q_s, 0, \dots, 0, q_{t+1}, \dots, q_n)$. Let

$$d_1 = \gcd(q_{s+1,\dots,q_n}), \quad d_2 = \gcd(q_{j+1}, \dots, q_s, q_{t+1}, \dots, q_n). \tag{5.1}$$

So from Theorem 4.3.5, $(q_{s+1}, \dots, q_n)$ and $(q_{j+1}, \dots, q_s, q_{t+1}, \dots, q_n)$ are the maximal subsets of $(q_0, \dots, q_n)$ that satisfy the condition (5.1). We conclude that

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$\gcd(q_{t+1},\dots,q_n) \geq \operatorname{lcm}(d_1,d_2)$ and that $(q_{t+1},\dots,q_n)$ must be the maximal subset of $(q_0,\dots,q_n)$ that satisfies this condition. So from Theorem 4.3.5, $\overline{O}_\tau$ is a twisted sector. ∎

Remark 5.2.3 From the above theorem, every 3-multisector of weighted projective space $\mathbf{P}(Q)$ is actually a twisted sector, so we can refer to Remark 4.3.6 to describe the orbifold structure of the 3-multisectors.

5.3 The Chen–Ruan Cohomology Ring of Weighted Projective Spaces

In this section we discuss the key point of computing the ring structure of Chen–Ruan cohomology of weighted projective space $\mathbf{P}(Q)$. The most important part for the orbifold cup product is the obstruction bundle which was constructed in Section 2.3.

Let $X_{(\mathbf{g})}$ be a 3-multisector of $X = \mathbf{P}(Q)$, $\mathbf{g} = (g_1, g_2, g_3) \in T_3^0$. Let $E_{(\mathbf{g})} \to X_{(\mathbf{g})}$ be the obstruction bundle defined in Section 2.3.

Proposition 5.3.1 *Let $\alpha \in H^*_{\mathrm{orb}}(X_{(g_1)};\mathbf{Q})$, $\beta \in H^*_{\mathrm{orb}}(X_{(g_2)};\mathbf{Q})$, if $\sum_{j=1}^{3} \iota_{(g_j)} > n$, then $\alpha \cup_{\mathrm{orb}} \beta = 0$.*

Proof From (2.2), we have $\Sigma_{j=1}^{3}\iota_{(g_j)} - n = \operatorname{rank}_{\mathbf{C}}(E_{(\mathbf{g})}) - \dim_{\mathbf{C}}(X_{(\mathbf{g})})$. If $\sum_{j=1}^{3} \iota_{(g_j)} > n$, then $\operatorname{rank}_{\mathbf{C}}(E_{(\mathbf{g})}) > \dim_{\mathbf{C}}(X_{(\mathbf{g})})$, so the integral (2.3) is zero, $\alpha \cup_{\mathrm{orb}} \beta = 0$. ∎

Now in the next three sections we concretely discuss how to compute the 3-point function defined in (2.3).

5.4 A Simple Case: $\mathbf{q_0}, \dots, \mathbf{q_n}$ Mutually Prime

Let $Q = (q_0, \dots, q_n)$, where the q_i's are mutually prime. Let $\mathbf{P}(Q)$ be the weighted projective space of type Q. Then the orbifold singularities are the $n+1$ isolated points: $p_i = [0, \dots, i, \dots 0]$ $(i = 0, 1, \dots, n)$ with local orbifold groups $\mathbf{Z}_{q_i}$ $(i = 0, 1, \dots, n)$. If we let $c_0, \dots, c_n$ be the generators of $\mathbf{Z}_{q_0}, \dots, \mathbf{Z}_{q_n}$, respectively, then we have $q_0 - 1$ twisted sectors isomorphic to $X_{(c_0)} = p_0, \dots, q_n - 1$ twisted sectors isomorphic to $X_{(c_n)} = p_n$. And we can also see that the 3-sectors are all isolated points.

If we have $\alpha \in H^*(X_{(g_1)};\mathbf{Q})$ and $\beta \in H^*(X_{(g_2)};\mathbf{Q})$, then $X_{(g_1,g_2,(g_1g_2)^{-1})} = \{pt\}$ if and only if g_1, g_2 belong to some $\mathbf{Z}_{q_i} (i = 0, 1, \dots, n)$. Without loss of generality, we assume $g_1, g_2 \in \mathbf{Z}_{q_0}$. Then from the formula [5, (4.1.7)],

$$\alpha \cup_{\mathrm{orb}} \beta = \sum_{\substack{(h_1,h_2)\\ h_i \in (g_i)}} (\alpha \cup_{\mathrm{orb}} \beta)_{(h_1,h_2)}, \tag{5.2}$$

where

$$\langle (\alpha \cup_{\mathrm{orb}} \beta)_{(h_1,h_2)}, \gamma \rangle_{\mathrm{orb}} = \int_{X_{(h_1,h_2)}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e(E_{(\mathbf{g})}), \tag{5.3}$$

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$e_i\colon X_{(\mathbf{g})} \to X_{(g_i)}$ is the map mentioned above, and $E_{(\mathbf{g})}$ is the obstruction bundle over $X_{(\mathbf{g})}$. From formula (2.2), the dimension of the bundle $E_{(\mathbf{g})}$ is

$$\dim(e(E_{(\mathbf{g})})) = 2(\iota_{(g_1)} + \iota_{(g_2)} + \iota_{(g_3)}) - 2n.$$

Because $X_{(\mathbf{g})}$ is a point, the integral (5.3) is nonzero if and only if $\alpha \in H^0(X_{(g_1)};\mathbf{Q})$, $\beta \in H^0(X_{(g_2)};\mathbf{Q})$, $\gamma \in H^0(X_{(g_3)};\mathbf{Q})$, and $\dim(e(E_{(\mathbf{g})})) = 0$. At this moment, $\iota_{(g_1)} + \iota_{(g_2)} + \iota_{(g_3)} = n$. Suppose α and β are the generators of $H^0(X_{(g_1)};\mathbf{Q})$ and $H^0(X_{(g_2)};\mathbf{Q})$ respectively. Let γ be the generator of $H^0(X_{(g_3)};\mathbf{Q})$. Then the integral (5.3) is

$$\langle(\alpha \cup_{\mathrm{orb}} \beta)_{(g_1,g_2)}, \gamma\rangle_{\mathrm{orb}} = \frac{1}{|\mathbf{Z}_{q_0}|}\int_{\{pt\}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma = \frac{1}{q_0}.$$

If we let δ be the generator of $H^0(X_{(g_1g_2)};\mathbf{Q})$, then

$$\langle\delta, \gamma\rangle_{\mathrm{orb}} = \int_{X_{(g_1g_2)}} \delta \wedge I^*\gamma = \frac{1}{q_0}.$$

So we have

$$\alpha \cup_{\mathrm{orb}} \beta = \delta. \tag{5.4}$$

Example 5.4.1 For $Q = (2,3,5)$, $\mathbf{P}(Q) = \mathbf{P}^2_{2,3,5}$, we have $q_0 = 2, q_1 = 3, q_2 = 5$. From Conrads [4], we compute that the fan of $\mathbf{P}(Q)$ is generated by $v_0 = (-3,-4)$, $v_1 = (2,1)$, $v_2 = (0,1)$. For $\sigma_2 = (v_0, v_1) = ((-3,-4),(2,1))$, we have $G_{\sigma_2} = N/N_{\sigma_2} = \mathbf{Z}_5$. We write the generator $g_{\sigma_2} = (\frac{1}{5}, \frac{4}{5}, 0)$ of $\mathbf{Z}_5$ as a 3-tuple as in Example 4.5.

For $\sigma_1 = ((-3,-4),(0,1))$, we have $G_{\sigma_1} = \mathbf{Z}_3$. We have the generator $g_{\sigma_1} = (\frac{1}{3}, 0, \frac{1}{3})$.

For $\sigma_0 = (v_1, v_2) = ((2,1),(0,1))$, we have $G_{\sigma_0} = \mathbf{Z}_2$ and the generator $g_{\sigma_0} = (0, \frac{1}{2}, \frac{1}{2})$.

Then we have the twisted sectors $p_0 = [1,0,0], p_1 = [0,1,0], p_2 = [0,0,1]$, $X_{(g_{\sigma_2})} = X_{(g^2_{\sigma_2})} = X_{(g^3_{\sigma_2})} = X_{(g^4_{\sigma_2})} = p_0$, $X_{(g_{\sigma_1})} = X_{(g^2_{\sigma_1})} = p_1$ and $X_{(g_{\sigma_0})} = p_3$. Then

$$\iota_{(g_1)} + \iota_{(g_1^4)} = 2, \quad \iota_{(g_1^2)} + \iota_{(g_1^3)} = 2,$$

$$\iota_{(g_2)} + \iota_{(g_2)} + \iota_{(g_2)} = 2, \quad \iota_{(g_2)} + \iota_{(g_2^2)} = 2, \quad \iota_{(g_3)} + \iota_{(g_3)} = 2.$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the generators of $H^0(X_{(g_{\sigma_2})};\mathbf{Q}), H^0(X_{(g^2_{\sigma_2})};\mathbf{Q}), H^0(X_{(g^3_{\sigma_2})};\mathbf{Q})$, $H^0(X_{(g^4_{\sigma_2})};\mathbf{Q})$, respectively. Then let β_1, β_2 be the generators of

$$H^0(X_{(g_{\sigma_1})};\mathbf{Q}) \quad \text{and} \quad H^0(X_{(g^2_{\sigma_1})};\mathbf{Q}),$$

and let γ be the generators of $H^0(X_{(g_{\sigma_0})};\mathbf{Q})$. We also let e_0 be the generator of $H^0(\mathbf{P}(Q);\mathbf{Q})$. So from the above discussion in Section 5.4. and the formula (5.4), we have

$$\alpha_1 \cup_{\mathrm{orb}} \alpha_4 = e_0, \quad \alpha_2 \cup_{\mathrm{orb}} \alpha_3 = e_0,$$

$$\beta_1 \cup_{\mathrm{orb}} \beta_1 = \beta_2, \quad \beta_1 \cup_{\mathrm{orb}} \beta_2 = e_0,$$

$$\gamma \cup_{\mathrm{orb}} \gamma = e_0.$$

5.5 The Obstruction Bundle

In this section we determine the obstruction bundle over any 3-multisector. Let $E_{(\mathbf{g}} \to X_{(\mathbf{g})}$ be the obstruction bundle over the 3-multisector $X_{(\mathbf{g})}$ constructed in Section 2.3. For a weighted projective space $\mathbf{P}(Q)$ of type $Q = (q_0, \dots, q_n)$, from Theorem 5.2.2, every 3-multisector $X_{(\mathbf{g})}$ is a twisted sector. Assume $X_{(\mathbf{g}))} = \overline{O}_\tau = \mathbf{P}(Q_\tau)$, $\tau = (v_0, \dots, v_{i-1})$ is a cone in the fan Σ, $Q_\tau = (q_i, \dots, q_n)$ and $\gcd(q_i, \dots, q_n) = d > 1$. From Theorem 5.2.2, we have $g_1, g_2, g_3 \in \mathbf{Z}_d$ and $g_1 g_2 g_3 = 1$. $\mathbf{P}(Q_\tau) = \mathbf{P}^n(0, \dots, 0, q_i, \dots, q_n)$ is a hyperplane of $\mathbf{P}(Q)$.

Let $U_j = \{[z]_Q \in \mathbf{P}(Q) : z_j \neq 0\} \subset \mathbf{P}(Q)$ for $j = 0, \dots, n$. Then since $Q_\tau = (q_i, \dots, q_n)$, we see that $X_{(\mathbf{g})} = \mathbf{P}(Q_\tau)$ can be covered by $X_{(\mathbf{g})} \cap U_i, \dots, X_{(\mathbf{g})} \cap U_n$. From Section 3.1, for $j \geq i$, we have a bijective map ϕ_j from U_j to $\mathbf{C}^n / \mu_{q_j}(Q_{q_j})$ given by

$$\phi_j([z]_Q) = \left(\frac{z_0}{(z_j)^{q_0/q_j}}, \dots, \frac{\hat{z}_j}{z_j}, \dots, \frac{z_n}{(z_j)^{q_n/q_j}} \right)_{q_j}.$$

So we choose the coordinates of $\mathbf{C}^n = V_j$ by

$$\left(\frac{z_0}{(z_j)^{q_0/q_j}}, \dots, 1, \dots, \frac{z_n}{(z_j)^{q_n/q_j}} \right).$$

If we let $x_0 = z_0/(z_j)^{q_0/q_j}, \dots, x_j = 1, \dots, x_n = z_n/(z_j)^{q_n/q_j}$, then let

$$p_j = [0, \dots, 1, \dots, 0]$$

be the point in V_j, $(TV_j)_{p_j}$ has the basis $\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n}$, and g_1 acts on $(TV_j)_{p_j}$ in the natural way. If we let $\{\overline{\omega}_1, \dots, \overline{\omega}_g\}$ be a basis of $H^1(C, \mathcal{O}_C)$, then on $(H^1(C, \mathcal{O}_C) \otimes (TV_j)_{p_j})$ we have a basis

$$\left\{ \frac{\partial}{\partial x_0} \otimes \overline{\omega}_1, \frac{\partial}{\partial x_1} \otimes \overline{\omega}_1, \dots, \frac{\partial}{\partial x_n} \otimes \overline{\omega}_1, \frac{\partial}{\partial x_0} \otimes \overline{\omega}_2, \dots, \frac{\partial}{\partial x_n} \otimes \overline{\omega}_g \right\},$$

and g_1 acts on $(H^1(C, \mathcal{O}_C) \otimes (TV_j)_{p_j})$.

Now assume $k \geq i$, and $U_k \cap X_{(\mathbf{g})}$ is another open subset of $X_{(\mathbf{g})}$. Let $U_k = V_k/\mathbf{Z}_k$, from the above discussion, we can choose the coordinates of V_k as:

$$\left(y_0 = \frac{z_0}{(z_k)^{q_0/q_k}}, \dots, y_k = 1, \dots, y_n = \frac{z_n}{(z_k)^{q_n/q_k}} \right).$$

So we have a basis on $(H^1(C, \mathcal{O}_C) \otimes (TV_k)_{p_k})$:

$$\left\{ \frac{\partial}{\partial y_0} \otimes \overline{\omega}_1, \frac{\partial}{\partial y_1} \otimes \overline{\omega}_1, \dots, \frac{\partial}{\partial y_n} \otimes \overline{\omega}_1, \frac{\partial}{\partial y_0} \otimes \overline{\omega}_2, \dots, \frac{\partial}{\partial y_n} \otimes \overline{\omega}_g \right\}.$$

Since $y_l = \frac{x_l}{(x_k)^{q_l/q_k}}$ on V_k, we have

$$\frac{\partial}{\partial x_0} = \Sigma \frac{\partial y_l}{\partial x_0} \frac{\partial}{\partial y_l} = \frac{\partial}{\partial x_0} \left(\frac{x_0}{(x_k)^{q_0/q_k}} \right) \frac{\partial}{\partial y_0} = \frac{1}{(x_k)^{q_0/q_k}} \frac{\partial}{\partial y_0} = (y_j)^{q_0/q_j} \frac{\partial}{\partial y_0}.$$

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From the result above and the similar computation, we have

$$
\begin{aligned}
\frac{\partial}{\partial x_0} &= (y_j)^{q_0/q_j}\frac{\partial}{\partial y_0};\\
\frac{\partial}{\partial x_1} &= (y_j)^{q_1/q_j}\frac{\partial}{\partial y_1};\\
&\vdots\\
\frac{\partial}{\partial x_{i-1}} &= (y_j)^{q_{i-1}/q_j}\frac{\partial}{\partial y_{i-1}}.
\end{aligned}
\tag{5.5}
$$

From Section 2.3, the obstruction bundle is $E_{(\mathbf{g})} = (e^*TX \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$. Consider the exact sequence $0 \to T(X_{(\mathbf{g})}) \to e^*TX \to N(X_{(\mathbf{g})}) \to 0$. From (5.5) we see that the normal bundle $N(X_{(\mathbf{g})})$ is the sum of i line bundles locally generated by $\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_{i-1}}$ on $U_j \cap X_{(\mathbf{g})}$ and $\frac{\partial}{\partial y_0}, \dots, \frac{\partial}{\partial y_{i-1}}$ on $U_k \cap X_{(\mathbf{g})}$, and for each line bundle the transition function can be described from (5.5). Now let $N(X_{(\mathbf{g})}) = \bigoplus_{l=0}^{i-1} L_l$, where L_l is locally generated by $\frac{\partial}{\partial x_l}$ on $U_j \cap X_{(\mathbf{g})}$ and $\frac{\partial}{\partial y_l}$ on $U_k \cap X_{(\mathbf{g})}$ for $0 \le l \le i-1$. Now from the above exact sequence, we have the exact sequence

$$T(X_{(\mathbf{g})}) \otimes H^1(C, \mathcal{O}_C) \to e^*TX \otimes H^1(C, \mathcal{O}_C) \to N(X_{(\mathbf{g})}) \otimes H^1(C, \mathcal{O}_C) \to 0.$$

We know that $\big(T(X_{(\mathbf{g})}) \otimes H^1(C, \mathcal{O}_C)\big)^{K(\mathbf{g})} = 0$, so $E_{(\mathbf{g})} = \big(e^*TX \otimes H^1(C, \mathcal{O}_C)\big)^{K(\mathbf{g})} \cong \big(N(X_{(\mathbf{g})}) \otimes H^1(C, \mathcal{O}_C)\big)^{K(\mathbf{g})}$. We obtain

$$E_{(\mathbf{g})} = \Big(\bigoplus_{l=0}^{i-1} L_l \otimes H^1(C, \mathcal{O}_C)\Big)^{K(\mathbf{g})}.$$

The following theorem completely determines the bundle $E_{(\mathbf{g})}$.

Theorem 5.5.1 *Let $X_{(\mathbf{g})} = X_{(g_1,g_2,g_3)}$ be a 3-mutisector of the weighted projective space $X = \mathbf{P}(Q)$. Suppose $X_{(\mathbf{g})} = \overline{O}_\tau$, where $\tau = (v_0, \dots, v_{i-1})$. If $g_1 + g_2 + g_3 = \sum_{l=0}^{i-1} a_l v_l$, then the obstruction bundle*

$$E_{(\mathbf{g})} \cong \bigoplus_{a_l = 2} L_l.$$

Proof From (2.2), the dimension of the obstruction bundle $E_{(\mathbf{g})}$ is

$$
\begin{aligned}
\dim_{\mathbf{C}}(E_{(\mathbf{g})}) &= \dim_{\mathbf{C}}(X_{(\mathbf{g})}) - n + \iota_{(g_1)} + \iota_{(g_2)} + \iota_{(g_3)}\\
&= \iota_{(g_1)} + \iota_{(g_2)} + \iota_{(g_3)} - (n - \dim_{\mathbf{C}}(X_{(\mathbf{g})}))\\
&= \iota_{(g_1)} + \iota_{(g_2)} + \iota_{(g_3)} - i.
\end{aligned}
$$

Because $g_1, g_2, g_3 \in K(\mathbf{g})$, from Section 2.4 we write $g_1 = \sum_{l=0}^{i-1} b_l v_l$, $g_2 = \sum_{l=0}^{i-1} c_l v_l$, $g_3 = \sum_{l=0}^{i-1} d_l v_l$, where $0 \le b_l, c_l, d_l < 1$. Then $g_1 + g_2 + g_3 = \sum_{l=0}^{i-1} a_l v_l$, so we have $b_l + c_l + d_l = a_l$. Because $g_1g_2g_3 = 1$, all a_l's are 1 or 2. There are a total of i numbers a_l, so from the formula of the dimension of obstruction bundle, $\dim_{\mathbf{C}}(E_{(\mathbf{g})})$ is the number of a_l that satisfy $a_l = 2$. For each $L_l \otimes H^1(C, \mathcal{O}_C)$, let $\mathcal{L}_l$ be the line bundle over C induced by L_l, with action of $K(\mathbf{g})$ on $\mathcal{L}_l$ the same as in L_l. Then from [3], when $(L_l \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$ is restricted to a point of $X_{(\mathbf{g})}$, we have $\dim(L_l \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})} = \dim H^1(C, \mathcal{L}_l)^{K(\mathbf{g})} = \dim H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-a_l))$. So

$$\dim_{\mathbf{C}}(L_l \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})} = \begin{cases} 0 & a_l = 1, \\ 1 & a_l = 2. \end{cases}$$

So we have $(L_l \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})} \cong L_l$. This completes the proof. ■

Now let $a_{t_1} = \cdots = a_{t_e} = 2$. Then the the transition function of the obstruction bundle can be written as:

$$h_{kj}(x; c_1, \dots, c_e) = (x; y_j^{q_{t_1}/q_j}(x) \cdot c_1, \dots, y_j^{q_{t_e}/q_j}(x) \cdot c_e). \tag{5.6}$$

So locally the transition matrix is

$$\operatorname{diag}(y_j^{q_{t_1}/q_j}(x), \dots, y_j^{q_{t_e}/q_j}(x)).$$

Every line bundle L_l is generated by ξ_l on the neighborhood $U_j \cap X_{(\mathbf{g})}$. The group $\mathbf{Z}_d$ acts diagonally on the obstruction bundle $E_{(\mathbf{g})}$, so it acts on every line bundle E_l naturally. Assume the matrix representation of the action of the generator of $\mathbf{Z}_d$ on the obstruction bundle $E_{(\mathbf{g})}$ is

$$\operatorname{diag}(e^{2\pi i \cdot \frac{m_1}{d}}, \dots, e^{2\pi i \cdot \frac{m_e}{d}}),$$

where $0 \le m_l < d, 1 \le l \le e$. Then we have the following facts: $e^{2\pi i \cdot \frac{m_l}{d}}$ is a d_l-root of 1 for $1 \le l \le e$, and it is clear that d_l is a divisor of d.

5.6 Computation of the 3-Point Function

In this section we use the localization technique [2, 8] to calculate the 3-point function defined in the orbifold cup product.

Let $X = \mathbf{P}(Q)$ be the weighted projective space of type $Q = (q_0, \dots, q_n)$ and $X_{(\mathbf{g})}$ be a 3-multisector. Then $X_{(\mathbf{g})}$ is a twisted sector from Theorem 5.2.2. Assume $X_{(\mathbf{g})} = \overline{O}_\tau = \mathbf{P}(Q_\tau)$, where $Q_\tau = (0, \dots, 0, q_i, \dots, q_n)$, $\mathbf{g} = (g_1, g_2, g_3) \in T_3^0$ and $\tau = (v_0, \dots, v_{i-1})$ is a cone of the fan $\Sigma = \{v_0, \dots, v_n\}$. The orbifold structure of $X_{(\mathbf{g})}$ can be described in Remark 4.3.6. From (2.3), the key calculation of the orbifold cup product is to calculate the 3-point function

$$\langle \eta_1, \eta_2, \eta_3 \rangle_{\mathrm{orb}} = \int_{X_{(\mathbf{g})}}^{\mathrm{orb}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge e_A(E_{(\mathbf{g})}), \tag{5.7}$$

where $\eta_j \in H^*(X_{(g_j)}; \mathbf{Q})$, for $j = 1, 2, 3$.

Now we analyze the integral (5.7). In order to compute conveniently, we prove that we always can suppose $q_0 = 1$. If $q_0 \neq 1$, let $\widetilde{Q} = (1, q_0, \dots, q_n)$, then $\mathbf{P}(Q) \subset \mathbf{P}(\widetilde{Q}) = \mathbf{P}^{n+1}_{1,q_0,\dots,q_n} = Y$ is a hypersurface which is obtained by letting the first homogeneous coordinate of $\mathbf{P}(\widetilde{Q})$ be zero. From Theorem 4.3.5 and Theorem 5.2.2, it is easy to see that $\mathbf{P}(Q)$ and $\mathbf{P}(\widetilde{Q})$ have the same twisted sectors and 3-multisectors. Suppose that the matrix representations in $\mathbf{P}(\widetilde{Q})$ corresponding to g_1, g_2, g_3 in $\mathbf{P}(Q)$ are $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3$. Then $Y_{(\widetilde{\mathbf{g}})} = Y_{(\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3)} = X_{(\mathbf{g})}$. The cohomological classes $e_1^*\eta_1, e_2^*\eta_2, e_3^*\eta_3$ are invariant when they are taken as the cohomological classes of $Y_{(\widetilde{\mathbf{g}})}$. Suppose the homogeneous coordinates of $\mathbf{P}(\widetilde{Q})$ are $\mathbf{z} = [z, z_0, \dots, z_n]$. Let $\widetilde{U}_j = \{z_j \neq 0 \mid \mathbf{z} \in \mathbf{P}(\widetilde{Q})\}$, $(0 \leq j \leq n)$, $\widetilde{U} = \{z \neq 0 \mid \mathbf{z} \in \mathbf{P}(\widetilde{Q})\}$. Then $Y_{(\widetilde{\mathbf{g}})}$ can be covered by $\bigsqcup_{j=i}^n \widetilde{U}_j \cap Y_{(\widetilde{\mathbf{g}})}$. For the local chart $\widetilde{U}_j$, let $\widetilde{U}_j = \widetilde{V}_j / \mathbf{Z}_{q_j}$ and choose the coordinates of $\widetilde{V}_j$ as

$$x = \frac{z}{(z_j)^{1/q_j}}, x_0 = \frac{z_0}{(z_j)^{q_0/q_j}}, \dots, x_n = \frac{z_n}{(z_j)^{q_n/q_j}},$$

so we have a base of $(T\widetilde{V}_j)_{p_j}$: $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n}\right)$. Because the invariant subspace $((TV_j)_{p_j} \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$ is generated by $\xi_1, \dots, \xi_e$, we see that

$$((TV_j)_{p_j} \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})} \subset ((T\widetilde{V}_j)_{\widetilde{p}_j} \otimes H^1(C, \mathcal{O}_C))^{K(\widetilde{\mathbf{g}})}.$$

We construct a new obstruction bundle $E_{(\widetilde{\mathbf{g}})}$ over $Y_{(\widetilde{\mathbf{g}})}$ as follows. On the local chart $\widetilde{U}_j \cap Y_{(\widetilde{\mathbf{g}})}$, this bundle is given by $\widetilde{V}_j \cap H \times ((TV_j)_{p_j} \otimes H^1(C, \mathcal{O}_C))^{K(\widetilde{\mathbf{g}})} \to \widetilde{V}_j \cap H$, where $H = \{\mathbf{x} \in \widetilde{V}_j \mid x = x_0 = \cdots = x_{i-1} = 0\}$ is a hypersurface of $\widetilde{V}_j$. It is easy to see that the transition function of this bundle is also given by (5.6), so it can also be split into the direct sum of line bundles. It is clear that $E_{(\widetilde{\mathbf{g}})} \cong E_{(\mathbf{g})}$, so we have

$$\int^{\mathrm{orb}}_{X_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge e_A(E_{(\mathbf{g})}) = \int^{\mathrm{orb}}_{Y_{(\widetilde{\mathbf{g}})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge e_{\widetilde{A}}(E_{(\widetilde{\mathbf{g}})}). \tag{5.8}$$

In the following analysis, we assume that $q_0 = 1$, and we give a formula to compute the integral (5.7).

First from Remark 4.3.6, we know that if $d = \gcd(q_i, \dots, q_n) \neq 1$, $X_{(\mathbf{g})}$ is a nonreduced orbifold. From the discussion of Section 5.5, let $E_{(\mathbf{g})} = \bigoplus_{l=1}^e E_l$. Then for every line bundle E_l, using the same method of Park and Poddar [16], consider the associated orbifold principal bundle P_l of E_l such that $E_l = P_l \times_{S^1} \mathbf{C}$. We know that there is a global action of $\mathbf{Z}_{d_l}$ on each fibre $F = S^1$. The quotient $P_l / \mathbf{Z}_{d_l}$ is again an orbifold principal bundle over the orbifold $X_{(\mathbf{g})}$. Let $\pi_l \colon P_l \to P_l / \mathbf{Z}_{d_l}$ be the quotient map, which extends to an orbifold bundle map. Choose an orbifold connection A_l that is the pullback $\pi_l^*(A_l')$, where A_l' is an orbifold connection on the associated bundle $E_l' = (P_l / \mathbf{Z}_{d_l}) \times_{S^1} \mathbf{C}$. The Lie algebra of F can be identified with $\mathbf{R}$, and then the induced map on the lie algebra $(\pi_l)_* \colon \mathbf{R} \to \mathbf{R}$ is just given by $a \mapsto d_l a$.

Let Ω_l and Ω'_l be the curvature 2-forms for A_l and A'_l. By [13, Proposition 6.2], $(\pi_l)^*(\Omega'_l) = d_l\Omega_l$. So

$$\begin{aligned}(5.9)\qquad \int^{\mathrm{orb}}_{X_{(\mathbf{g})}} & e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge e_A(E_{(\mathbf{g})}) \\ &= \int^{\mathrm{orb}}_{X_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge \Pi_{l=1}^e e_{A_l}(E_l) \\ &= \frac{1}{\Pi_l d_l}\int^{\mathrm{orb}}_{X_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge \Pi_{l=1}^e e_{A'_l}(E'_l).\end{aligned}$$

Since the action of $\mathbf{Z}_{d_l}$ in any uniformizing system of E'_l is trivial, E'_l induces an orbifold bundle E''_l over the reduced orbifold $X'_{(\mathbf{g})}$ which has an induced connection A''_l. The connections A'_l and A''_l may be represented by the same 1-form over V for $(V \times \mathbf{C}, G'/\mathbf{Z}_{d_l}, \widetilde{\pi}''_1)$ of E'_l and E''_l, respectively. By Chern–Weil theory, Ω'_l and Ω''_l can therefore be represented by the same 2-form on V. We know that $e_1^*(\eta_1), e_2^*(\eta_2)$ and $e_3^*(\eta_3)$ are invariant when taken as the cohomology classes of $X'_{(\mathbf{g})}$. Since $K(\mathbf{g})$ acts on $X_{(\mathbf{g})}$ trivially, from the definition of the integral on orbifold we have
(5.10)

$$\int^{\mathrm{orb}}_{X_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge \Pi_{l=1}^e e_{A'_l}(E'_l) = \frac{1}{|\mathbf{Z}_d|}\int^{\mathrm{orb}}_{X'_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge \Pi_{l=1}^e e_{A''_l}(E'_l).$$

Next we mainly discuss the method to calculate the integral in (5.10). From above, $X_{(\mathbf{g})} = \mathbf{P}(Q_\tau)$, $Q_\tau = (0,\dots,0,q_i,\dots,q_n)$. So we have $X'_{(\mathbf{g})} = \mathbf{P}(Q_\tau/d)$, $Q_\tau/d = (q_i/d,\dots,q_n/d), d = \gcd(q_i,\dots,q_n)$. Let N_τ be the sublattice of N generated by $\tau = (v_0,\dots,v_{i-1})$, and let $N(\tau) = N/N_\tau$ be the quotient lattice. The fan of $X'_{(\mathbf{g})}$ is given by the projection of Σ to $N(\tau)\otimes\mathbf{R}$. The dual lattice of $N(\tau)$ is $M(\tau) = \tau^\perp \cap M$. The torus $T = \operatorname{spec}(\mathbf{C}[M(\tau)]) = O_\tau$. The characters χ^m correspond to rational functions on $X'_{(\mathbf{g})}$ when $m \in M(\tau)$. Then we can use the localization technique of [2], when reduced to orbifolds, to calculate the integral (5.10).

We know $X'_{(\mathbf{g})}$ is a toric variety, T acts on $X'_{(\mathbf{g})}$, and this action has $n - i + 1$ fixed points p_j for $i \le j \le n$. We let $\{\rho_i\}$ be the basic characters of action T and $\{\lambda_i\}$ the parameters of the Lie algebra $t_{\mathbf{C}}$ of T corresponding to the above base $\{\rho_i\}$. Because $q_0 = 1$, from Conrads [4] we compute that the fan of the weighted projective space $\mathbf{P}(Q)$ is generated by $v_0,\dots,v_n$, where $v_0 = (-q_1,\dots,-q_n)$, $v_j = e_j$ for $1 \le j \le n$. Let $\{m_1,\dots,m_n\}$ be the standard basis of M. We calculate the base of $M(\tau)$ as

$$\left\{\rho_1 = \frac{q_n}{d}m_i - \frac{q_i}{d}m_n, \dots, \rho_{n-i} = \frac{q_n}{d}m_{n-1} - \frac{q_{n-1}}{d}m_n\right\}.$$

We first study the action of T on the normal bundle of p_j, *i.e.*, the orbifold tangent space $(TX'_{(\mathbf{g})})_{p_j}$.

Consider the fixed points p_j $(i \le j \le n-1)$. Denote the local coordinates on a uniformizing system of $X'_{(\mathbf{g})}$ around p_j by $[x_i,\dots,1,\dots,x_n]$. Let $m^1 = c_1\rho_1 + \cdots + c_{n-i}\rho_{n-i}$, and $\langle m^1, v_i\rangle = 1$, $\langle m^1, v_k\rangle = 0$, for $k > i, k \ne j$. Then we have

$c_r = -dq_i/q_nq_j$ and $c_1 = d/q_n$, so $\chi^{m^1} = x_i$. Similarly, we compute $\chi^{m^t} = x_t$ for $t \neq r, n-i$ and $m^t = (d/q_n)\rho_t - (dq_{i+t-1}/q_nq_j)\rho_r$. Using the same method, let $m^{n-i} = a_1\rho_1 + \cdots + a_{n-i}\rho_{n-i}$, and $\langle m^{n-i}, v_k\rangle = 0$ for $k \geq i, k \neq j, n$, $\langle m^{n-i}, v_n\rangle = 1$. We have $c_r = -d/q_j$, $m^{n-i} = -(d/q_j)\rho_r$, $\chi^{m^{n-i}} = x_n$. So the T-equivariant Euler class of normal bundle of $p_j (i \leq j \leq n)$ is given by

$$e_T(\nu_{p_j}) = \Big(-\frac{d}{q_j}\lambda_r\Big)\prod_{k\neq r}\frac{d}{q_n}\Big(\lambda_k - \frac{q_{i+k-1}}{q_j}\lambda_r\Big). \tag{5.11}$$

Now we consider the fixed point p_n with local coordinates $[w_i, \dots, w_{n-1}, 1]$ in the uniformizing system of $X'_{(\mathbf{g})}$. Using the same method, we find that the T-equivariant Euler class of normal bundle of $p_n (i \leq j \leq n)$ is given by

$$e_T(\nu_{p_n}) = \prod_{k=1}^{n-i}\frac{d}{q_n}\lambda_k. \tag{5.12}$$

Since $e_1^*(\eta_1)$, $e_2^*(\eta_2)$, and $e_3^*(\eta_3)$ all belong to $H^*(X'_{(\mathbf{g})}, \mathbf{Q})$, from the ordinary ring structure of weighted projective space in Section 5.1, we only consider the generator β of $H^2(X'_{(\mathbf{g})}, \mathbf{Q}) = \mathbf{Q}$. Suppose $L \to X'_{(\mathbf{g})}$ is the canonical line bundle whose first chern-class is ξ_1. The corresponding Cartier divisor is

$$D = D_1D_2\cdots D_{i-1}D_i + \cdots + D_1D_2\cdots D_{i-1}D_n,$$

where $D_j = \{z_j = 0\} \subset \mathbf{P}(Q)$ is the basic divisor. Then from Oda [14], in the neighborhood $U_j \cap X'_{(\mathbf{g})}$, $(i \leq j \leq n-1)$, let $m = u_1\rho_1 + \cdots + u_{n-i}\rho_{n-i}$, and $\langle -m, v_i\rangle = 1, \dots, \langle -m, v_{j-1}\rangle = 1, \langle -m, v_{j+1}\rangle = 1, \dots, \langle -m, v_n\rangle = 1$. Then we calculate

$$-m = \sum_{k\neq r}\frac{d}{q_n}\rho_k - \frac{d}{q_n}\Big(\sum_{k\neq r}\frac{q_{k+i-1}}{q_j}\Big)\rho_r,$$

where $j = i + r - 1$. So the divisor D is given by the rational function χ^{-m} on $U_j \cap X'_{(\mathbf{g})}$. Similarly, it is given by the rational function $\chi^{-m} = \chi^{\frac{d}{q_n}(\rho_1+\cdots+\rho_{n-i})}$ on $U_n \cap X'_{(\mathbf{g})}$. Hence the action of T on the corresponding line bundle of D at the fixed points p_j has weights

$$\sum_{k\neq r}\frac{d}{q_n}\lambda_k - \frac{d}{q_n}\Big(\sum_{k\neq r}\frac{q_{k+i-1}}{q_j}\Big)\lambda_r \quad \text{at } p_j, (i \leq j \leq n-1), \tag{5.13}$$

$$\frac{d}{q_n}\sum_k \lambda_k \qquad \text{at } p_n. \tag{5.14}$$

On the other hand, we also can write $e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 = a(\xi_1)^s$, $a \in \mathbf{Q}$, where s is an integer.

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Now we analyze the Euler form $e(E''_{(\mathbf{g})})$. From Section 5.5, we compute the local generating vectors of the obstruction bundle $E_{(\mathbf{g})}$, where $E_{(\mathbf{g})} = \bigoplus_{l=1}^{e} E_l$. For each line bundle E_l, from (5.6), we have the transition function of E_l as

$$h_{jn}(x,c) = (x, x_n^{q_{t_l}/q_n}(x)\cdot c), \quad (i \le j \le n-1).$$

Because the line bundle E''_l is the reduction of E_l under the $\mathbf{Z}_{d_l}$-invariant homomorphism, the transition function of the line bundle E''_l,

$$(U_n \cap X'_{(\mathbf{g})}) \times c \supset (U_n \cap U_j \cap X'_{(\mathbf{g})}) \times c \to (U_n \cap U_j \cap X'_{(\mathbf{g})}) \times c \subset (U_j \cap X'_{(\mathbf{g})}) \times c,$$

is given by

$$h''_{jn}(x,c) = \big(x, (x_n^{q_{t_l}/q_n})^{d_l}(x)\cdot c\big), \quad (i \le j \le n-1).$$

So we define the action of T on $E''_{(\mathbf{g})} = \bigoplus_{l=1}^{e} E''_l$ as follows: on the line bundle E''_l for $t \in T$ and $(x,c) \in X'_{(g)} \times C$,

$$t(x,c) = \begin{cases} (tx, c) = (tx, \chi^0(t)c) & \text{if } x \in U_n, \\ \big(tx, (x_n^{q_{t_l}/q_n})^{d_l}(t)c\big) = \big(tx, (\chi^{-\frac{d}{q_j}\rho_r})^{\frac{q_{t_l}d_l}{q_n}}(t)c\big) & \text{if } x \in U_j, \end{cases}$$

where for $i \le j \le n-1$, $j = i+r-1$. Then the action of T on E''_l at the fixed points $p_n, p_j (i \le j \le n-1)$ has weights

$$0, \quad -\frac{q_{t_l}d_l}{q_n}\cdot\frac{d}{q_j}\lambda_r. \tag{5.15}$$

So using the localization formula for the orbifold $X'_{(\mathbf{g})}$, see [8, Corollary 9.1.4], we have the integral

(5.16)
$$\begin{aligned}
&\int^{\mathrm{orb}}_{X'_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge \Pi_{l=1}^{e} e_{A''_l}(E''_l) \\
&\qquad = \frac{a\big(\frac{d}{q_n}\sum_{k=1}^{n-i}\lambda_k\big)^s \cdot 0}{a_n \cdot \prod_{k=1}^{n-i}\frac{d}{q_n}\lambda_k} \\
&\qquad\quad + \sum_{j=i}^{n-1} \frac{a\big[\sum_{k\ne r}\frac{d}{q_n}\lambda_k - \frac{d}{q_n}\big(\sum_{k\ne r}\frac{q_{k+i-1}}{q_j}\big)\lambda_r\big]^s \cdot \prod_{l=1}^{e}\big(-\frac{q_{t_l}d_l}{q_n}\cdot\frac{d}{q_j}\lambda_r\big)}{|G_{p_j}|\cdot\big(-\frac{d}{q_j}\lambda_r\big)\cdot\prod_{k\ne r}\frac{d}{q_n}\big(\lambda_k - \frac{q_{i+k-1}}{q_j}\lambda_r\big)},
\end{aligned}$$

where $j = i+r-1$ and $|G_{p_j}|$ is the order of the local cyclic group of p_j in the orbifold $X'_{(\mathbf{g})}$.

5.7 Example

In this example we use the methods of the above sections to calculate the 3-point functions. Let $Q = (1, 2, 2, 3, 3, 3)$ and $\mathbf{P}(Q) = \mathbf{P}^5_{1,2,2,3,3,3}$ be the weighted projective space of type Q; then $q_0 = 1, q_1 = q_2 = 2, q_3 = q_4 = q_5 = 3$. From Conrads [4], we compute that the fan Σ is generated by the vectors $v_1 = e_1, v_2 = e_2, v_3 = e_3, v_4 = e_4, v_5 = e_5, v_0 = -\Sigma \frac{q_i}{q_0} v_i = (-2, -2, -3, -3, -3)$.

- For $\sigma_5 = (v_0, v_1, v_2, v_3, v_4)$, we have $G_{\sigma_5} = N/N_{\sigma_5} = \mathbf{Z}_3$. So $g_{\sigma_5} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0, 0)$ is the generator of $\mathbf{Z}_3$.
- For $\sigma_4 = (v_0, v_1, v_2, v_3, v_5)$ and $\sigma_3 = (v_0, v_1, v_2, v_4, v_5)$, we have $G_{\sigma_4} = G_{\sigma_3} = \mathbf{Z}_3$. The generators $g_{\sigma_4}, g_{\sigma_3}$ are the same as above.
- For $\sigma_2 = (v_0, v_1, v_3, v_4, v_5)$, we have the generator $g_{\sigma_2} = (\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of $\mathbf{Z}_2$.
- For $\sigma_1 = (v_0, v_2, v_3, v_4, v_5)$, we have $G_{\sigma_1} = \mathbf{Z}_2$. The generator g_{σ_1} is the same as above.
- For $\sigma_0 = (v_1, v_2, v_3, v_4, v_5)$, $G_{\sigma_0} = 1$, and the action is trivial.

Then we have the twisted sectors $X_{(g_{\sigma_5})} = X_{(g^2_{\sigma_5})} = \mathbf{P}(Q_\tau)$, where $Q_\tau = (0, 0, 0, 3, 3, 3)$, $\tau = (v_0, v_1, v_2)$; $X_{(g_{\sigma_2})} = \mathbf{P}(Q_\delta)$, $Q_\delta = (0, 2, 2, 0, 0, 0)$, $\delta = (v_0, v_3, v_4, v_5)$. The degree-shifting numbers are $\iota_{(g_{\sigma_5})} = \frac{5}{3}$, $\iota_{(g^2_{\sigma_5})} = \frac{4}{3}$, $\iota_{(g_{\sigma_2})} = 2$. So the Chen–Ruan cohomology group of $\mathbf{P}(Q)$ is

$$H^d_{\mathrm{orb}}(\mathbf{P}(Q); \mathbf{Q}) = H^d(\mathbf{P}(Q); \mathbf{Q}) \oplus H^{d-\frac{10}{3}}(\mathbf{P}(Q_\tau); \mathbf{Q}) \oplus H^{d-\frac{8}{3}}(\mathbf{P}(Q_\tau); \mathbf{Q}) \oplus H^{d-4}(\mathbf{P}(Q_\delta); \mathbf{Q}).$$

All the 3-multisectors are $X_{(g_{\sigma_5}, g_{\sigma_5}, g_{\sigma_5})} = X_{(g^2_{\sigma_5}, g^2_{\sigma_5}, g^2_{\sigma_5 1})} = \mathbf{P}(Q_\tau)$, $X_{(g_{\sigma_5}, g^2_{\sigma_5}, 1)} = \mathbf{P}(Q_\tau)$, and $X_{(g_{\sigma_2}, g_{\sigma_2}, 1)} = \mathbf{P}(Q_\delta)$. In $X_{(g_{\sigma_5}, g^2_{\sigma_5}, 1)}$ and $X_{(g_{\sigma_2}, g_{\sigma_2}, 1)}$, from (2.2), the dimensions of the obstruction bundles of these two 3-multisectors are all zero, so the integral (2.3) is the usual integral on orbifolds. The orbifold cup product can be described easily:

For the 3-multisector $X_{(g^2_{\sigma_5}, g^2_{\sigma_5}, g^2_{\sigma_5})}$, the dimension of the obstruction bundle $E_{(\mathbf{g})}$ is one. Let $X_{(\mathbf{g})} = X_{(g^2_{\sigma_5}, g^2_{\sigma_5}, g^2_{\sigma_5})}$, $\eta_j \in H^*(X_{(g^2_{\sigma_5})}; \mathbf{Q})$, $(j = 1, 2, 3)$. Then

$$\langle \eta_1, \eta_2, \eta_3 \rangle_{\mathrm{orb}} = \int^{\mathrm{orb}}_{X_{(\mathbf{g})}} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_3^* \eta_3 \wedge e_A(E_{(\mathbf{g})}). \tag{5.17}$$

Next we use the localization formula (5.16) to compute the 3-point function (5.17). Because $\tau = (v_0, v_1, v_2)$, we have $\overline{O}_\tau = X_{(\mathbf{g})}$. Suppose $U_i = \{z_i \neq 0\}$. Then $X_{(\mathbf{g})} = (X_{(\mathbf{g})} \cap U_3) \cup (X_{(\mathbf{g})} \cap U_4) \cup (X_{(\mathbf{g})} \cap U_5)$. Let $U_3 = V_3/\mathbf{Z}_3$, $U_4 = V_4/\mathbf{Z}_3$, $U_5 = V_5/\mathbf{Z}_3$. From Section 5.5, the coordinates of V_3, V_4 and V_5 are

$$V_5 : \Big\{ x_0 = \frac{z_0}{(z_5)^{1/3}}, x_1 = \frac{z_1}{(z_5)^{2/3}}, x_2 = \frac{z_2}{(z_5)^{2/3}}, x_3 = \frac{z_3}{z_5}, x_4 = \frac{z_4}{z_5}, x_5 = 1 \Big\},$$

$$V_4 : \Big\{ y_0 = \frac{z_0}{(z_4)^{1/3}}, y_1 = \frac{z_1}{(z_4)^{2/3}}, y_2 = \frac{z_2}{(z_4)^{2/3}}, y_3 = \frac{z_3}{z_4}, y_4 = 1, y_5 = \frac{z_5}{z_4} \Big\},$$

$$V_3 : \Big\{ w_0 = \frac{z_0}{(z_3)^{1/3}}, w_1 = \frac{z_1}{(z_3)^{2/3}}, w_2 = \frac{z_2}{(z_3)^{2/3}}, w_3 = 1, w_4 = \frac{z_4}{z_3}, w_5 = \frac{z_5}{z_3} \Big\}.$$

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For the chart $(V_5, \mathbf{Z}_3, \pi_5)$, $TV_5|_{p_5}$ has framing $\{\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\}$. As $o(g^2_{\sigma_5})=3$, the genus of C is one from the Riemann–Hurwitz formula. So $TV_5|_{p_5} \otimes H^1(C, \mathcal{O}_C)$ has framing

$$\Big\{ \frac{\partial}{\partial x_0} \otimes \overline{\omega}, \frac{\partial}{\partial x_1} \otimes \overline{\omega}, \frac{\partial}{\partial x_2} \otimes \overline{\omega}, \frac{\partial}{\partial x_3} \otimes \overline{\omega}, \frac{\partial}{\partial x_4} \otimes \overline{\omega} \Big\},$$

where $\overline{\omega}$ is the basis of $H^1(C, \mathcal{O}_C)$. From Theorem 5.5.1, since $g_{\sigma_5} = \big(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0, 0\big)$ and $g^2_{\sigma_5} = \big(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0\big)$, the generator of $(TV_5|_{p_5} \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$ is $\xi_0 \otimes \overline{\omega} = \frac{\partial}{\partial x_0} \otimes \overline{\omega}$. Similarly, we find that the generator of $(TV_4|_{p_4} \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$ is $\xi'_0 \otimes \overline{\omega} = \frac{\partial}{\partial y_0} \otimes \overline{\omega}$ and the generator of $(TV_3|_{p_3} \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$ is $\xi''_0 \otimes \overline{\omega} = \frac{\partial}{\partial w_0} \otimes \overline{\omega}$.

Now we describe the local uniformizing charts for $E_{(\mathbf{g})}$. If $x \in X_{(\mathbf{g})}$, then $C(\mathbf{g}) = G_x = K(\mathbf{g}) = \mathbf{Z}_3$, and $(V^{\mathbf{g}}_x \times \mathbf{C}, K(\mathbf{g}), \widetilde{\pi})$ is a uniformizing system for $E_{(\mathbf{g})}$, where $K(\mathbf{g})$ acts on $V^{\mathbf{g}}_x \times \mathbf{C}$ by $g^2_1(u, v) = (u, e^{2\pi i \cdot \frac{2}{3}} v)$.

The bundle $E_{(\mathbf{g})}$ is a line bundle and $\dim_{\mathbf{C}} X_{(\mathbf{g})} = 2$, so the 3-point function (5.17) is nonzero only if there is some $\eta_i \in H^2(X_{(g_{\sigma_5})}; \mathbf{Q})$. Without loss of generality, assume $\eta_1 \in H^2(X_{(g^2_{\sigma_5})}; \mathbf{Q})$, $\eta_2 \in H^0(X_{(g^2_{\sigma_5})}; \mathbf{Q})$, and $\eta_3 \in H^0(X_{(g^2_{\sigma_5})}; \mathbf{Q})$. In this case

$$\langle \eta_1, \eta_2, \eta_3 \rangle_{\mathrm{orb}} = \eta_2 \eta_3 \int^{\mathrm{orb}}_{X_{(\mathbf{g})}} \eta_1 \wedge e_A(E_{(\mathbf{g})}). \tag{5.18}$$

From the first part of this section, we see that in this case, the orbifold principal S^1 bundle is $P_{(\mathbf{g})}$. Let $E'_{(\mathbf{g})} = (P_{(\mathbf{g})}/K(\mathbf{g})) \times_{S^1} \mathbf{C}$ over $X_{(\mathbf{g})}$. $\pi_{K(\mathbf{g})} \colon P_{(\mathbf{g})} \to P_{(\mathbf{g})}/K(\mathbf{g})$. Note that $\pi_{K(\mathbf{g})}$ on each fibre is given by $z \mapsto z^3$. The Lie algebra of $F = S^1$ can be identified with $\mathbf{R}$. Hence the induced map on the Lie algebra $(\pi_{K(\mathbf{g})})_* \colon \mathbf{R} \to \mathbf{R}$ is just multiplication by 3, so from (5.9)

$$\int^{\mathrm{orb}}_{X_{(\mathbf{g})}} \eta_1 \wedge e_A(E_{(\mathbf{g})}) = \frac{1}{3} \int^{\mathrm{orb}}_{X_{(\mathbf{g})}} \eta_1 \wedge e_{A'}(E'_{(\mathbf{g})}),$$

where A and A' are the connections of $E_{(\mathbf{g})}$ and $E'_{(\mathbf{g})}$ such that $\pi^*_{K(\mathbf{g})}(A') = A$. Then $E'_{(\mathbf{g})}$ induces an orbifold bundle $E''_{(\mathbf{g})}$ over the reduced orbifold $X'_{(\mathbf{g})}$, and from (5.10),

$$\int^{\mathrm{orb}}_{X_{(\mathbf{g})}} \eta_1 \wedge e_{A'}(E'_{(\mathbf{g})}) = \frac{1}{3} \int^{\mathrm{orb}}_{X'_{(\mathbf{g})}} \eta_1 \wedge e_{A''}(E''_{(\mathbf{g})}),$$

where A'' is the connection of $E''_{(\mathbf{g})}$ induced from $E'_{(\mathbf{g})}$. So we obtain

$$\int^{\mathrm{orb}}_{X_{(\mathbf{g})}} \eta_1 \wedge e_A(E_{(\mathbf{g})}) = \frac{1}{9} \int^{\mathrm{orb}}_{X'_{(\mathbf{g})}} \eta_1 \wedge e_{A''}(E''_{(\mathbf{g})}). \tag{5.19}$$

We now compute the integral $\int^{\mathrm{orb}}_{X'_{(\mathbf{g})}} \eta_1 \wedge e_{A''}(E''_{(\mathbf{g})})$ in (5.19).

The uniformizing system of $E''_{(\mathbf{g})}$ over $X'_{(\mathbf{g})}$ can be described as follows. If $x \in X'_{(\mathbf{g})}$, then $C(\mathbf{g})/K(\mathbf{g}) = 1$ is the trivial group.

Now we use the localization technique to calculate the integral (5.19). Note that $X'_{(\mathbf{g})} = O_\tau$, $\tau = (v_0, v_1, v_2)$, so $N(\tau) = N/N_\tau$, where N_τ is the sublattice generated by τ, and $M(\tau) = \tau^\perp \cap M$. The 2-torus associated to $X'_{(\mathbf{g})}$ is $T = \operatorname{spec}(\mathbf{C}[M(\tau)]) = O_\tau$. The characters χ^m correspond to rational functions on $X'_{(\mathbf{g})}$ when $m \in M(\tau)$. If $\{m_1, m_2, m_3, m_4, m_5\}$ is the standard basis of M, then $\{\rho_1 = m_3 - m_5, \rho_2 = m_4 - m_5\}$ is a basis for $M(\tau)$. The T-action on $X'_{(\mathbf{g})}$ has three fixed points p_3, p_4, p_5. First we study the action of T on the normal bundle of p_3, p_4 and p_5, *i.e.,* the orbifold tangent space of p_3, p_4 and p_5.

From (5.11) and (5.12), of course, we can compute using the same method as in Section 5.6. We see that the T-equivariant Euler class of the normal bundle of p_5 is given by $e_T(\nu_{p_5}) = \lambda_1\lambda_2$, and we also have $e_T(\nu_{p_4}) = (\lambda_1 - \lambda_2)(-\lambda_2)$, $e_T(\nu_{p_3}) = (-\lambda_1 + \lambda_2)(-\lambda_1)$. In particular, we have $\chi^{-\rho_2} = y_5$ in the neighborhood V_4, and $\chi^{-\rho_1} = \omega_5$ in the neighborhood V_5.

The orbifold line bundle $E''_{(\mathbf{g})}$ is trivialized by the generator $\frac{\partial}{\partial x_0} \otimes \overline{\omega}$ on $U_5 \cap X'_{(\mathbf{g})}$, $\frac{\partial}{\partial y_0} \otimes \overline{\omega}$ on $U_4 \cap X'_{(\mathbf{g})}$, and $\frac{\partial}{\partial w_0} \otimes \overline{\omega}$ on $U_3 \cap X'_{(\mathbf{g})}$. So from (5.15), the action of T on $E'_{(\mathbf{g})}$ at the fixed points p_5, p_4, p_3 has weights $0, (-\lambda_2), (-\lambda_1)$, respectively.

Since $\eta_1 \in H^2(X_{(g_1^2)}; \mathbf{Q})$, we take η_1 as the generator. So let $\eta_1 = D = D_0D_1D_2D_3 + D_0D_1D_2D_4 + D_0D_1D_2D_5$, from (5.11) and (5.12), the action of T on the corresponding line bundle of η_1 at the fixed points p_5, p_4 and p_3 has weights $\lambda_1 + \lambda_2$, $\lambda_1 - 2\lambda_2$ and $-2\lambda_1 + \lambda_2$, respectively. So using the localization formula (5.16), we have

$$\int^{\mathrm{orb}}_{X'_{(\mathbf{g})}} \eta_1 \wedge e_{A''}(E''_{(\mathbf{g})}) = \frac{(\lambda_1 + \lambda_2)\cdot 0}{\lambda_1\lambda_2} + \frac{(\lambda_1 - 2\lambda_2)(-\lambda_2)}{(\lambda_1 - \lambda_2)(-\lambda_2)} + \frac{(-2\lambda_1 + \lambda_2)(-\lambda_1)}{(-\lambda_1 + \lambda_2)(-\lambda_1)} = 3.$$

So from (5.19),

$$\int^{\mathrm{orb}}_{X_{(\mathbf{g})}} \eta_1 \wedge e_A(E_{(\mathbf{g})}) = \frac{1}{3}.$$

From (5.18),

$$\langle \eta_1, \eta_2, \eta_3 \rangle_{\mathrm{orb}} = \frac{1}{3}\eta_2\eta_3.$$

For the 3-multisector $X_{(g_{\sigma_5}, g_{\sigma_5}, g_{\sigma_5})} = \mathbf{P}(Q_\tau)$, the dimension of the obstruction bundle $E_{(\mathbf{g})}$ is 2. Let $X_{(\mathbf{g})} = X_{(g_{\sigma_5}, g_{\sigma_5}, g_{\sigma_5})}$, $\eta_j \in H^*(X_{(g_{\sigma_5})}; \mathbf{Q})$, $(j = 1, 2, 3)$. Then

$$\langle \eta_1, \eta_2, \eta_3 \rangle_{\mathrm{orb}} = \int^{\mathrm{orb}}_{X_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge e_A(E_{(\mathbf{g})}). \tag{5.20}$$

Next we use the localization technique to calculate the 3-point function (5.20). We know that $\overline{O}_\tau = X_{(\mathbf{g})}$. Assume $U_i = \{z_i \neq 0\}$. Then $X_{(\mathbf{g})} = (X_{(\mathbf{g})} \cap U_3) \cup (X_{(\mathbf{g})} \cap U_4) \cup (X_{(\mathbf{g})} \cap U_5)$. Write $U_3 = V_3/\mathbf{Z}_3$, $U_4 = V_4/\mathbf{Z}_3$, $U_5 = V_5/\mathbf{Z}_3$. Then we can choose the coordinates of the open set V_3, V_4 and V_5 the same as before. For the coordinate neighborhood $(V_5, \mathbf{Z}_3, \pi_5)$, $TV_5|_{p_5}$ has a framing $\{\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\}$, so $TV_5|_{p_5} \otimes H^1(C, \mathcal{O}_C)$ has a framing $\{\frac{\partial}{\partial x_0} \otimes \overline{\omega}, \frac{\partial}{\partial x_1} \otimes \overline{\omega}, \frac{\partial}{\partial x_2} \otimes \overline{\omega}, \frac{\partial}{\partial x_3} \otimes \overline{\omega}, \frac{\partial}{\partial x_4} \otimes \overline{\omega}\}$. From Theorem 5.5.1 we see that the invariant subspace $(TV_5|_{p_5} \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$ has generators $\xi_1 \otimes \overline{\omega} = \frac{\partial}{\partial x_1} \otimes \overline{\omega}$ and $\xi_2 \otimes \overline{\omega} = \frac{\partial}{\partial x_2} \otimes \overline{\omega}$. Similarly, $(TV_4|_{p_4} \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$ has generators $\xi_1' \otimes \overline{\omega} = \frac{\partial}{\partial y_1} \otimes \overline{\omega}$ and $\xi_2' \otimes \overline{\omega} = \frac{\partial}{\partial y_2} \otimes \overline{\omega}$, and $(TV_3|_{p_3} \otimes H^1(C, \mathcal{O}_C))^{K(\mathbf{g})}$ has generators $\xi_1'' \otimes \overline{\omega} = \frac{\partial}{\partial w_1} \otimes \overline{\omega}$ and $\xi_2'' \otimes \overline{\omega} = \frac{\partial}{\partial w_2} \otimes \overline{\omega}$.

We describe the uniformizing system of $E_{(\mathbf{g})}$ as follows: if $x \in X_{(\mathbf{g})}$, then $C(\mathbf{g}) = G_x = K(\mathbf{g}) = \mathbf{Z}_3$, and if $(V_x^{\mathbf{g}} \times \mathbf{C}^2, K(\mathbf{g}), \widetilde{\pi})$ is a uniformizing system of the bundle $E_{(\mathbf{g})}$, then $K(\mathbf{g})$ acts on $V_x^{\mathbf{g}} \times \mathbf{C}^2$ through $g_1(u, v_1, v_2) = (u, e^{2\pi i \cdot \frac{2}{3}} v_1, e^{2\pi i \cdot \frac{2}{3}} v_2)$.

The obstruction bundle $E_{(\mathbf{g})}$ is a plane bundle. And $\dim_{\mathbf{C}} X_{(\mathbf{g})} = 2$, so the 3-point function (5.20) is nonzero only if $\eta_j \in H^0(X_{(g_1)}; \mathbf{Q})$, $j = 1, 2, 3$. In this case

$$\langle \eta_1, \eta_2, \eta_3 \rangle_{\mathrm{orb}} = \eta_1 \eta_2 \eta_3 \int_{X_{(\mathbf{g})}}^{\mathrm{orb}} e_A(E_{(\mathbf{g})}). \tag{5.21}$$

From Section 5.5., the obstruction bundle $E_{(\mathbf{g})}$ is the direct sum of two orbifold line bundles. Let $E_{(\mathbf{g})} = E_1 \oplus E_2$. Then E_1 is generated by $\xi_1 \otimes \overline{\omega}$ on the neighborhood $U_5 \cap X_{(\mathbf{g})}$, and E_2 is generated by $\xi_2 \otimes \overline{\omega}$ on $U_5 \cap X_{(\mathbf{g})}$. So from the first part of this section, consider the orbifold principal S^1-bundle P_l of $E_l (l = 1, 2)$. From Section 5.5, we can see that $\mathbf{Z}_{d_l} = K(\mathbf{g}) = \mathbf{Z}_3$, so let $E_l' = (P_l / K(\mathbf{g})) \times_{S^1} \mathbf{C}$ be the orbifold bundle over $X_{(\mathbf{g})}$. Then $\pi_{K(\mathbf{g})} \colon P_l \to P_l / K(\mathbf{g})$ is the projective map. Note that on every fibre, $\pi_{K(\mathbf{g})}$ is given by $z \mapsto z^3$. The Lie algebra of $F = S^1$ is $\mathbf{R}$. So the induced map on the Lie algebra is $(\pi_{K(\mathbf{g})})_* \colon \mathbf{R} \to \mathbf{R}$, $a \mapsto 3a$. From (5.9),

$$\int_{X_{(\mathbf{g})}}^{\mathrm{orb}} e_A(E_{(\mathbf{g})}) = \frac{1}{9} \int_{X_{(\mathbf{g})}}^{\mathrm{orb}} \Pi_{l=1}^2 e_{A_l'}(E_l'),$$

where A_l and A_l' are the connections on bundles E_l and E_l' such that $\pi_{K(\mathbf{g})}^*(A_l') = A_l$. In this moment, the group $K(\mathbf{g})$ acts on the bundle E_l' trivially, so E_l' induces an orbifold bundle E_l'' over the reduced orbifold $X_{(\mathbf{g})}'$. From (5.10),

$$\int_{X_{(\mathbf{g})}}^{\mathrm{orb}} \Pi_{l=1}^2 e_{A_l'}(E_l') = \frac{1}{3} \int_{X_{(\mathbf{g})}'}^{\mathrm{orb}} \Pi_{l=1}^2 e_{A_l''}(E_l''),$$

where A_l'' is the connection on the bundle E_l'' induced from the bundle E_l'. Thus,

$$\int_{X_{(\mathbf{g})}}^{\mathrm{orb}} e_A(E_{(\mathbf{g})}) = \frac{1}{27} \int_{X_{(\mathbf{g})}'}^{\mathrm{orb}} \Pi_{l=1}^2 e_{A_l''}(E_l''). \tag{5.22}$$

We now calculate the integral $\int_{X_{(\mathbf{g})}'}^{\mathrm{orb}} \Pi_{l=1}^2 e_{A_l''}(E_l'')$.

The unformizing system of the bundle E_l'' over the reduced orbifold $X_{(\mathbf{g})}'$ can be described as follows: if $x \in X_{(\mathbf{g})}'$, then $C(\mathbf{g})/K(\mathbf{g}) = 1$ is the trivial group, and the action is trivial.

Now we use the localization technique to compute the integral (5.22). We know that $X_{(\mathbf{g})}' = \overline{O}_\tau$ is a toric variety with $\tau = (v_0, v_1, v_2)$. The three fixed points by the

T-action on $X'_{(\mathbf{g})}$ are p_3, p_4, p_5. We already found that the T-equivariant Euler class of the point p_5 at the normal bundle $e_T(\nu_{p_5}) = \lambda_1\lambda_2$. Similarly, the T-equivariant Euler classes of the points p_4 and p_3 at the normal bundles are

$$e_T(\nu_{p_4}) = (\lambda_1 - \lambda_2)(-\lambda_2), \quad e_T(\nu_{p_3}) = (-\lambda_1 + \lambda_2)(-\lambda_1).$$

The orbifold line bundle $E_l'', (l = 1, 2)$ is trivialized by $\frac{\partial}{\partial x_l} \otimes \overline{\varpi}, (l = 1, 2)$ on $U_5 \cap X'_{(\mathbf{g})}$, $\frac{\partial}{\partial y_l} \otimes \overline{\varpi}, (l = 1, 2)$ on $U_4 \cap X'_{(\mathbf{g})}$, and $\frac{\partial}{\partial w_l} \otimes \overline{\varpi}, (l = 1, 2)$ on $U_3 \cap X'_{(\mathbf{g})}$. So from (5.15), the action of T on E_l'' at the fixed points p_5, p_4, p_3 has weights $0, (-2\lambda_2), (-2\lambda_1)$, respectively. From (5.16), we have

$$\int_{X'_{(\mathbf{g})}}^{\mathrm{orb}} \Pi_{t=1}^2 e_{A_t''}(E_t'') = \frac{0}{\lambda_1\lambda_2} + \frac{(-2\lambda_2)^2}{(\lambda_1 - \lambda_2)(-\lambda_2)} + \frac{(-2\lambda_1)^2}{(-\lambda_1 + \lambda_2)(-\lambda_1)} = 4.$$

From (5.22),

$$\int_{X_{(\mathbf{g})}}^{\mathrm{orb}} e_A(E_{(\mathbf{g})}) = \frac{4}{27}.$$

And by (5.21),

$$\langle \eta_1, \eta_2, \eta_3 \rangle_{\mathrm{orb}} = \frac{4}{27}\eta_1\eta_2\eta_3.$$

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