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HOLOMORPHIC AUTOMORPHISMS OF THE UNIT BALLS OF HILBERT C*-MODULES

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Abstract. We show that every Hilbert C*-module E is a JB*-triple in a canonical way, establish an explicit expression for the holomorphic automorphisms of the unit ball of E, discuss the existence of fixed points for these automorphisms and give sufficient conditions for E to have the density property.

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1.1. Introduction. Hilbert C*-modules first appeared in 1953 in a paper of Kaplansky [**7**] who worked only with modules over commutative unital C*-algebras. In 1973 Paschke [**14**] proved that most of the properties of Hilbert C*-modules were valid for modules over an arbitrary C*-algebra. About the same time Rieffel independently developed much of the same theory and used it to study representations of C*-algebras. Since then the subject has grown and spread rapidly and now there is an extensive literature on the topic; (see [**12**] or [**13**] for a systematic introduction). Many interesting developments have been made by Kasparov, who used Hilbert C*-modules as the framework for K-theory. More recently Hilbert C*-modules have been a useful tool in the C*-algebraic approach to quantum groups. The geometry of Hilbert C*-modules has been investigated by Solel in [**15**], where the isometries of these Banach spaces have been characterized.

On the other hand Kaup, searching for a metric-algebraic setting in which he could make the study of bounded symmetric domains in complex Banach spaces, introduced a class of complex Banach spaces called JB*-triples. In 1983 he proved that, except for a biholomorphic bijection, every such domain is the open unit ball of a JB*-triple [**8**]. In 1981 he made the complete analytic classification of bounded symmetric domains in reflexive Banach spaces [**9**]. Since then the study of JB*-triples has grown and spread considerably.

These two theories have developed independently from one another. Here we show that every Hilbert C*-module E is in a canonical way a JB*-triple, a bridge between the two theories that may be useful in the study of the geometry of Hilbert C*-modules. In particular the open unit ball B_E of E is a bounded symmetric domain (see [**8**]). We establish an explicit expression of the holomorphic automorphisms of the unit ball B_E of a Hilbert C*-module E and a sufficient condition for a Hilbert module E to have the density property (see [**2**]). Especial interest has been paid to the case of *selfdual* standard Hilbert modules $E = \ell_2(S, A)$. For this type of module we prove: (i) that E

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always has the density property [2], (ii) that every holomorphic automorphism of B_E has at least one fixed point in $\overline{B_E}$, which extends some results in [4].

1.2. Hilbert C*-modules. We now introduce formally the objects we shall be studying. Let A be a C*-algebra (not necessarily unital or commutative), where the product is denoted by juxtaposition xy, the norm is $\|\cdot\|_A$ and the mapping $x \mapsto x^*$ is conjugation. An *inner product* A-module is a complex linear space E with two laws of composition $E \times A \to E$ (denoted by $(x, a) \mapsto x \cdot a$) and $E \times E \to A$ (denoted by $(x, y) \mapsto \langle x, y\rangle$) such that the following properties hold.

1. With respect to the operation $(x, a) \mapsto x \cdot a$, E is a right A-module with a compatible scalar multiplication; that is, $\lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a)$, for all $x \in E$, $a \in A$ and $\lambda \in \mathbb{C}$.
2. The inner product $(x, y) \mapsto \langle x, y\rangle$ satisfies

$$\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle, \tag{1}$$
$$\langle x, y \cdot a\rangle = \langle x, y\rangle a, \tag{2}$$
$$\langle y, x\rangle = \langle x, y\rangle^*, \tag{3}$$
$$\langle x, x\rangle \geq 0 \quad \text{and if} \quad \langle x, x\rangle = 0, \quad \text{then} \quad x = 0. \tag{4}$$

Here $x, y, z \in E, \alpha, \beta \in \mathbb{C}$ and $a \in A$.

Note that, in particular, the inner product is complex linear in the second variable while it is conjugate linear in the first. This convention is in line with the recent research literature. Let E be an inner product A-module; then the Cauchy-Schwarz inequality

$$\langle y, x\rangle\, \langle x, y\rangle \leq \|\langle x, x\rangle\|\, \langle y, y\rangle$$

holds. Hence $\|x\|_E^2 := \|\langle x, x\rangle\|_A$ is a norm in E with respect to which the inner product and the module product are continuous; that is

$$\|\langle x, y\rangle\|_A \leq \|x\|_E\, \|y\|_E, \qquad \|x \cdot a\| \leq \|x\|_E\, \|a\|_A.$$

To simplify notation we shall use the same symbol $\|\cdot\|$ to denote the norms on A and E. As usual we set $|x| := \langle x, x\rangle^{\frac{1}{2}}$, for $x \in E$. Then

$$|\langle x, y\rangle| \leq \|x\|\, |y|, \qquad |\langle x, y\rangle| \leq |x|\, \|y\|,$$

and the module product satisfies

$$\|x \cdot a\| \leq \|x\|\, |a|.$$

An inner product A-module E, which is a Banach space with respect to $\|\cdot\|$, is called a *Hilbert C*-module* over A.

Example 1.1. Let S and A be a non empty set and a C*-algebra, respectively, and denote by $\ell_2(S, A)$ the set of all indexed A-valued families $x = (x_s)_{s\in S}$ such that $\sum_{s\in S} x_s^* x_s$ converges in A. With the pointwise operations and the inner product

$$\langle y, x\rangle := \sum_{s\in S} y_s^* x_s, \qquad y,\ x \in \ell^2(S,\ A),$$

$M := \ell^2(S, A)$ becomes a Hilbert A-module. We refer to it as the *standard* Hilbert module over A; see [13, Ex.1.3.5]. If S is a countably infinite set we write

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$\ell^2(A) := \ell^2(\mathbb{N}, A)$. When S consists of a single point we get $M = A$ with the natural module operation $x \cdot a := xa$ and the inner product $\langle a, b\rangle := a^* b$, for $x, a, b \in A$.

LEMMA 1.2. *Let $M = \ell^2(A)$ be the standard Hilbert module over the C^*-algebra A. Then for every fixed $y = (y_n) \in M$ the series $\langle y, x\rangle = \sum_n y_n^* x_n$ converges uniformly over the bounded subsets of M.*

Proof. It is known [**13**, Ex.1.3.5] that the above series converges in A for every fixed pair y, x in M. Let $S \subset M$ be a bounded set and put $K := \sup_{x\in S} \|\langle x, x\rangle\| < \infty$. Let $\epsilon > 0$ be given. Since $|y|^2 = \langle y, y\rangle = \sum_1^\infty y_n^* y_n$ is convergent, there is an index N such that

$$\left\| \sum_{N+1}^{\infty} y_n^* y_n \right\| \leq \frac{\epsilon}{K},$$

and therefore by the Cauchy-Schwarz inequality

$$\begin{aligned}\left\| \langle y, x\rangle - \sum_1^N y_k^* x_k \right\| &= \left\| \sum_{N+1}^{\infty} y_k^* x_k \right\| \leq \left\| \sum_{N+1}^{\infty} y_k^* y_k \right\| \left\| \sum_{N+1}^{\infty} x_k^* x_k \right\| \\ &\leq \|\langle x, x\rangle\| \left\| \sum_{N+1}^{\infty} y_k^* y_k \right\| \leq K \left\| \sum_{N+1}^{\infty} y_k^* y_k \right\| \leq \epsilon,\end{aligned}$$

which completes the proof. □

A linear map $f: E \to E$ is called an *A-map* if $f(x \cdot a) = f(x) \cdot a$ holds, for all $x \in E$ and $a \in A$, and we say that f is *adjointable* if there exists an A-map $f^*: E \to E$ such that

$$\langle f(x), y\rangle = \langle x, f^*(y)\rangle, \qquad x, y \in E.$$

In such a case f is continuous (though the converse is not true!), f^* is adjointable and $(f^*)^* = f$. We let $\mathcal{L}(E)$ denote the Banach algebra of all bounded complex linear operators on E and $\mathcal{A}(E) \subset \mathcal{L}(E)$ is the vector space of all adjointable A-module maps on E. In fact $\mathcal{A}(E)$ is a C*-algebra in the operator norm since $\|f^* f\| = \|f\|^2$ holds, for all $f \in \mathcal{A}(E)$. For $x,\ y \in E$ we define $\theta_{x,y}$ (also denoted $x \otimes y^*$) by

$$\theta_{x,y}(z) := x \cdot \langle y, z\rangle \qquad (z \in E).$$

Then $\theta_{x,y}$ is adjointable and $\theta_{x,y}^* = \theta_{y,x}$ (see [**12**, p. 9]). For later reference we state the following result.

LEMMA 1.3. *Let E be a Hilbert C^*-module and let $f: E \to E$ be a bounded A-module map. Then f is a positive element in the C^*-algebra $\mathcal{A}(E)$ if and only if $\langle x, f(x)\rangle \geq 0$, for all $x \in E$.*

We refer to [**12**] or [**13**] for background on Hilbert C*-modules and for the proofs of the results above.

1.3. JB*-triples. A complex Banach space Z with a continuous mapping $(a, b, c) \mapsto \{a, b, c\}$ from $Z \times Z \times Z$ to Z is called a *JB*-triple* if the following

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conditions are satisfied for all $a, b, c, d \in Z$, where the operator $a \square b \in \mathcal{L}(Z)$ is defined by $z \mapsto \{abz\}$ and $[\, ,\,]$ is the commutator product.

1. $\{abc\}$ is symmetric complex linear in a, c and conjugate linear in b.
2. $[a \square b,\ c \square d] = \{a, b, c\} \square d - c \square \{d, a, b\}$ (called the *Jordan identity*).
3. $a \square a$ is hermitian and has non-negative spectrum.
4. $\|\{a, a, a\}\| = \|a\|^3$.

If a complex vector space Z admits a JB*-triple structure, then the norm and the triple product determine each other. An *automorphism* is a linear bijection $\phi \in \mathcal{L}(Z)$ such that $\phi\{z, z, z\} = \{(\phi z), (\phi z), (\phi z)\}$ for $z \in Z$, which occurs if and only if ϕ is a surjective linear isometry of Z. For $z \in Z$, the conjugate linear operator $Q(z)\colon Z \to Z$ is defined by $Q(z)x := \{z, x, z\}$ where $x \in Z$. For x, y in Z, the operator $x \square y$ is sometimes denoted by $D(x, y)$ and the *Bergmann operator* $B(x, y) \in \mathcal{L}(Z)$ is defined by

$$B(x, y)z := z - 2(x \square y)z + Q(x)Q(y)z \qquad (z \in Z).$$

An element $e \in Z$ is called a *tripotent* if $\{eee\} = e$. In this case the set of eigenvalues of $e \square e \in \mathcal{L}(Z)$ is contained in $\{0, \frac{1}{2}, 1\}$ and we have the topological direct sum decomposition, called the *Peirce decomposition* of Z, given by

$$Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e).$$

Here $Z_k(e)$, the *Peirce subspaces of* e, are the k-eigenspaces of $e \square e$ and the *Peirce projections* are given by

$$P_1(e) = Q^2(e), \qquad P_{1/2}(e) = 2(e \square e - Q^2(e)), \qquad P_0(e) = \mathsf{Id} - 2e \square e + Q^2(e).$$

A closed subspace $J \subset Z$ is called a *subtriple* of Z if $\{J, J, J\} \subset J$ and an *ideal* if $\{Z, J, Z\} \subset J$, $\{J, Z, Z\} \subset J$. The Peirce subspaces of a tripotent e are subtriples and $P_1(e) \square P_0(e) = \{0\}$.

Recall that every C*-algebra Z is a JB*-triple with respect to the triple product $2\{abc\} := (ab^*c + cb^*a)$. In this case, every projection in Z is a tripotent and more generally the tripotents are precisely the partial isometries in Z. C*-algebra derivations and C*-automorphisms are derivations and automorphisms of Z as a JB*-triple although the converse is not true.

We refer to **[8]**, **[16]** and the references therein for the background of JB*-triples theory.

1.4. Hilbert C*-modules are JB*-triples. Let E be a Hilbert module over the C*-algebra A. For $a \in A$ fixed, we denote by $R_a \in \mathcal{L}(E)$ the operator $x \mapsto x \cdot a$ of right multiplication by a.

THEOREM 1.4. *Every Hilbert C*-module E is a JB*-triple in a canonical way.*

Proof. Let E be a Hilbert C*-module over the C*-algebra A. Define a triple product in E by

$$2\,\{x, y, z\} := x \cdot \langle y, z\rangle + z \cdot \langle y, x\rangle \qquad (x, y, z \in E). \tag{5}$$

It is clear that $\{\cdot, \cdot, \cdot\}$ is symmetric complex linear in the external variables, and complex conjugate linear in the middle variable. It is a matter of routine calculation to check

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that the triple product satisfies the Jordan identity. On the other hand, for fixed $x \in E$ we have

$$2\,(x \,\square\, x)\, z = x \cdot \langle x, z\rangle + z \cdot \langle x, x\rangle \qquad (z \in E),$$

which can be written in the form $x \,\square\, x = \frac{1}{2}(\theta_{x,x} + R_{|x|^2})$. We show that the summands in the right hand side of the latter are hermitian elements in the algebra $\mathcal{L}(E)$. Since $\mathcal{A}(E)$ is a closed complex subalgebra of $\mathcal{L}(E)$ and contains the unit element, it suffices to consider the numerical ranges of $\theta_{x,x}$ and $R_{|x|^2}$, viewed as elements in the C*-algebra $\mathcal{A}(E)$. We have seen before that $\theta_{x,x}$ is selfadjoint. Clearly

$$\big(\exp\, itR_{|x|^2}\big)\,(w) = w \cdot (\exp\, it|x|^2) \qquad (w \in E),$$

and, as $\exp\, it|x|^2$ is a unitary element in A, the operator $\exp\, itR_{|x|^2}$ is an isometry of E, for all $t \in \mathbb{R}$, which shows that $R_{|x|^2}$ is hermitian. For $y \in E$ we have

$$\langle y, \theta_{x,x}(y)\rangle = \langle y, x \cdot \langle x, y\rangle\rangle = \langle y, x\rangle\langle x, y\rangle \geq 0,$$

which by (1.3) proves that $\theta_{x,x} \geq 0$ in $\mathcal{A}(E)$ and hence also in $\mathcal{L}(E)$. Clearly $|x|^2 \geq 0$ in A, and so its spectrum satisfies $\sigma_A(|x|^2) \subset [0, \infty)$. Therefore

$$\sigma_{\mathcal{L}(A)}\big(R_{|x|^2}\big) \subset \sigma_A(|x|^2) \subset [0, \infty).$$

Since the numerical range is the convex hull of the spectrum, $R_{|x|^2} \geq 0$ as we wanted to check.

Let us set $y := \langle x, x\rangle \in A$ for every $x \in E$. The definition of the norm in E and the properties of the norm in the C*-algebra A yield

$$\begin{aligned}\|\{x, x, x\}\|^2 &= \|x \cdot \langle x, x\rangle\|^2 = \|\langle x \cdot \langle x, x\rangle, x \cdot \langle x, x\rangle\rangle\| = \|\langle x, x\rangle\langle x, x\rangle\langle x, x\rangle\| \\ &= \|\{y, y, y\}\| = \|y\|^3 = \|\langle x, x\rangle\|^3 = \|x\|^6,\end{aligned}$$

which establishes property (4) in the definition of a JB*-triple. Finally, this is the unique JB*-triple structure on E, since the triple product is determined by the norm of E. □

1.5. Submodules and subtriples. Let E be a Hilbert module over the C*-algebra A. For subsets $F \subset E$ and $B \subset A$, we set

$$F^{\bullet} := \{a \in A : F \cdot a = 0\}, \qquad {}^{\bullet}B := \{x \in E : x \cdot B = 0\}.$$

PROPOSITION 1.5. *Let E be a Hilbert module over a C*-algebra A, and assume that $E^{\bullet} = \{0\}$. Then for every $c \in E$ the following conditions are equivalent.*

(i) *$R_{|c|^2}$ is a module map.*

(ii) *$D(c, c)$ is a module map.*

(iii) *$|c|^2$ is a central element of A.*

If these conditions hold, then $Q(c)^2$ and $B(c, c)$ are module maps.

Proof. (i) $\Longleftrightarrow$ (iii). We have

$$R_{|c|^2}(x \cdot a) = (x \cdot a) \cdot |c|^2, \qquad \big(R_{|c|^2}(x)\big) \cdot a = (x \cdot |c|^2) \cdot a.$$

Thus $R_{|c|^2}$ is a module map if and only if $x \cdot (a|c|^2 - |c|^2 a) = 0$, for all $x \in E$ and $a \in A$. Hence the result follows from $E^{\bullet} = 0$.

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(iii) $\Longleftrightarrow$ (ii). We have $2D(c, c) = c \otimes c^* + R_{|c|^2}$. Since $c \otimes c^*$ is a module map, $D(c, c)$ is a module map if and only if $R_{|c|^2}$ is too, which occurs if and only if $|c|^2$ is a central element of A. The conclusion follows from the expressions $Q(c)^2 = c \otimes c^* \circ R_{|c|^2}$ and $B(c, c) = \mathsf{Id} - 2D(c, c) + Q(c)^2$. $\square$

Let $e \in E$ be a tripotent. By the above, in general neither the Peirce projectors of e are module maps nor are the Peirce spaces of e submodules of E.

Recall that the odd powers of an element x in a JB*-triple Z are defined inductively by $x^1 := x$ and $x^{2k+1} := D(x, x)^k x$ for $k \geq 1$, and that the closed JB*-subtriple generated by x is the closed linear span of the set $\{x^{2k+1} : k \in \mathbb{N}\}$.

For $x \in E$, we denote by E_x^M and E_x^J the closed submodule and the closed subtriple generated by x in E.

Lemma 1.6. *We have* $E_x^J \subset E_x^M = \overline{x \cdot A}$.

Proof. Since E_x^M is a module that contains x, it is a triple that contains x and so $E_x^J \subset E_x^M$. Clearly E_x^M must contain $\overline{x \cdot A}$ and, as the latter is a closed module that contains x, the result follows. Note that the inclusion $E_x^J \subset E_x^M$ in general is strict. Indeed, $\{x, x, x\} = x \cdot \langle x, x\rangle = x \cdot |x|^2$ and an induction argument gives $x^{2k+1} = x \cdot |x|^{2k}$, for all $k \in \mathbb{N}$. Since E_x^J is the closed $\mathbb{C}$-linear span of the set of odd powers of x, we have $E_x^J \subset \overline{x \cdot B}$, where B is the C*-algebra generated by $|x|^2$ in A, which obviously may be strictly smaller than A. $\square$

The following characterizes those Hilbert C*-modules that can be identified with a C*-algebra in the sense that they are associated to a C*-algebra via Example 1.1.

Proposition 1.7. *Let E be a Hilbert module over a unital C*-algebra A. Then E is the module associated to A if and only if there exists an element $x \in E$ such that $|x|^2 = \mathbf{1}$ and $x \cdot A = E$.*

Proof. The element $x := \mathbf{1}$ clearly satisfies the conditions above, and so the forward implication holds. Suppose now that $x \in E$ satisfies $|x|^2 = \mathbf{1}$ and $x \cdot A = E$; let $E_x^M = \overline{x \cdot A}$ be the closed submodule generated by x in E. The map $\Phi\colon A \to \overline{x \cdot A}$ given by $a \mapsto \Phi(a) := x \cdot a$ is a JB*-triple homomorphism since

$$\Phi(a)^3 = \{x \cdot a, x \cdot a, x \cdot a\} = (x \cdot a) \cdot \langle x \cdot a, x \cdot a\rangle = x \cdot aa^* \langle x, x\rangle a = x \cdot aa^* a = \Phi(a^3).$$

It is injective since $\Phi(a) = 0$ gives $x \cdot a = 0$. Taking the scalar product with x we get $0 = \langle x, x \cdot a\rangle = \langle x, x\rangle a = |x|^2 a = a$ so that $a = 0$. By assumption $\Phi(A) = x \cdot A = E$. Thus $\Phi\colon A \to E$ is a surjective linear isometry which is also a module map, since

$$\Phi(wa) = x \cdot (wa) = (x \cdot w) \cdot a = \Phi(w) \cdot a, \qquad w \in A, \ a \in A.$$

This shows that Φ is a Hilbert C*-module isomorphism. $\square$

1.6. Holomorphic automorphisms of the unit ball. Motivated by the deep formal analogy between Hilbert C*-modules E and Hilbert spaces H, we shall establish an explicit formula for the holomorphic automorphisms of the unit ball of E. Set $Q_c := Q(c)$ and recall **[8]** that

$$B(c, c)(x) := x - 2(c \,\square\, c)(x) + Q_c^2(x) \qquad (x \in E).$$

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In our case

$$\begin{aligned}2(c\,\Box\, c)(x) &= 2\{c, c, x\} = c\cdot\langle c, x\rangle + x\cdot|c|^2 = c\otimes c^*(x) + x\cdot|c|^2,\\ Q_c^2(x) &= \{c, Q_c(x), c\} = \{c, c\cdot\langle x, c\rangle, c\}\\ &= c\cdot\langle c\cdot\langle x, c\rangle, c\rangle = c\cdot\langle c, x\rangle|c|^2 = (c\otimes c^*)(x\cdot|c|^2).\end{aligned}$$

Therefore

$$\begin{aligned}B(c, c)(x) &= x\cdot(\mathbf{1} - |c|^2) - (c\otimes c^*)(x\cdot(\mathbf{1} - |c|^2))\\ &= (\mathbf{1} - c\otimes c^*)(x\cdot(\mathbf{1} - |c|^2)).\end{aligned}$$

Clearly $\mathbf{1} - c\otimes c^*$ and $\mathbf{1} - |c|^2$ are selfadjoint elements in the C*-algebras $\mathcal{A}(E)$ and A, respectively, and for c in the open unit ball of E they are positive. Hence they have well defined square roots. We show that the operator B_c defined by

$$B_c(x) := (\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\left(x\cdot(\mathbf{1} - |c|^2)^{\frac{1}{2}}\right) \qquad (x\in E),$$

satisfies $B_c^2 = B(c, c)$. Indeed, since $\mathbf{1} - c\otimes c^*$ is an A-linear map, so is its square root and we have

$$\begin{aligned}B_c(B_c(x)) &= (\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\left(B_c(x)\cdot(\mathbf{1} - |c|^2)^{\frac{1}{2}}\right)\\ &= \left((\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}B_c(x)\right)\cdot\left(\mathbf{1} - |c|^2)^{\frac{1}{2}}\right)\\ &= \left((\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\left[(\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}x\cdot(\mathbf{1} - |c|^2)^{\frac{1}{2}}\right]\right)\cdot(\mathbf{1} - |c|^2)^{\frac{1}{2}}\\ &= (\mathbf{1} - c\otimes c^*)x(\mathbf{1} - |c|^2) = B(c, c)(x),\end{aligned}$$

as we wanted to check.

For c and x in the open unit ball of E, $\mathbf{1} + \langle c, x\rangle$ is an invertible element in A, since $\|\langle c, x\rangle\| \le \|c\|\,\|x\| < 1$. In particular, $x\cdot(\mathbf{1} + \langle c, x\rangle)^{-1} = (\mathbf{1} + x\,\Box\, c)^{-1}x$ is well defined in A. Recall [**8**] that for c in the open unit ball of E, the *transvection* g_c is the holomorphic automorphism of the open ball of E given by

$$g_c(x) := c + B_c((\mathbf{1} + x\,\Box\, c)^{-1}x) \qquad (\|x\| < 1).$$

Replacing the expressions of B_c and $(\mathbf{1} + x\,\Box\, c)^{-1}x$ we get

$$g_c(x) = c + (\mathbf{1} - c\otimes c^*)^{\frac{1}{2}}\left[x\cdot(\mathbf{1} + \langle c, x\rangle)^{-1}(\mathbf{1} - |c|^2)^{\frac{1}{2}}\right].$$

In case E is a Hilbert space, the above can be stated in terms of projections and coincides with the formula for the transvections of the ball given in [**3**, p. 21]. By [**8**], every holomorphic automorphism h of the unit ball of E can be represented in the form $h = L\circ g_c$, for some surjective linear isometry of E and some $c\in E$ with $\|c\| < 1$.

For a complex Banach space E, the set $\operatorname{Extr} B_E$ of extreme points in the unit ball $\overline{B_E}$ of E plays an important role in the study of the geometry of E. Obviously, we can replace a Hilbert C*-module E with its associated JB*-triple in order to study the extreme points of the ball $\overline{B_E}$. By [**10**, Proposition 3.5] we have

$$\operatorname{Extr} B_E = \{c\in E\colon B(c, c) = 0\}.$$

Thus, for $c \in E$ the condition $c \in \mathsf{Extr}\, B_E$ is equivalent to

$$(\mathbf{1} - c \otimes c^*)(x \cdot (\mathbf{1} - |c|^2)) = 0 \quad \forall x \in E.$$

Therefore we have two obvious families of extreme points given by

$$E \cdot (\mathsf{1} - |c|^2) = \{0\} \Longrightarrow c \in \mathsf{Extr}\, B_E,$$
$$(\mathsf{Id} - c \otimes c^*)\, E = \{0\} \Longrightarrow c \in \mathsf{Extr}\, B_E.$$

These two families may coincide (as it occurs when E is Hilbert space) but in general they are different. We do not know whether every extreme point lies in one of the above families. Every extreme point is a tripotent; that is, it satisfies $c = \{c, c, c\} = c \cdot \langle c, c\rangle = c \cdot |c|^2$.

It might be interesting to characterize Hilbert C*-modules within the category of JB*-triples.

1.7. The density property for Hilbert C*-modules. Suppose that Z is a Hilbert C*-module over the C*-algebra A. Then $B(x, y)$ can be written in the form

$$\begin{aligned} B(x, y)z &= z - (z \cdot \langle y, x\rangle + x \cdot \langle y, z\rangle) + x \cdot \langle y, z\rangle\langle y, x\rangle \\ &= z \cdot (\mathsf{1} - \langle y, x\rangle) + x \cdot \langle y, z\rangle(\langle y, x\rangle - \mathsf{1}\,) = (\mathsf{Id} - x \otimes y^*)z \cdot (\mathsf{1} - \langle y, x\rangle). \end{aligned}$$

Thus, in particular,

$$B(x, y) = (\mathsf{Id} - x \otimes y^*) \circ R_{\mathsf{1} - \langle y,x\rangle}. \tag{6}$$

We recall that a pair (x, y) of elements in a JB*-triple is said to be *quasi-invertible* if $B(x, y)$ is invertible in $\mathcal{L}(Z)$. In this case the quasi-inverse of x relative to y is defined by

$$x^y := B(x, y)^{-1}(x - Q(x)y).$$

A JB*-triple Z has the density property if the set of all quasi-invertible pairs in Z is dense in $Z \times Z$, see [**2**].

THEOREM 1.8. *Let Z be a Hilbert module over the C*-algebra A and let (x, y) be any pair in $Z \times Z$.*

(i) *If (x, y) is quasi-invertible, then $R_{\mathsf{1}-\langle y,x\rangle}$ is invertible in $\mathcal{L}(Z)$.*

(ii) *If A is unital and $\mathsf{1} - \langle x, y\rangle$ is invertible in A, then (x, y) is quasi-invertible in Z and*

$$x^y = x \cdot (\mathsf{1} - \langle y, x\rangle)^{-1}.$$

Proof. (i) is obvious. Assume that $\mathsf{1} - \langle y, x\rangle$ is invertible in A. Set $a = (\mathsf{1} - \langle y, x\rangle)^{-1}$. Then $R_{\mathsf{1} - \langle y,x\rangle}$ is invertible in $\mathcal{L}(Z)$ and $R_a^{-1} = R_{a^{-1}}$. Hence by (6) we have to show that $\mathsf{Id} - x \otimes y^*$ is invertible in $\mathcal{L}(Z)$ and so it suffices to prove the existence of a *linear* operator which is the inverse of $\mathsf{Id} - x \otimes y^*$, since then such an inverse is automatically continuous. That is, we have to show that, for every $w \in Z$, the equation

$$z - x \cdot \langle y, z\rangle = w \tag{7}$$

has a unique solution $z = z(w) \in Z$. Suppose that it has a solution. Taking the inner product with y we get

$$\langle y, z\rangle - \langle y, x\rangle\langle y, z\rangle = \langle y, w\rangle;$$

that is $(1 - \langle y, x\rangle)\langle y, z\rangle = \langle y, w\rangle$. Since $1 - \langle y, x\rangle$ is invertible, this gives $\langle y, z\rangle = a\langle y, w\rangle$ which replaced in (7) yields

$$z = w + x \cdot a\langle y, w\rangle \tag{8}$$

as the only possible solution. We check now that (8) actually is a solution of (7). We must verify that

$$w + x \cdot a\langle y, w\rangle - x \cdot \langle y, w + x \cdot a\langle y, w\rangle\rangle = w$$

holds in Z. After obvious cancellations, the above becomes, by the properties of the inner product,

$$x \cdot a\langle y, w\rangle = x \cdot \langle y, w\rangle + x \cdot \langle y, x\rangle a\langle y, w\rangle,$$

and it suffices to check that

$$a\langle y, w\rangle = \langle y, w\rangle + \langle y, x\rangle a\langle y, w\rangle$$

holds in A. Multiplying on the left by a^{-1} and using the fact that $\langle y, x\rangle$ commutes with $a = (1 - \langle y, x\rangle)^{-1}$, the latter becomes

$$\langle y, w\rangle = (1 - \langle y, x\rangle)\langle y, w\rangle + \langle y, x\rangle\langle y, w\rangle,$$

which is true. Thus

$$(\mathrm{Id} - x \otimes y^*)^{-1} w = w + x \cdot (1 - \langle y, x\rangle)^{-1}\langle y, w\rangle \qquad (w \in Z). \tag{9}$$

Finally, since $x - Q(x)y = x \cdot (1 - \langle y, x\rangle)$ and $(\mathrm{Id} - x \otimes y^*)^{-1}$ is a module map,

$$\begin{aligned} x^y &= B(x, y)^{-1}(x - Q(x)y) = B(x, y)^{-1} x \cdot (1 - \langle y, x\rangle) \\ &= \left(R_{1 - \langle y, x\rangle}\right)^{-1}[(\mathrm{Id} - x \otimes y^*)^{-1} x \cdot (1 - \langle y, x\rangle)] \\ &= [(\mathrm{Id} - x \otimes y^*)^{-1} x \cdot (1 - \langle y, x\rangle)] \cdot (1 - \langle y, x\rangle)^{-1} \\ &= (\mathrm{Id} - x \otimes y^*)^{-1} x, \end{aligned}$$

and, by (9), for $w = x$ we have

$$\begin{aligned} (\mathrm{Id} - x \otimes y^*)^{-1} x &= x + x \cdot a\langle y, x\rangle = x \cdot [1 + (1 - \langle y, x\rangle)^{-1}\langle y, x\rangle] \\ &= x \cdot (1 - \langle y, x\rangle)^{-1}[(1 - \langle y, x\rangle) + \langle y, x\rangle] = x \cdot (1 - \langle y, x\rangle)^{-1}, \end{aligned}$$

as we wanted to prove. □

COROLLARY 1.9. *Let Z be a Hilbert module over a C^*-algebra A. Assume that the set $S(Z) := \{(x, y) \in Z \times Z : 1 - \langle y, x\rangle$ is invertible in $A\}$ is dense in $Z \times Z$. Then Z has the density property.*

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For a Hilbert C*-module E we set

$$\langle E, E\rangle := \{\langle x, y\rangle : x, y \in E\}.$$

Then E is said to be *strongly full* if $\langle E, E\rangle = A$. Note that $\langle E, E\rangle$ is usually defined as the complex linear span in A of the above set of inner products. The Hilbert module associated to a unital C*-algebra A and the standard Hilbert module $\ell^2(A)$ are strongly full since $\mathbf{1} = \mathbf{1}^*\mathbf{1} = \langle \mathbf{1}, \mathbf{1}\rangle$; hence $a = \langle \mathbf{1}, \mathbf{1} \cdot a\rangle \in \langle E, E\rangle$, for all $a \in A$. We also recall that whenever A is unital, the set $\mathsf{Inv}(A)$ of invertible elements is a non void open set.

THEOREM 1.10. *Let E be a strongly full Hilbert C*-module over a C*-algebra A such that $\mathsf{Inv}(A)$ is a dense subset. Let $(a, b) \in E \times E$ be given. Then for every $\epsilon > 0$ there are elements $(a', b') \in E \times E$ such that*

$$\|a' - a\| < \epsilon, \qquad \|b' - b\| < \epsilon, \qquad \mathbf{1} - \langle a', b'\rangle \in \mathsf{Inv}(A).$$

Proof. The inner product $p: (x, y) \mapsto \langle y, x\rangle$ is clearly a continuous homogeneous *real* polynomial $p := E \times E \to A$ of degree one on each variable. Clearly

$$\begin{aligned} A = \langle E, E\rangle = p(E \times E) &= p\left[\bigcup_{n=1}^{\infty}(n\,B_\epsilon(a) \times n\,B_\epsilon(b))\right] \\ &= \bigcup_{n=1}^{\infty} n^2 p(B_\epsilon(a) \times B_\epsilon(b)) \subset \bigcup_{n=1}^{\infty} n^2\overline{p(B_\epsilon(a) \times B_\epsilon(b))} \subset A, \end{aligned}$$

and therefore

$$A = \bigcup_{n=1}^{\infty} n^2\, \overline{p(B_\epsilon(a) \times B_\epsilon(b))}.$$

For $S \subset A$ we let $\mathsf{int}\, S$ denote its interior. Since A is a complete metric space, by Baire's category theorem the above ensures that $\mathsf{int}\,\overline{p(B_\epsilon(a) \times B_\epsilon(b))} \neq \emptyset$, whence applying a translation we get $\mathbf{1} - \mathsf{int}\,\overline{p(B_\epsilon(a) \times B_\epsilon(b))} \neq \emptyset$. Since $\mathsf{Inv}(A) \subset A$ is a dense set, it must intersect every non void open subset in A and in particular

$$[\mathbf{1} - \overline{p(B_\epsilon(a) \times B_\epsilon(b))}] \cap \mathsf{Inv}(A) \neq \emptyset.$$

Hence there are a non void open set $\Omega \subset \mathsf{Inv}\, A$ and a point $x_0 \in \Omega$ such that

$$x_0 \in \Omega \subset \mathbf{1} - \overline{p(B_\epsilon(a) \times B_\epsilon(b))} = \overline{\mathbf{1} - p(B_\epsilon(a) \times B_\epsilon(b))}$$

which in turn, by the definition of closure, yields

$$\Omega \cap (\mathbf{1} - p(B_\epsilon(a) \times B_\epsilon(b)) \neq \emptyset.$$

Thus we can find a point $y_0 \in \Omega \subset \mathsf{Inv}(A)$ with $y_0 \in \mathbf{1} - p(B_\epsilon(a) \times B_\epsilon(b))$, or in other words there are $a' \in B_\epsilon(a)$ and $b' \in B_\epsilon(b)$ such that $y_0 = \mathbf{1} - \langle a', b'\rangle$ is invertible. □

COROLLARY 1.11. *If E is a strongly full Hilbert module over a C*-algebra A such that $\mathsf{Inv}(A)$ is a dense subset of A, then E has the density property.*

COROLLARY 1.12. *Assume that $E := \ell^2(A)$ is a selfdual Hilbert module over the C*-algebra A. Then E has the density property.*

Proof. The selfduality of E entails $\dim A < \infty$. Hence A is unital and $\mathsf{Inv}(A)$ is a dense set in A. □

1.8. Fixed points in the unit ball. Despite the formal analogy between the definitions of Hilbert C*-modules and Hilbert spaces, there are deep differences in the analytic behaviour of these two structures. It is well known [**19**] that every holomorphic automorphism h of the open unit ball B_Z of a JB*-triple Z extends uniquely by continuity to the closure $\overline{B}_Z$. If Z is a Hilbert space, then h has at least one fixed point in $\overline{B}_Z$ [**4**], a result that is no longer true for a general Hilbert C*-module, as proved by the following counterexample due to Stachó [**17**].

Counterexample 1.13. Let Ω be the closed unit disk of the complex plane $\mathbb{C}$ and denote by $A := \mathcal{C}(\Omega)$ the Banach space of all continuous complex valued functions on Ω, with the supremum norm. Then A is a unital abelian C*-algebra with the usual operations; hence it is a Hilbert A-module in the inner product $\langle f, g\rangle := \overline{f}g, f, g \in A$. The transformation $F\colon B_A \to B_A$ of the unit ball of A defined by

$$F : f \mapsto F(f), \qquad F(f)(z) = \frac{f(z) + \frac{1}{2}z}{1 + \frac{1}{2}\overline{z}f(z)} \qquad (z \in \Omega),$$

is a holomorphic automorphism of B_A that has no fixed point in $\overline{B}_A$ since $F(f_0) = f_0$ implies that $f_0(z)^2 = z/\overline{z}$, for $z \in \Omega\backslash\{0\}$, which contradicts the continuity of f_0 at the origin.

Theorem 1.14. *Let $M := \ell_2(A)$ be the standard Hilbert module over a W^*-algebra A. Assume that M is selfdual. Then every holomorphic automorphism of the unit ball B_M has at least one fixed point in $\overline{B}_Z$.*

Proof. By [**13**, Proposition 3.3.3] M is a dual Banach space so that M is a JBW*-triple and has a unique predual M_*. Thus the weak*-topology (or the $\sigma(M, M_*)$ topology) is well defined in M. As proved in [**13**, p. 181], this topology is generated by the family of seminorms

$$p_{\phi,u}(x) := \phi(\langle u, x\rangle) \qquad (x \in M),$$

where ϕ ranges over the set of states in the W^*-algebra A and u ranges over M. Let $h = \lambda \circ g_c$ be a holomorphic automorphism of B_M, where λ is a surjective linear isometry of M and g_c is a transvection of the ball B_M. It is well known that surjective linear isometries of a JBW*-triple are $w^* - w^*$-continuous and we shall prove below that g_c is also $w^* - w^*$-continuous on the ball. Since $\overline{B}_M$ is a convex w^*-compact set, we can apply the Schauder- Tychonoff fixed point theorem to get the conclusion. We need some preliminary results.

Proposition 1.15. *Let $M := \ell^2(A)$ be the standard Hilbert module over the W^*-algebra A. For every fixed $c \in M$ the map $M \to A$ defined by $x \mapsto \langle c, x\rangle$ is w^*-w^* continuous on bounded subsets of M.*

Proof. Let $S \subset M$ be a bounded set and let (x_α) (where $\alpha \in I$) and x be respectively a net and a point in S such that $\lim_\alpha x_\alpha = x$ in the w^* topology of M. We have to show that $a_\alpha := \langle c, x_\alpha\rangle$ converges to $a := \langle c, x\rangle$ in the w^* topology of A.

Recall that the family of sets

$$W(\phi,\epsilon) := \{b \in A : |\phi(b-a)| < \epsilon\},$$

where ϕ is a state of A and $\epsilon > 0$, is a basis of neighbourhoods of a in the w^* topology of A. Let one of these neighbourhoods be given, and let $c = (c_n)$ and $x_\alpha = (x_{\alpha,n})$ be the expressions of c and x_α as elements in $\ell^2(A)$. Since S is bounded, by (1.2) the series $\langle c, x\rangle = \sum_1^\infty c_n^* x_n$ is uniformly convergent for $x \in S$ and so there is an index N (not depending on $\alpha \in I$) such that

$$\left\| \sum_{N+1}^{\infty} c_n^* x_{\alpha,n} \right\| < \frac{\epsilon}{3\|\phi\|}, \qquad \left\| \sum_{N+1}^{\infty} c_n^* x_n \right\| < \frac{\epsilon}{3\|\phi\|}.$$

By assumption the net (x_α) converges to x in the w^* topology of M, so that in particular $\lim_\alpha \phi(\langle c_n, x_{\alpha,n}\rangle) = \phi(\langle c_n, x_n\rangle)$, for every $n \in \mathbb{N}$. Therefore we can find an index $\alpha_0 \in I$ such that

$$|\phi(\langle c_n, x_n - x_{\alpha,n}\rangle) \leq \frac{\epsilon}{3N},$$

for all $\alpha \geq \alpha_0$ and all n with $1 \leq n \leq N$. Hence

$$\begin{aligned} |\phi(\langle c, x_\alpha - x\rangle)| &\leq \left|\phi\left(\sum_1^N \langle c_n, x_{\alpha,n} - x_n\rangle\right)\right| + \left|\phi\left(\sum_{N+1}^\infty \langle c_n, x_{\alpha,n} - x_n\rangle\right)\right| \\ &\quad + \left|\phi\left(\sum_{N+1}^\infty \langle c_n, x_n\rangle\right)\right| \leq \sum_1^N |\phi(\langle c_n, x_{\alpha,n} - x_n\rangle)| + \|\phi\| \left\|\sum_{N+1}^\infty c_n^* x_{\alpha,n}\right\| \\ &\quad + \|\phi\| \left\|\sum_{N+1}^\infty c_n^* x_n\right\| \leq N\frac{\epsilon}{3N} + \frac{2\epsilon}{3} = \epsilon \end{aligned}$$

for all $\alpha \geq \alpha_0$, which shows that $a_\alpha = \langle c, x_\alpha\rangle$ lies in the given neighbourhood $W(\phi,\epsilon)$ of $a = \langle c, x\rangle$ and completes the proof. □

REMARK 1.16. The module action $M \times A \to M$, $(x, a) \mapsto x \cdot a$ is not *jointly* w^*–w^* continuous at the origin and as a consequence the map $M \to M$, $x \mapsto \{x, c, x\} = x \cdot \langle c, x\rangle$ is not jointly w^*–w^* continuous at $x = 0$. This forces us to introduce some restriction on A in order to have good w^*–w^* continuity properties of the triple product.

PROPOSITION 1.17. *Let $M := \ell_2(A)$ be the standard Hilbert module over a W^*-algebra A. Assume that M is selfdual. Then for every fixed $c \in M$, the map $x \mapsto x \cdot \langle c, x\rangle$ is w^*–w^* continuous on bounded sets of M.*

Proof. For the standard Hilbert module $\ell^2(A)$ the condition of being selfdual is equivalent to the property $\dim A < \infty$; see [**18**, Theorem 1.1.6]. Let $S \subset M$ be bounded. Let (x_α) and x be respectively a net and a point in S such that $w^* \lim_\alpha x_\alpha = x$. Choose a basis $(a_1, \dots, a_n)$ in A. The inner product $\langle c, x_\alpha\rangle$ and $\langle c, x\rangle$ are elements in A. Hence they can be expressed in terms of the basis in the form

$$\langle c, x_\alpha\rangle = \sum_1^n \xi_{\alpha,k} a_k, \qquad \langle c, x\rangle = \sum_1^n \xi_k a_k,$$

for some nets $(\xi_{\alpha,k})_{\alpha\in I}$ and some ξ_k $(1 \le k \le n)$ in $\mathbb{C}$. By (1.15) the map $x \mapsto \langle c, x\rangle$ is w^*–w^* continuous on bounded sets of M which amounts to saying that $\lim_\alpha \xi_{\alpha,k} = \xi_k$ for $1 \le k \le n$. But then from

$$\{x_\alpha, c, x_\alpha\} = x_\alpha \cdot \langle c, x_\alpha\rangle = \sum_{i,j} \xi_{\alpha,i}\xi_{\alpha,j}a_i a_j,$$
$$\{x, c, x\} = x \cdot \langle c, x\rangle = \sum_{i,j} \xi_i\xi_j a_i a_j$$

it clearly follows that $w^* \lim_\alpha\{x_\alpha, c, x_\alpha\} = \{x, c, x\}$. □

We can now proceed to prove the main theorem. By polarization in the last result we get the joint w^*–w^*-continuity on bounded sets of the map $M \times M \to M$ defined by $(x, y) \mapsto \{x, c, y\}$ and an induction argument shows that

$$w^* \lim_\alpha (x_\alpha \,\square\, c)^n\, x_\alpha = (x \,\square\, c)^n x \tag{10}$$

holds for every exponent $n \in \mathbb{N}$. The expression of the Bergmann operator

$$B_c(z) = (\mathbf{1} - c \otimes c^*)^{\frac{1}{2}}\big(z \cdot (\mathbf{1} - |c|)^{\frac{1}{2}}\big) \qquad (z \in M)$$

shows that it is the composition of the operator of multiplication on the right by a fixed element in A (hence w^*–w^*-continuous) with the operator $(\mathbf{1} - c \otimes c^*)^{\frac{1}{2}}$. But $z \mapsto c \otimes c^*(z) = c \cdot \langle c, z\rangle$ is also w^*–w^*-continuous; hence so is B_c. Thus in order to show that the transvection $g_c(x) = c + B_c((\mathrm{Id} + x\square c)^{-1}x)$ is w^*–w^*-continuous we only need to show the w^*–w^*-continuity of the mapping $x \mapsto f_c(x) := (\mathrm{Id} + x \,\square\, c)^{-1}x$ on the closed ball $\overline{B}_M$. From the usual power series development we get for $\|x\| \le 1$ and $\|c\| \le r < 1$,

$$\|(\mathrm{Id} + x \,\square\, c)^{-1}\| = \left\| \sum_1^\infty (-1)^n (x \,\square\, c)^n \right\| \le \sum_1^\infty \|x\|^n \|c\|^n \le \sum r^n < \infty.$$

Let $\epsilon > 0$ be given and fix any N such that $\sum_{N+1}^\infty r^n < \epsilon/2$. We simplify by writing

$$y_\alpha := \sum_{N+1}^\infty (x_\alpha \,\square\, c)^n x_\alpha, \qquad y := \sum_{N+1}^\infty (x \,\square\, c)^n x.$$

Then for $\phi \in A_*$ and $v \in M$ we have

$$|\phi(\langle v, y_\alpha\rangle) - \phi(\langle v, y\rangle)| = |\phi(\langle v, y_\alpha - y\rangle)| \le \|\phi\|\,\|v\|\,\|y_\alpha - y\| \le \epsilon\,\|\phi\|\,\|v\|\ (\alpha \in I).$$

As proved before, for every k with $0 \le k \le N$ we have $w^* \lim_\alpha (x_\alpha \,\square\, c)^k x_\alpha = (x\square c)^n x$, so that there is an index α_0 such that for $\alpha \ge \alpha_0$ we have

$$|\phi(\langle v, (x_\alpha \,\square\, c)^k x_\alpha\rangle) - \phi(\langle v, (x\square c)^k x\rangle)| = |\phi(\langle v, (x_\alpha \,\square\, c)^k x_\alpha - (x \,\square\, c)^k x)| \le \frac{\epsilon}{2N},$$

which combined with the above yields $w^* \lim_\alpha f_c(x_\alpha) = f_c(x)$ and completes the proof. □

With minor changes in the proof given above one can prove the next result.

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THEOREM 1.18. *Let A and M respectively be a W^*-algebra and a Hilbert A-module which is a dual Banach space, and let $c \in M$ be such that the mapping $x \mapsto x \cdot \langle c, x \rangle$ is $w^* - w^*$-continuous on $\overline{B}_M$.*

(i) *For every surjective linear isometry λ of M, the automorphism $h = \lambda g_c$ is $w^* - w^*$-continuous on $\overline{B}_M$. In particular, h has at least one fixed point in $\overline{B}_M$.*

(ii) *The set N of those $c \in M$ for which $x \mapsto x \cdot \langle c, x \rangle$ is $w^* - w^*$-continuous on $\overline{B}_M$ is a norm closed submodule of M. If $M = \ell_2(A)$ and* $\dim A < \infty$*, then $N = M$.*

Proof. The arguments in [**5**] and [**6**] show that N is a closed triple ideal in M and the proof of the main theorem actually shows that N is a submodule in M. □

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