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The Closure Ordering of Nilpotent Orbits of the Complex Symmetric Pair $(\mathrm{SO}_{p+q}, \mathrm{SO}_p \times \mathrm{SO}_q)$

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Abstract. The main problem that is solved in this paper has the following simple formulation (which is not used in its solution). The group $K = \mathrm{O}_p(\mathbf{C}) \times \mathrm{O}_q(\mathbf{C})$ acts on the space $M_{p,q}$ of $p \times q$ complex matrices by $(a, b) \cdot x = axb^{-1}$, and so does its identity component $K^0 = \mathrm{SO}_p(\mathbf{C}) \times \mathrm{SO}_q(\mathbf{C})$. A K-orbit (or K^0-orbit) in $M_{p,q}$ is said to be nilpotent if its closure contains the zero matrix. The closure, $\overline{\mathcal{O}}$, of a nilpotent K-orbit (resp. K^0-orbit) $\mathcal{O}$ in $M_{p,q}$ is a union of $\mathcal{O}$ and some nilpotent K-orbits (resp. K^0-orbits) of smaller dimensions. The description of the closure of nilpotent K-orbits has been known for some time, but not so for the nilpotent K^0-orbits. A conjecture describing the closure of nilpotent K^0-orbits was proposed in [11] and verified when $\min(p, q) \leq 7$. In this paper we prove the conjecture. The proof is based on a study of two prehomogeneous vector spaces attached to $\mathcal{O}$ and determination of the basic relative invariants of these spaces.

The above problem is equivalent to the problem of describing the closure of nilpotent orbits in the real Lie algebra $\mathfrak{so}(p, q)$ under the adjoint action of the identity component of the real orthogonal group $\mathrm{O}(p, q)$.

1 Introduction

Let G_0 be an almost simple real Lie group (not necessarily connected), G_0^0 its identity component, and $\mathfrak{g}_0$ its Lie algebra. The orbits of G_0^0 in $\mathfrak{g}_0$ under the adjoint action are known as the adjoint orbits. If they consist of ad-nilpotent elements, then we refer to them as the nilpotent adjoint orbits. It is well known that there are only finitely many of them. In the representation theory of such groups it is important to know the closures of the nilpotent adjoint orbits (see *e.g.* [2, Chapter 10]). The description of these closures is known in almost all cases. We refer the reader to the monograph [2] for many of the known facts and the important references. The cases where G_0 is of exceptional type have been all treated by the first author in a series of papers [5]–[10].

Surprisingly, there is an infinite series of classical groups for which the answer was not known so far. This is the case of the groups $G_0 = \mathrm{O}(p, q)$, with $p, q \geq 1$. More precisely, one should assume here that p or q is even. The point is that if one replaces the group $G_0^0 = \mathrm{SO}(p, q)^0$ with the full orthogonal group $G_0 = \mathrm{O}(p, q)$, then the description of the closure of nilpotent $\mathrm{O}(p, q)$-orbits in the Lie algebra $\mathfrak{g}_0 = \mathfrak{so}(p, q)$

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is known (see *e.g.* [11] for more details), and if both p and q are odd, then the $\mathrm{O}(p,q)$-orbits in $\mathfrak{g}_0$ are the same as the adjoint orbits. A conjecture describing the closures of the *adjoint* nilpotent orbits in $\mathfrak{so}(p,q)$ has been proposed in *loc. cit.* and was verified for $\min(p,q) \leq 7$. The main objective of this paper is to prove this conjecture.

Choose a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{so}(p,q) = \mathfrak{k}_0 + \mathfrak{p}_0$, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its complexification, and let $G = \mathrm{O}_n(\mathbf{C})$, $n = p + q$, be the complexification of $G_0 = \mathrm{O}(p,q)$. Let $K = \mathrm{O}_p(\mathbf{C}) \times \mathrm{O}_q(\mathbf{C})$ be the normalizer of $\mathfrak{k}$ in K, and $K^0 = \mathrm{SO}_p(\mathbf{C}) \times \mathrm{SO}_q(\mathbf{C})$ its identity component. By using the Kostant-Sekiguchi bijection (see [16], [2], [17]) from nilpotent G_0^0-orbits in $\mathfrak{g}_0$ to nilpotent K^0-orbits in $\mathfrak{p}$, which is known to preserve the closure relation (see [1]), the closure ordering problem for nilpotent G_0^0-orbits in $\mathfrak{g}_0$ is translated into the same problem for nilpotent K^0-orbits in $\mathfrak{p}$. The conjecture in [11] was formulated in this latter setting, and our proof will also use the same setting. As a K-module, $\mathfrak{p}$ is isomorphic to the module $M_{p,q}$, mentioned in the abstract, but this simple reformulation of the problem is not conducive to its solution.

In the next section we recall known facts about the parametrization of nilpotent orbits of K^0 in $\mathfrak{p}$. Almost all of this material is taken from [11].

In Section 3, which is very short, we recall the main results of [11] and state the conjecture made there.

In Section 4 we describe two important prehomogeneous vector spaces (PVs) (4.1) and (4.4) attached to a nilpotent K^0-orbit in $\mathfrak{p}$. We also describe the method, developed in the papers of the first author [5, Proposition 3.1], [8, Proposition 7.1] and the joint paper [11, Proposition 1], which was used to determine the closure ordering of nilpotent adjoint orbits in noncompact real forms of exceptional complex Lie algebras.

In order to be able to apply the same method, it is imperative to obtain the full list of the basic relative invariants of the two PVs mentioned above. This is accomplished in Section 5, with most of the proofs delegated to the appendix.

In Section 6 we give our proof of the conjecture.

In Section 7 we define, following a recipe of V. Kac [13], a relative invariant of the prehomogeneous vector space (4.1) and obtain the prime factorization of it. The zero set of this particular relative invariant is exactly the singular set of the PV just mentioned.

In Section 8 we give several illustrative examples and state some open problems.

2 Parametrization of Nilpotent Orbits

In this section we give the basic definitions and state some well known facts which will be used throughout the paper. For more details, examples, and motivation see [2], [11].

Let V be an n-dimensional complex vector space, $f\colon V \times V \to \mathbf{C}$ a nondegenerate symmetric bilinear form and $G = \mathrm{O}(V,f)$ the orthogonal group of (V,f). Fix an involution $\theta \in G$ ($\theta \neq 1$), and denote by V_a (resp. V_b) the $+1$-eigenspace (resp. -1-eigenspace) of θ. Let $p = \dim(V_a)$ and $q = \dim(V_b)$. As V_a and V_b are orthogonal to each other, the restriction f_a (resp. f_b) of f to $V_a \times V_a$ (resp. $V_b \times V_b$) is nondegenerate. Denote by K the centralizer of θ in G, and by K^0 its identity component. We have $K = K_a \times K_b$ with $K_a = \mathrm{O}(V_a, f_a)$ and $K_b = \mathrm{O}(V_b, f_b)$, and $K^0 = K_a^0 \times K_b^0$ with $K_a^0 = \mathrm{SO}(V_a, f_a)$ and $K_b^0 = \mathrm{SO}(V_b, f_b)$.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(V, f)$ of G consists of the linear operators $u\colon V \to V$ satisfying $f\big(u(x), y\big) + f\big(x, u(y)\big) = 0$ for all $x, y \in V$. This condition can be also expressed as $u^* = -u$, where u^* denotes the adjoint of u with respect to the form f. The Lie algebra $\mathfrak{k}$ of K is the centralizer of θ in $\mathfrak{g}$, *i.e.*, $\mathfrak{k} = \{u \in \mathfrak{g} : u(V_a) \subset V_a, u(V_b) \subset V_b\}$. Thus $\mathfrak{k} = \mathfrak{k}_a \oplus \mathfrak{k}_b$ where $\mathfrak{k}_a = \mathfrak{so}(V_a, f_a)$ and $\mathfrak{k}_b = \mathfrak{so}(V_b, f_b)$. We denote by Ad (resp. ad) the adjoint representation of G (resp. $\mathfrak{g}$) on $\mathfrak{g}$. As a K-module (under the restriction of Ad), $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} = \{u \in \mathfrak{g} : u(V_a) \subset V_b, u(V_b) \subset V_a\}$.

Remark 2.1 Note that if θ is replaced by $-\theta$, then $\mathfrak{k}$ and $\mathfrak{p}$ remain the same, but V_a and V_b get interchanged, p and q get interchanged, *etc.* In order to be able to use the symmetry argument, *i.e.*, to replace θ by $-\theta$, we *do not assume* that $p \geq q$.

The nilpotent G-orbits in $\mathfrak{g}$ are parametrized by the Young diagrams with n cells (or, equivalently, partitions of n) such that each row of even length occurs an even number of times. We refer to such diagrams (resp. partitions) as the *orthogonal Young diagrams* (resp. *orthogonal partitions*). An orthogonal Young diagram (or the corresponding partition) is *very even* if it contains no rows of odd length. The nilpotent G-orbits in $\mathfrak{g}$ are connected, except those which correspond to the very even partitions, in which case they have two connected components. The nilpotent G^0-orbits in $\mathfrak{g}$ are exactly the connected components of the nilpotent G-orbits.

An *orthogonal ab-diagram* is an orthogonal Young diagram whose cells are filled with letters a and b subject to the following two conditions:

(i) in each row the letters a and b alternate,
(ii) the rows of even length are split into pairs of rows of equal length and, in each pair, one of the rows begins with the letter a and the other with b.

As we only use orthogonal ab-diagrams, from now on we shall refer to them as the *ab-diagrams*. Two ab-diagrams are said to be *equivalent* if one can be obtained from the other by permuting the rows. The collection of the equivalence classes of ab-diagrams will be denoted by $\mathcal{D}$. The subcollection consisting of (the equivalence classes of) ab-diagrams having exactly n cells with p a's and q b's is denoted by $\mathcal{D}(p, q)$.

We consider equivalent ab-diagrams as being the same, *i.e.*, we identify an ab-diagram with its equivalence class. We write a concrete ab-diagram as a sequence of its rows. A row of length $2k + 1$ with a (resp. b) in the first cell is written as $(ab)^k a$ (resp. $(ba)^k b$). The pair of rows of even length $2k$, one starting with a and the other with b, is written as $(ab)^k, (ba)^k$. If X and Y are arbitrary ab-diagrams, then $X + Y$ denotes the ab-diagram obtained by writing Y below X and rearranging the rows of this extended diagram. We say that two ab-diagrams are *disjoint* if they have no common rows.

Denote by $\mathcal{N}$ the nilpotent variety in $\mathfrak{p}$, *i.e.*, $\mathcal{N} = \{u \in \mathfrak{p} : u^n = 0\}$. The K-orbits in $\mathcal{N}$ are parametrized by $\mathcal{D}(p, q)$ and those of K^0 are precisely the connected components of the K-orbits. Since $[K : K^0] = 4$, a K-orbit in $\mathcal{N}$ may have 1, 2 or

4 connected components. Let $\mathcal{N}/K$ (resp. $\mathcal{N}/K^0$) denote the set of K-orbits (resp. K^0-orbits) in $\mathcal{N}$.

We now describe the correspondence between $\mathcal{D}(p,q)$ and $\mathcal{N}/K$. Let $u \in \mathcal{N}$. Define a *Jordan chain* for u to be a sequence of nonzero vectors $v_1, v_2, \dots, v_k$ such that $u(v_i) = v_{i+1}$ for $1 \leq i < k$ and $u(v_k) = 0$. We say that k is the *length* of this chain, and that v_1 (resp. v_k) is the *top* (resp. *bottom*) *vector* of this chain. If moreover each $v_i \in V_a \cup V_b$ then we say that this Jordan chain is *graded*. By replacing each v_i by the letter a if $v_i \in V_a$ and by b if $v_i \in V_b$, we obtain an alternating sequence of these letters to which we refer as the *type* of this graded Jordan chain. A Jordan chain for u is said to be *maximal* if it cannot be extended to a larger one. This is the case if and only if the top vector of the chain is not contained in the image of u. A *graded Jordan basis* for u is a basis of V consisting of graded Jordan chains for u (necessarily maximal). They always exist. Let us choose one of them. Then we form the Young diagram by creating a row of length k for each maximal Jordan chain of length k, say $v_1, \dots, v_k$, contained in this basis. We temporarily fill the cells of this row (successively from the left to the right) by the vectors $v_1, \dots, v_k$. Finally we replace each of the vectors, say v, in the resulting diagram by the letter a if $v \in V_a$ and by b if $v \in V_b$. We obtain an ab-diagram, say X, which is independent (up to equivalence) of the choice of the graded Jordan basis for u. Then the K-orbit containing u is denoted by $\mathcal{O}_X$. The trivial orbit $\{0\}$ corresponds to the ab-diagram consisting of n rows of length 1, with cells filled with p a's and q b's.

If $\mathcal{O}_1, \mathcal{O}_2$ are members of $\mathcal{N}/K$ (or $\mathcal{N}/K^0$) and $\mathcal{O}_1$ is contained in the closure of $\mathcal{O}_2$, then we write $\mathcal{O}_1 \leq \mathcal{O}_2$. This defines a partial order on $\mathcal{N}/K$ (resp. $\mathcal{N}/K^0$) called the *closure ordering*. The closure ordering "$\leq$" on $\mathcal{N}/K$ corresponds to a natural combinatorially defined partial order on $\mathcal{D}(p,q)$, which we denote again by "$\leq$". This combinatorial partial order is defined as follows.

If $X \in \mathcal{D}(p,q)$ denote by X' the diagram obtained from X by deleting the first column. Set $X^{(0)} = X$ and define recursively $X^{(k+1)} = (X^{(k)})'$ for $k \geq 0$. In particular, $X^{(1)} = X'$. For any such diagram Y, denote by $n_a(Y)$ (resp. $n_b(Y)$) the number of a's (resp. b's) in Y. For $X, Y \in \mathcal{D}(p,q)$ we write $X \leq Y$ if $n_a(X^{(k)}) \leq n_a(Y^{(k)})$ and $n_b(X^{(k)}) \leq n_b(Y^{(k)})$ for all $k \geq 0$. The relation "$\leq$" makes $\mathcal{D}(p,q)$ into a partially ordered set.

If $X, Y \in \mathcal{D}(p,q)$ are distinct and $X \leq Y$ then we write $X < Y$. If $X < Y$ and there is no $Z \in \mathcal{D}(p,q)$ such that $X < Z < Y$, then we write $Y \to X$ and say that X is a *child* of Y, or that Y is a *parent* of X. We shall also refer to a pair (X, Y), with $X \to Y$, as an *elementary move* $X \to Y$. We define similarly the relation "$<$", the children, parents, and elementary moves in the partially ordered sets $(\mathcal{N}/K, \leq)$ and $(\mathcal{N}/K^0, \leq)$.

We point out that if $X \to Y$ is an elementary move and $X = P + Z$, $Y = Q + Z$, then $P \to Q$ is also an elementary move. On the other hand, the converse of this statement fails (we leave to the reader to supply a counter-example).

It is a known fact that the partially ordered sets $(\mathcal{D}(p,q), \leq)$ and $(\mathcal{N}/K, \leq)$ are isomorphic, and that an isomorphism is provided by the map that sends X to $\mathcal{O}_X$. The Hasse diagram of these two partially ordered sets is denoted by $\Gamma(p,q)$. (See Section 8 for an example.) Each $X \in \mathcal{D}(p,q)$ is represented by a node in $\Gamma(p,q)$. If $X \to Y$ for some $X, Y \in \mathcal{D}(p,q)$, then the node X is placed in $\Gamma(p,q)$ higher than the node Y and these two nodes are joined by a line. The Hasse diagram of $(\mathcal{N}/K, \leq)$ is essentially the same as $\Gamma(p,q)$. We just have to replace each node $X \in \mathcal{D}(p,q)$ by the corresponding node $\mathcal{O}_X \in \mathcal{N}/K$.

The set $\mathcal{N}/K$ (and $\mathcal{N}/K^0$) can be also parametrized by using the characteristics introduced by Dynkin. In order to explain this, we fix a basis $\{a_1, a_2, \dots, a_p\}$ of V_a and a basis $\{b_1, b_2, \dots, b_q\}$ of V_b such that $f(a_i, a_j) = \delta_{i+j,p+1}$ for $1 \leq i, j \leq p$ and $f(b_i, b_j) = \delta_{i+j,q+1}$ for $1 \leq i, j \leq q$, where δ_{ij} is Kronecker's delta and we identify linear operators on V with their matrices with respect to this basis. Denote by $\mathfrak{h}_a$ the Cartan subalgebra of $\mathfrak{k}_a$ consisting of the diagonal matrices. These diagonal matrices $\operatorname{diag}(h_1, h_2, \dots, h_p)$ satisfy $h_i + h_{p+1-i} = 0$ for $1 \leq i \leq p$. The centralizer of $\mathfrak{h}_a$ in K_a^0 is the maximal torus T_a which consists of all diagonal matrices in K_a^0. Denote by N_a the normalizer of T_a (or $\mathfrak{h}_a$) in K_a. The Weyl group of $(\mathfrak{k}_a, \mathfrak{h}_a)$ is $W_a = (N_a \cap K_a^0)/T_a$. We set $W_a^* = N_a/T_a$. Clearly W_a is a normal subgroup of W_a^* and the quotient group W_a^*/W_a is trivial if p is odd, and has order 2 if p is even. We introduce the real form $(\mathfrak{h}_a)_{\mathbf{R}}$ of $\mathfrak{h}_a$ consisting of the diagonal matrices as above with $h_i \in \mathbf{R}$. Define the closed Weyl chamber $C_a \subset (\mathfrak{h}_a)_{\mathbf{R}}$ by the inequalities

$$h_i \geq h_{i+1}, \quad 1 \leq i \leq k, \tag{2.1}$$

if $p = 2k+1$ is odd, and by

$$h_i \geq h_{i+1}, \quad 1 \leq i \leq k-2 \tag{2.2}$$

and

$$h_{k-1} \geq |h_k|, \tag{2.3}$$

if $p = 2k$ is even. If p is odd we set $C_a^* = C_a$ while if $p = 2k$ is even we define $C_a^* \subset (\mathfrak{h}_a)_{\mathbf{R}}$ by inequalities (2.2) above and

$$h_{k-1} \geq h_k \geq 0. \tag{2.4}$$

Define similarly $\mathfrak{h}_b$, T_b, *etc.*, and set $\mathfrak{h} = \mathfrak{h}_a \times \mathfrak{h}_b$, $T = T_a \times T_b$, *etc.*

If $\mathcal{O}_X$ is nontrivial, there exists a unique element $H_X \in C^* = C_a^* \times C_b^*$ such that $[H_X, E_X] = 2E_X$ for some nonzero element $E_X \in \mathcal{O}_X$. If $\mathcal{O}_X$ is the trivial orbit, define $H_X = 0$.

Definition 2.2 We refer to this element H_X as the *characteristic* of X (or of $\mathcal{O}_X$).

It is well known that different K-orbits in $\mathcal{N}$ have different characteristics. Denote by $(H_X)_a$ (resp. $(H_X)_b$) the component of H_X in $\mathfrak{h}_a$ (resp. $\mathfrak{h}_b$).

The eigenvalues (*i.e.*, the diagonal entries) of $(H_X)_a$ and $(H_X)_b$ can be easily determined. For this purpose insert in each cell of X an integer as follows: if a row has length k then we insert successively in the cells of that row the integers

$$k-1, k-3, k-5, \dots, 5-k, 3-k, 1-k.$$

Then the integers written in all a-cells (resp. b-cells) are the eigenvalues of $(H_X)_a$ (resp. $(H_X)_b$). The order in which these eigenvalues occur on the diagonal is determined uniquely by the condition that $H_X \in C^*$.

The group W^*/W permutes transitively the connected components of $\mathcal{O}_X$. The element E_X, as described above, is not unique but all such elements lie in the same connected component of $\mathcal{O}_X$. We refer to this particular connected component as the *basic component* of $\mathcal{O}_X$.

If $p = 2k$ is even let $x_a \in N_a$ be the linear operator which interchanges the vectors a_k and a_{k+1} and fixes all the other a_i's. If q is even define $x_b \in N_b$ similarly. We use the left and/or right superscripts, roman I and II, to label the connected components of $\mathcal{O}_X$. The precise rule that governs the use of these superscripts is given in the next definition and is identical to the one used in [11].

Definition 2.3 We introduce the following notation for the connected components of the K-orbit $\mathcal{O}_X \subset \mathcal{N}$ by considering four possibilities:

(i) Both $(H_X)_a$ and $(H_X)_b$ have 0 eigenvalues: Then $\mathcal{O}_X$ is connected and we do not need any new notation.

(ii) $(H_X)_a$ has no 0 eigenvalue but $(H_X)_b$ does: Then p is even and $\mathcal{O}_X$ has two connected components. The basic component will be denoted by ${}^{\mathrm{I}}\mathcal{O}_X$, and the other one by ${}^{\mathrm{II}}\mathcal{O}_X = \mathrm{Ad}(x_a)({}^{\mathrm{I}}\mathcal{O}_X)$.

(iii) $(H_X)_a$ has a 0 eigenvalue but $(H_X)_b$ does not: Then q is even and again $\mathcal{O}_X$ has two connected components. The basic component will be denoted by $\mathcal{O}_X^{\mathrm{I}}$, and the other one by $\mathcal{O}_X^{\mathrm{II}} = \mathrm{Ad}(x_b)(\mathcal{O}_X^{\mathrm{I}})$.

(iv) H_X has no 0 eigenvalues: Then both p and q must be even and $\mathcal{O}_X$ has four connected components. The basic component will be denoted by ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$, and the remaining three are ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}} = \mathrm{Ad}(x_a)({}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}})$, ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}} = \mathrm{Ad}(x_b)({}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}})$, and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}} = \mathrm{Ad}(x_a x_b)({}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}})$. Furthermore, in this case we set

$$
{}^{\mathrm{I}}\mathcal{O}_X = {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}} \cup {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}, \quad {}^{\mathrm{II}}\mathcal{O}_X = {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}} \cup {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}},
$$
$$
\mathcal{O}_X^{\mathrm{I}} = {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}} \cup {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}, \quad \mathcal{O}_X^{\mathrm{II}} = {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}} \cup {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}.
$$

If p (resp. q) is odd then the left (resp. right) superscripts I, II are not used. In particular if p and q are odd then all K-orbits in $\mathcal{N}$ are connected.

Let us also introduce the *characteristics* for the K^0-orbits in $\mathcal{N}$. The characteristic H_X of $\mathcal{O}_X$ is also the characteristic of the basic component of $\mathcal{O}_X$. In case (ii), the characteristic of ${}^{\mathrm{II}}\mathcal{O}_X$ is $\mathrm{Ad}(x_a)(H_X)$. In case (iii), the characteristic of $\mathcal{O}_X^{\mathrm{II}}$ is $\mathrm{Ad}(x_b)(H_X)$. Finally, in case (iv), the characteristics of the orbits ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}$, ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}$, ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}$ are

$$
\mathrm{Ad}(x_a)(H_X), \quad \mathrm{Ad}(x_b)(H_X), \quad \mathrm{Ad}(x_a x_b)(H_X),
$$

respectively. All these characteristics belong to the closed Weyl chamber $C = C_a \times C_b$, and different orbits have different characteristics.

The left (resp. right) superscripts I and II depend on the choice of the basis $\{a_i\}$ of V_a (resp. $\{b_i\}$ of V_b). If $p = 2k$ is even then there are exactly two K_a^0-orbits of maximal isotropic subspaces of V_a and the left superscripts I, II depend on the orbit to which the subspace spanned by $\{a_1, \dots, a_k\}$ belongs. If this subspace is chosen from a different orbit, then the left superscripts I and II get interchanged. The same phenomenon occurs with the right superscripts when q is even.

3 The Closure Ordering Conjecture

In this section we state the main conjecture from [11] and recall some basic results proved there.

A conjecture describing the closure ordering in $\mathcal{N}/K^0$ was proposed in [11] and verified when $\min(p, q) \leq 7$. In order to state this conjecture, we introduce a few more terms and then give the basic definition of the diagram $\Delta(p, q)$.

A vertex X of $\Gamma(p, q)$ is *stable* (resp. *unstable*) if the K-orbit $\mathcal{O}_X$ is connected (resp. disconnected). An unstable vertex X is an *a-vertex* (resp. *b-vertex*) if the linear operator $(H_X)_a$ (resp. $(H_X)_b$) is nonsingular. Equivalently, X is an a-vertex (resp. b-vertex) if the middle letter of each row of odd length (if any) in X is a b (resp. a). If X is both an a-vertex and a b-vertex, then we say that it is an *ab-vertex*. Thus X is an ab-vertex if and only if it has no rows of odd length, *i.e.*, the corresponding partition is very even. An a-vertex that is not a b-vertex will be called a *proper a-vertex*. One defines similarly a *proper b-vertex*.

We remark that if X and Y are ab-vertices, then $X \not\to Y$ (*i.e.*, $X \to Y$ does not hold). The same is true if X is a stable vertex and Y an ab-vertex.

Definition 3.1 $\Delta(p, q)$ is the diagram which is obtained from $\Gamma = \Gamma(p, q)$ by the following modifications.

Step 1 For every vertex pair (X, Y) such that $X \to Y$ and X or Y is unstable erase the line in Γ joining X to Y.

Step 2 Replace each node X by as many nodes as there are connected components in $\mathcal{O}_X$ and label them by these components.

Step 3 Insert two or four lines for each line that was erased in Step 1. For this purpose reconsider all pairs (X, Y) from Step 1 and perform the indicated action:

(i) X stable and Y unstable: Join $\mathcal{O}_X$ to each of the nodes corresponding to the connected components of $\mathcal{O}_Y$.

(ii) X unstable and Y stable: Join each of the nodes corresponding to the connected components of $\mathcal{O}_X$ to $\mathcal{O}_Y$.

(iii) X a proper a-vertex and Y a proper b-vertex: Join each of ${}^{\mathrm{I}}\mathcal{O}_X$, ${}^{\mathrm{II}}\mathcal{O}_X$ to each of $\mathcal{O}_Y^{\mathrm{I}}$, $\mathcal{O}_Y^{\mathrm{II}}$.

(iv) X a proper b-vertex and Y a proper a-vertex: Join each of $\mathcal{O}_X^{\mathrm{I}}$, $\mathcal{O}_X^{\mathrm{II}}$ to each of ${}^{\mathrm{I}}\mathcal{O}_Y$, ${}^{\mathrm{II}}\mathcal{O}_Y$.

(v) X and Y proper a-vertices: Join ${}^{\mathrm{I}}\mathcal{O}_X$ to ${}^{\mathrm{I}}\mathcal{O}_Y$, and ${}^{\mathrm{II}}\mathcal{O}_X$ to ${}^{\mathrm{II}}\mathcal{O}_Y$.

(vi) X and Y proper b-vertices: Join $\mathcal{O}_X^{\mathrm{I}}$ to $\mathcal{O}_Y^{\mathrm{I}}$, and $\mathcal{O}_X^{\mathrm{II}}$ to $\mathcal{O}_Y^{\mathrm{II}}$.

(vii) X a proper a-vertex and Y an ab-vertex: Join ${}^{\mathrm{I}}\mathcal{O}_X$ to ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}}$ and ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}}$, and ${}^{\mathrm{II}}\mathcal{O}_X$ to ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}$ and ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}}$.

(viii) X a proper b-vertex and Y an ab-vertex: Join $\mathcal{O}_X^{\mathrm{I}}$ to ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}}$ and ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}$, and $\mathcal{O}_X^{\mathrm{II}}$ to ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}}$ and ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}}$.

(ix) X an ab-vertex and Y a proper a-vertex: Join ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$ and ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}$ to ${}^{\mathrm{I}}\mathcal{O}_Y$, and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}$ and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}$ to ${}^{\mathrm{II}}\mathcal{O}_Y$.

(x) X an ab-vertex and Y a proper b-vertex: Join ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$ and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}$ to $\mathcal{O}_Y^{\mathrm{I}}$, and ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}$ and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}$ to $\mathcal{O}_Y^{\mathrm{II}}$.

We can now state the conjecture.

Conjecture 3.2 (**[11]**) The diagram $\Delta(p,q)$ is the Hasse diagram of the partially ordered set $(\mathcal{N}/K^0, \leq)$.

In addition to the closure ordering "$\leq$" on $\mathcal{N}/K^0$, introduce the new partial order "$\preceq$" on the same set $\mathcal{N}/K^0$. It is defined by postulating that its Hasse diagram is $\Delta(p,q)$. The conjecture can be reformulated as follows: The two partial orders "$\leq$" and "$\preceq$" are the same. We quote the following basic result:

Theorem 3.3 (**[11, Theorem 1]**) *If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}/K^0$ and $\mathcal{O}_1 \preceq \mathcal{O}_2$ then $\mathcal{O}_1 \leq \mathcal{O}_2$.*

Next we introduce the concept of pure pairs of nodes in $\Gamma(p,q)$.

Definition 3.4 If $X, Y \in \mathcal{D}(p,q)$ are a-vertices with $Y < X$, and if $Y < Z < X$ implies that Z is an a-vertex, then we say that (X,Y) is a *pure a-pair*. One defines similarly pure b-pairs. A *pure pair* is either a pure a-pair or a pure b-pair.

We remark that if (X,Y) is a pure pair, then X and Y cannot be both ab-vertices.

In the next important definition we introduce the concept of splitting for pure pairs.

Definition 3.5 A pure a-pair (X,Y) *splits* if $\overline{{}^{\mathrm{I}}\mathcal{O}_X} \cap \mathcal{O}_Y = {}^{\mathrm{I}}\mathcal{O}_Y$ (or, equivalently, $\overline{{}^{\mathrm{II}}\mathcal{O}_X} \cap \mathcal{O}_Y = {}^{\mathrm{II}}\mathcal{O}_Y$). Similarly, a pure b-pair (X,Y) *splits* if $\overline{\mathcal{O}_X^{\mathrm{I}}} \cap \mathcal{O}_Y = \mathcal{O}_Y^{\mathrm{I}}$.

Remark 3.6 Let (X,Y) be a pure b-pair. Theorem 3.3 implies that $\overline{\mathcal{O}_X^{\mathrm{I}}} \supset \mathcal{O}_Y^{\mathrm{I}}$ and $\overline{\mathcal{O}_X^{\mathrm{II}}} \supset \mathcal{O}_Y^{\mathrm{II}}$. Hence, the condition $\overline{\mathcal{O}_X^{\mathrm{I}}} \cap \mathcal{O}_Y = \mathcal{O}_Y^{\mathrm{I}}$ is equivalent to $\overline{\mathcal{O}_X^{\mathrm{I}}} \cap \mathcal{O}_Y \subset \mathcal{O}_Y^{\mathrm{I}}$. If X is an ab-vertex, then it is also equivalent to $\overline{{}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}} \cap \mathcal{O}_Y \subset \mathcal{O}_Y^{\mathrm{I}}$. Indeed, we have $K_a \cdot {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}} = \mathcal{O}_X^{\mathrm{I}}$ and $K_a \cdot \mathcal{O}_Y^{\mathrm{I}} = \mathcal{O}_Y^{\mathrm{I}}$. Analogous statements are valid for pure a-pairs.

Our proof of the conjecture will be based on the following reduction result.

Theorem 3.7 (**[11, Theorem 2]**) *In order to prove the above conjecture, it suffices to show that every pure pair in $\Gamma(p,q)$ splits.*

4 Two Prehomogeneous Vector Spaces Attached to an ab-Diagram

In this section we describe two important prehomogeneous vector spaces (PV) attached to an ab-diagram $X \in \mathcal{D}(p,q)$. We refer the reader to [14], [15] for an exposition of the theory of PVs.

Let $H = H_X$, the characteristic of X, and let H_a (resp. H_b) be its restriction to V_a (resp. V_b). In this section, we denote by $\mathcal{O}$ the basic connected component of $\mathcal{O}_X$. We use the standard notation for the centralizers and for the connected component of the identity. Thus $Z_K(H)$ denotes the centralizer of H in K, and $Z_K(H)^0$ its identity component. We have

$$Z_K(H) = Z_{K_a}(H_a) \times Z_{K_b}(H_b), \quad Z_K(H)^0 = Z_{K_a}(H_a)^0 \times Z_{K_b}(H_b)^0.$$

Introduce the eigenspaces of H in V_a and V_b:

$$V_a(H,k) = \{v \in V_a : H(v) = kv\}, \quad k \in \mathbf{Z},$$
$$V_b(H,k) = \{v \in V_b : H(v) = kv\}, \quad k \in \mathbf{Z}.$$

Each nonzero $V_a(H,k)$ is a simple $Z_K(H)^0$-module with one exception: If p is even, say $p = 2m$, and the dimension of $V_a(H,0)$ is 2, then this 2-dimensional module is not simple. Each of the 1-dimensional subspaces spanned by a_m or a_{m+1} is a submodule. The analogous assertions are valid for the subspaces $V_b(H,k)$.

Next introduce the eigenspaces of $\mathrm{ad}(H)$ in $\mathfrak{k}$ and $\mathfrak{p}$:

$$\mathfrak{g}_H(0,j) = \{X \in \mathfrak{k} : [H,X] = jX\}, \quad j \in \mathbf{Z},$$
$$\mathfrak{g}_H(1,j) = \{X \in \mathfrak{p} : [H,X] = jX\}, \quad j \in \mathbf{Z}.$$

The first prehomogeneous vector space attached to $\mathcal{O}$ is

$$\left(Z_K(H)^0, \mathfrak{g}_H(1,2)\right). \tag{4.1}$$

This PV is regular and its generic orbit coincides with $\mathcal{O}_X \cap \mathfrak{g}_H(1,2) = \mathcal{O} \cap \mathfrak{g}_H(1,2)$. We say that an element E is a *generic element* of $\mathfrak{g}_H(1,2)$ if it belongs to this generic orbit. Denote by $f_1, \dots, f_r$ the basic relative invariants of this PV and let S_i be the hypersurface in $\mathfrak{g}_H(1,2)$ defined by f_i. The singular set (*i.e.*, the complement of the generic orbit) S of this PV is the union of these hypersurfaces.

We now introduce the subspaces

$$\mathfrak{g}_H(i, \geq k) = \bigoplus_{j \geq k} \mathfrak{g}_H(i,j), \quad i = 0,1; k \geq 0. \tag{4.2}$$

For $i = 0$, these subspaces are subalgebras of $\mathfrak{k}$. Denote by Q_H the parabolic subgroup of K^0 whose Lie algebra is $\mathfrak{q}_H = \mathfrak{g}_H(0, \geq 0)$. When $i = 1$ and $k = 2$, the above decomposition gives the natural projection

$$\pi_X \colon \mathfrak{g}_H(1, \geq 2) \to \mathfrak{g}_H(1,2). \tag{4.3}$$

The second PV attached to X is

$$\left(Q_H, \mathfrak{g}_H(1, \geq 2)\right). \tag{4.4}$$

Its generic orbit is $\mathcal{O}_X \cap \mathfrak{g}_H(1, \geq 2) = \mathcal{O} \cap \mathfrak{g}_H(1, \geq 2)$ and its singular set $\hat{S}$ is the union of the hypersurfaces $\hat{S}_i$, $i = 1, \dots, r$, defined by the polynomial functions $\hat{f}_i = f_i \circ \pi_X$.

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The $Z_K(H)^0$-varieties S_i are all quasi-homogeneous (*i.e.*, they possess an open dense orbit). This follows from the finiteness of the set of orbits of $Z_H(K)^0$ in $\mathfrak{g}_H(1,2)$. On the other hand the Q_H-varieties $\hat{S}_i$ are not necessarily quasi-homogeneous (see Section 8 for examples). Nevertheless (see [8, Proposition 7.1]) for each i there exists a unique K^0-orbit $\mathcal{O}_i$ such that $\hat{S}_i \cap \mathcal{O}_i$ is open and dense in $\hat{S}_i$. The orbits $\mathcal{O}_i$, $i = 1, \dots, r$, are not necessarily distinct (again see Section 8).

Definition 4.1 The *dominating orbit* of the hypersurface $\hat{S}_i$ is the unique K^0-orbit $\mathcal{O}_i$ whose intersection with $\hat{S}_i$ is open and dense in $\hat{S}_i$.

Recall that $\mathcal{O}$ is the basic component of $\mathcal{O}_X$. The closure of $\mathcal{O}$ can be expressed as

$$\overline{\mathcal{O}} = K^0 \cdot \mathfrak{g}_H(1, \geq 2), \tag{4.5}$$

see [5, Theorem 3.1], and $\partial\mathcal{O} = \overline{\mathcal{O}} \setminus \mathcal{O}$ (the boundary of $\mathcal{O}$) is given by

$$\partial\mathcal{O} = \bigcup_{i=1}^{r} K^0 \cdot \hat{S}_i.$$

Each child of $\mathcal{O}$ occurs among the $\mathcal{O}_i$'s but an $\mathcal{O}_i$ does not have to be a child of $\mathcal{O}$ (see Section 8 for an example).

There is a natural direct decomposition of (4.1) which we are going to describe.

First of all, the H-eigenspace decompositions of V_a and V_b induce a direct decomposition of $Z_K(H)^0$. Let $z \in Z_K(H)^0$ be arbitrary. The eigenspace $V_a(H,k)$ is z-invariant and let $z_{a,k}$ be the restriction of z to this subspace. Then $z_{a,-k}z^*_{a,k} = 1$ for $k \in \mathbf{Z}$, where * denotes the adjoint operation with respect to a suitable restriction of the form f_a. In particular, $z_{a,0} \in \mathrm{SO}\big(V_a(H,0)\big)$. One defines similarly the restriction $z_{b,k}$ of z to $V_b(H,k)$, and these restrictions have similar properties as $z_{a,k}$. We now introduce the subgroups

$$Z_a(H,k) = \{z \in K_a^0 : z_{a,i} = 1,\ i \neq \pm k\},$$
$$Z_b(H,k) = \{z \in K_b^0 : z_{b,i} = 1,\ i \neq \pm k\},$$

for $k \geq 0$. Note that $Z_a(H,0) = \mathrm{SO}\big(V_a(H,0)\big)$, and that, for $k > 0$, the natural projection map $Z_a(H,k) \to \mathrm{GL}\big(V_a(H,k)\big)$ is an isomorphism. Similar assertions hold for the groups $Z_b(H,k)$. The direct decomposition that we need is:

$$Z_K(H)^0 = Z(H)_a \times Z(H)_e \times Z(H)_b,$$

where

$$Z(H)_a = Z_a(H,0) \times Z_b(H,2) \times Z_a(H,4) \times \cdots, \tag{4.6}$$

$$Z(H)_e = Z(H,1) \times Z(H,3) \times Z(H,5) \times \cdots, \tag{4.7}$$

$$Z(H)_b = Z_b(H,0) \times Z_a(H,2) \times Z_b(H,4) \times \cdots, \tag{4.8}$$

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and $Z(H, 2k+1) = Z_a(H, 2k+1) \times Z_b(H, 2k+1)$.

Next, the H-eigenspace decompositions of V_a and V_b also induce a direct decomposition of $\mathfrak{g}_H(1,2)$. Let $u \in \mathfrak{g}_H(1,2)$ be arbitrary. Then u maps $V_a(H, k-1)$ into $V_b(H, k+1)$ and $V_b(H, k-1)$ into $V_a(H, k+1)$ for each $k \in \mathbf{Z}$. We denote the first of these restrictions by $u_{a,k}$ and the second one by $u_{b,k}$. Since $u^* = -u$, we have

$$u_{b,-k} = -u^*_{a,k}. \tag{4.9}$$

All these restrictions of u can be distributed over four infinite sequences as shown in the following diagram. (For simplicity, we write $V_a(k)$ and $V_b(k)$ instead of $V_a(H, k)$ and $V_b(H, k)$, respectively.)

$$\cdots \longrightarrow V_a(-4) \xrightarrow{u_{a,-3}} V_b(-2) \xrightarrow{u_{b,-1}} V_a(0) \xrightarrow{u_{a,1}} V_b(2) \xrightarrow{u_{b,3}} V_a(4) \longrightarrow \cdots$$

$$\cdots \longrightarrow V_b(-4) \xrightarrow{u_{b,-3}} V_a(-2) \xrightarrow{u_{a,-1}} V_b(0) \xrightarrow{u_{b,1}} V_a(2) \xrightarrow{u_{a,3}} V_b(4) \longrightarrow \cdots$$

$$\cdots \longrightarrow V_b(-3) \xrightarrow{u_{b,-2}} V_a(-1) \xrightarrow{u_{a,0}} V_b(1) \xrightarrow{u_{b,2}} V_a(3) \longrightarrow \cdots$$

$$\cdots \longrightarrow V_a(-3) \xrightarrow{u_{a,-2}} V_b(-1) \xrightarrow{u_{b,0}} V_a(1) \xrightarrow{u_{a,2}} V_b(3) \longrightarrow \cdots$$

For $k \geq 0$, denote by $\mathfrak{g}_H(1,2,2k+1)_a$ the subspace of $\mathfrak{g}_H(1,2)$ consisting of all u for which all of the above restrictions are 0 except for $u_{a,2k+1}$ and $u_{b,-2k-1}$. Similarly, for $k \geq 0$, denote by $\mathfrak{g}_H(1,2,2k+1)_b$ the subspace of $\mathfrak{g}_H(1,2)$ consisting of all u for which all of the above restrictions are 0 except for $u_{b,2k+1}$ and $u_{a,-2k-1}$. Finally, for $k \in \mathbf{Z}$, denote by $\mathfrak{g}_H(1,2,2k)$ the subspace of $\mathfrak{g}_H(1,2)$ consisting of all u for which all of the above restrictions are 0 except for $u_{a,2k}$ and $u_{b,-2k}$. All these subspaces of $\mathfrak{g}_H(1,2)$ are $Z_K(H)^0$-submodules and the projection maps

$$\mathfrak{g}_H(1,2,2k+1)_a \to \operatorname{Hom}_{\mathbf{C}}\big(V_a(H,2k), V_b(H,2k+2)\big),$$
$$\mathfrak{g}_H(1,2,2k+1)_b \to \operatorname{Hom}_{\mathbf{C}}\big(V_b(H,2k), V_a(H,2k+2)\big),$$
$$\mathfrak{g}_H(1,2,2k) \to \operatorname{Hom}_{\mathbf{C}}\big(V_a(H,2k-1), V_b(H,2k+1)\big)$$

are isomorphisms of $Z_K(H)^0$-modules.

We obtain the direct decomposition:

$$\mathfrak{g}_H(1,2) = \mathfrak{g}_H(1,2)_a \oplus \mathfrak{g}_H(1,2)_e \oplus \mathfrak{g}_H(1,2)_b, \tag{4.10}$$

where

$$\mathfrak{g}_H(1,2)_a = \mathfrak{g}_H(1,2,1)_a \oplus \mathfrak{g}_H(1,2,3)_b \oplus \mathfrak{g}_H(1,2,5)_a \oplus \cdots, \tag{4.11}$$

$$\mathfrak{g}_H(1,2)_e = \sum_{k \in \mathbf{Z}} \mathfrak{g}_H(1,2,2k), \tag{4.12}$$

$$\mathfrak{g}_H(1,2)_b = \mathfrak{g}_H(1,2,1)_b \oplus \mathfrak{g}_H(1,2,3)_a \oplus \mathfrak{g}_H(1,2,5)_b \oplus \cdots. \tag{4.13}$$

Finally we can write down the desired direct decomposition of (4.1):

$$(4.14)\quad \left(Z_K(H)^0, \mathfrak{g}_H(1,2)\right) = \left(Z(H)_a, \mathfrak{g}_H(1,2)_a\right) \times \left(Z(H)_e, \mathfrak{g}_H(1,2)_e\right) \times \left(Z(H)_b, \mathfrak{g}_H(1,2)_b\right).$$

In order to construct the basic relative invariants of (4.1), it suffices to consider separately the three factors in this decomposition.

5 Construction of Basic Relative Invariants

Let $X \in \mathcal{D}(p,q)$ and let $H = H_X$, the characteristic of X. We denote by $r(X)$ the number r of basic relative invariants $f_1, \ldots, f_r$, of $\left(Z_K(H)^0, \mathfrak{g}_H(1,2)\right)$. Our objective in this section is to compute $r = r(X)$ and construct the basic relative invariants. We obtain an explicit formula for r whose proof is based on the following result, a special case of [15, §4, Proposition 19], [14, Proposition 1.2.4].

Proposition 5.1 *If $E \in \mathfrak{g}_H(1,2)$ is generic, then $r = \dim(Z/Z'Z_E)$, where $Z = Z_K(H)^0$, Z' is the derived subgroup of Z, and Z_E the stabilizer of E in Z.*

As mentioned in the previous section, it suffices to consider the following three cases:

(i) $\mathfrak{g}_H(1,2) = \mathfrak{g}_H(1,2)_a$, *i.e.*, X is a b-vertex and has no rows of even length,
(ii) $\mathfrak{g}_H(1,2) = \mathfrak{g}_H(1,2)_e$, *i.e.*, the partition associated to X is very even,
(iii) $\mathfrak{g}_H(1,2) = \mathfrak{g}_H(1,2)_b$, *i.e.*, X is an a-vertex and has no rows of even length.

Due to symmetry of (i) and (iii), we can dismiss case (iii).

This suggests that we split X into three pieces:

$$(5.1)\qquad X = X(a) + X(e) + X(b),$$

where $X(a)$ consists of the rows of X of odd length having the letter a in the middle cell, $X(e)$ consists of the rows of X of even length, and $X(b)$ of the remaining rows of X. Thus X is an a-vertex iff $X(a) = \varnothing$, and a b-vertex iff $X(b) = \varnothing$. In case (ii), all eigenvalues of H are odd integers. In the other two cases, they are even integers. Moreover, the eigenvalues of H_a are congruent to 0 (resp. 2) modulo 4 in case (i) (resp. (iii)).

Definition 5.2 We attach to X the following three integers:

(i) $\rho_a(X) = m_a + \delta_a$, where $2m_a + 1$ is the length of the first row of $X(a)$ and $\delta_a = 0$ except when $X(a)$ has exactly two rows and these two rows have different lengths, in which case $\delta_a = 1$.
(ii) $\rho_e(X) = 2m_e - d_e$, where $2m_e$ is the length of the first row of $X(e)$ and d_e is the number of different row-lengths of $X(e)$.
(iii) $\rho_b(X) = m_b + \delta_b$, where $2m_b + 1$ is the length of the first row of $X(b)$ and $\delta_b = 0$ except when $X(b)$ has exactly two rows and these two rows have different lengths, in which case $\delta_b = 1$.

If, say $X(a) = \varnothing$, then $\rho_a(X) = 0$.

We can now state the main result of this section.

Theorem 5.3 *Let $X \in \mathcal{D}(p,q)$ and $H = H_X$. Then the number of basic relative invariants of $\big(Z_K(H)^0, \mathfrak{g}_H(1,2)\big)$ is given by $r(X) = \rho_a(X) + \rho_e(X) + \rho_b(X)$.*

Since $r(X) = r\big(X(a)\big) + r\big(X(e)\big) + r\big(X(b)\big)$, it suffices to prove that $r\big(X(a)\big) = \rho_a(X)$, $r\big(X(e)\big) = \rho_e(X)$ and $r\big(X(b)\big) = \rho_b(X)$. These proofs will be given in the next two propositions. Moreover we shall list the basic relative invariants for the cases $X(a)$ and $X(e)$.

Proposition 5.4 *If $X = X(a)$, then $r(X) = \rho_a(X)$.*

Proof In this case, $\big(Z_K(H)^0, \mathfrak{g}_H(1,2)\big)$ is of the type considered in the first subsection of the appendix. Hence we need only apply Proposition 9.2 and the remark following its proof.

Denote by s the number of rows of X and by $2m+1$ the length of its first row. The subspaces $\mathfrak{g}_H(1,2,4k+1)_a$ are nonzero for $0 \le k \le m/2$ and the subspaces $\mathfrak{g}_H(1,2,4k+3)_b$ are nonzero for $0 \le k \le (m-1)/2$. Hence there are exactly m nonzero summands in the direct decomposition (4.11). It is shown in the appendix (see Proposition 9.2 and the remark following it) that $r(X) = m$, except when $s = 2$ and the two rows of X have different lengths, in which case $r(X) = m+1$. Hence, $r(X) = \rho_a(X)$ holds. ■

We shall now list the basic relative invariants for this case by using the results proved in the appendix.

Since we assume that $X = X(a)$, we have $V(H,2i) = V_a(H,2i)$ for i even and $V(H,2i) = V_b(H,2i)$ for i odd. Denote by d_{2i} the dimension of $V(H,2i)$. Let $u \in \mathfrak{g}_H(1,2)$ be arbitrary. Define the linear transformations $u_1, u_3, u_5, \dots$ and $v_1, v_3, v_5, \dots$ by

$$u_1 = u_{a,1}, \quad u_3 = u_{b,3}, \quad u_5 = u_{a,5}, \dots$$
$$v_1 = u_{a,1}, \quad v_3 = u_{b,3}u_{a,1}, \quad v_5 = u_{a,5}u_{b,3}u_{a,1}, \dots$$

and the polynomial functions $F_{a,2k} \colon \mathfrak{g}_H(1,2) \to \mathbf{C}$, $k \ge 1$, by

$$F_{a,2k}(u) = \det(v_{2k-1}v_{2k-1}^*),$$

where the determinant is taken with respect to some fixed bases of $V(H,-2k)$ and $V(H,2k)$. If these spaces are 0-dimensional, then we adopt the convention that $F_{a,2k}$ is the constant 1. It is also convenient to define $F_{a,0} = 1$. The polynomial $F_{a,2k}$ is a relative invariant of (4.1) and the associated character is $\chi_{2k}(z) = \det(z_{2k})^2$, where $z \in Z(H)_a = Z_K(H)^0$ and $z_0, z_2, z_4, \dots$ are its components with respect to the direct decomposition (4.6).

If $d_{2k-2} > d_{2k}$, $k \geq 1$, then $F_{a,2k}$ is irreducible, with an exception which will be discussed below. On the other hand, if $d_{2k-2} = d_{2k} \geq 1$, $k \geq 1$, then $F_{a,2k}$ is the product of $F_{a,2k-2}$ and the square of the determinant function $\det(u_{2k-1})$.

First assume that if $s = 2$ then the two rows of X have the same length. Then the basic relative invariants are:

(i) $F_{a,2k}$ if $d_{2k-2} > d_{2k}$.
(ii) $f_{a,2k}(u) = \det(u_{2k-1})$ if $d_{2k-2} = d_{2k} \geq 1$.

Let us now consider the excluded case: $s = 2$ and the second row of X has length $2k+1$, $k < m$. In this case both p and q are even and the space $V_a(H,0)$ has dimension 2 with the basis $\{a_{p/2}, a_{p/2+1}\}$. Let $v_{2k+1}(a_{p/2}) = \xi c$ and $v_{2k+1}(a_{p/2+1}) = \eta c$, where c is a basis vector of the 1-dimensional space $V(H, 2k+2)$. Then a simple computation shows that $F_{a,2k+2}(u) = 2\xi\eta$. The basic relative invariants are the same as above in the general case except that $F_{a,2k+2}$ has to be replaced by the two multilinear relative invariants ξ and η.

Proposition 5.5 *If $X = X(e)$, then $r(X) = \rho_e(X)$.*

Proof In this case, $\big(Z_K(H)^0, \mathfrak{g}_H(1,2)\big)$ is of the type considered in the second subsection of the appendix. Hence we need only apply Proposition 9.5.

All rows of X have even length, the underlying partition is very even, $p = q$ is even, and all eigenvalues of H are odd integers. In particular, the number of rows of X is even, say $2s$. Denote by $2m$ the length of the first row of X and by d the number of different row-lengths in X. Thus $\rho_e(X) = 2m - d$. On the other hand, it is shown in the appendix (see Proposition 9.5) that $r(X) = 2m - d$. ■

Let us list the basic relative invariants for this case. Let $u \in \mathfrak{g}_H(1,2)$ be arbitrary and define linear transformations w_{2k}, $k \geq 0$, by

$$w_0 = u_{a,0}, \quad w_2 = u_{b,2}u_{a,0}u_{b,-2}, \quad w_4 = u_{a,4}u_{b,2}u_{a,0}u_{b,-2}u_{a,-4}, \dots .$$

Note that for k even, $w_{2k}\colon V_a(H, -2k-1) \to V_b(H, 2k+1)$, and for k odd, $w_{2k}\colon V_b(H, -2k-1) \to V_a(H, 2k+1)$. The four spaces $V_a\big(H, \pm(2i-1)\big)$, $V_b\big(H, \pm(2i-1)\big)$ have the same dimension, which we denote by d_{2i-1}, $i \geq 1$.

Define the polynomial functions $F_{e,2k+1}\colon \mathfrak{g}_H(1,2) \to \mathbf{C}$, $k \geq 0$, by $F_{e,2k+1}(u) = \det(w_{2k})$, where the determinant is taken with respect to some fixed bases of the domain and co-domain of w_{2k}. If these spaces are 0-dimensional, then we adopt the convention that $F_{e,2k+1}$ is the constant 1. The polynomial $F_{e,2k+1}$ is a relative invariant of (4.1) with the character χ_{2k+1} given by $\chi_{2k+1}(z) = \det(z_{a,2k+1})\det(z_{b,2k+1})$.

If $d_{2k-1} > d_{2k+1}$, $k \geq 1$, then $F_{e,2k+1}$ is irreducible. On the other hand, if $d_{2k-1} = d_{2k+1} \geq 1$, $k \geq 1$, then $F_{e,2k+1}$ is the product of $F_{e,2k-1}$ and the two determinant functions, $\det(u_{a,2k})$ and $\det(u_{a,-2k})$ if k is even, or $\det(u_{b,2k})$ and $\det(u_{b,-2k})$ if k is odd.

The basic relative invariants are:

(i) $F_{e,2k+1}$ if $d_{2k-1} > d_{2k+1}$ $(1 \leq k \leq m-1)$ or $k = 0$;

No.	P	Q	
1	$(ab)^{2k}a, (ba)^{2k-1}b$	$(ab)^{2k}, (ba)^{2k}$	$k \geq 1$
2	$(ba)^{2k+1}b, (ab)^{2k}a$	$(ab)^{2k+1}, (ba)^{2k+1}$	$k \geq 0$
3	$(ab)^{2k}, (ba)^{2k}, (ab)^{2m}, (ba)^{2m}$	$\left((ba)^{2k-1}b\right)^2, \left((ab)^{2m}a\right)^2$	$k > m \geq 0$
4	$(ab)^{2k+1}, (ba)^{2k+1}, (ab)^{2m+1}, (ba)^{2m+1}$	$\left((ab)^{2k}a\right)^2, \left((ba)^{2m+1}b\right)^2$	$k > m \geq 0$
5	$(ab)^{2k}a, (ba)^{2m-1}b$	$(ba)^{2k-1}b, (ab)^{2m}a$	$k > m \geq 1$
6	$(ba)^{2k+1}b, (ab)^{2m}a$	$(ab)^{2k}a, (ba)^{2m+1}b$	$k > m \geq 0$
7	$(ab)^{2k}a, (ab)^{2m}, (ba)^{2m}$	$(ba)^{2k-1}b, \left((ab)^{2m}a\right)^2$	$k > m \geq 0$
8	$(ba)^{2k+1}b, (ab)^{2m+1}, (ba)^{2m+1}$	$(ab)^{2k}a, \left((ba)^{2m+1}b\right)^2$	$k > m \geq 0$
9	$(ab)^{2k}, (ba)^{2k}, (ba)^{2m-1}b$	$\left((ba)^{2k-1}b\right)^2, (ab)^{2m}a$	$k > m \geq 1$
10	$(ab)^{2k+1}, (ba)^{2k+1}, (ab)^{2m}a$	$\left((ab)^{2k}a\right)^2, (ba)^{2m+1}b$	$k > m \geq 0$

Table 1: Disjoint minimal pure b-pairs (P, Q) in $\Gamma(p, q)$.

(ii) $f^a_{e,2k+1}(u) = \det(u_{a,-2k})$ and $f^b_{e,2k+1}(u) = \det(u_{a,2k})$ if $d_{2k-1} = d_{2k+1}$ and k is even $(1 \leq k \leq m)$;

(iii) $f^a_{e,2k+1}(u) = \det(u_{b,2k})$ and $f^b_{e,2k+1}(u) = \det(u_{b,-2k})$ if $d_{2k-1} = d_{2k+1}$ and k is odd $(1 \leq k \leq m)$.

The basic relative invariants of PV (4.1) when $X = X(b)$ are defined analogously to the case when $X = X(a)$, but we will then use the subscript b instead of a. In the general case, $X = X(a) + X(e) + X(b)$, the basic relative invariants of PV (4.1) are just the relative invariants of the three pieces $X(a)$, $X(e)$ and $X(b)$. This is a consequence of the decomposition (4.14).

The basic relative invariants of PV (4.4) are obtained from those of (4.1) by composing the latter with the projection map π_X. We shall use the hat to denote these relative invariants. For instance, we have $\hat{F}_{a,2i} = F_{a,2i} \circ \pi_X$, $\hat{f}_{a,2i} = f_{a,2i} \circ \pi_X$.

6 Proof of the Closure Ordering Conjecture

The objective of this section is to prove the closure ordering conjecture (3.2).

We say that a pure pair (X, Y) is *minimal* if $X \to Y$, *i.e.*, Y is a child of X in the diagram $\Gamma(p, q)$. Every minimal pure b-pair (X, Y) has the form

$$(X, Y) = (P + Z, Q + Z),$$

where P, Q, Z are orthogonal ab-diagrams and (P, Q) is a disjoint minimal pure b-pair. These latter pairs have been enumerated in [11, Table 8]. For the convenience of the reader, we reproduce it here (Table 1).

We say that a pure pair (X, Y) is *maximal* if there is no pure pair (P, Q) such that $Q \leq Y < X \leq P$ and $(P, Q) \neq (X, Y)$.

The diagrams Q are written in this table in abbreviated form. For instance, the symbol $\left((ba)^{2k-1}b\right)^2$ in the row for type 3 stands for two identical rows of type $(ba)^{2k-1}b$.

We need some additional notation. Let $X \in \mathcal{D}(p,q)$. The eigenspaces $V_a(H_X,i)$ and $V_b(H_X,i)$ were introduced in Section 4. We now set $V(H_X,i) = V_a(H_X,i) + V_b(H_X,i)$ and define

$$V^{\downarrow}(H_X,k) = \sum_{i\le k} V(H_X,i), \quad V^{\uparrow}(H_X,k) = \sum_{i\ge k} V(H_X,i),$$
$$V_a^{\downarrow}(H_X,k) = \sum_{i\le k} V_a(H_X,i), \quad V_a^{\uparrow}(H_X,k) = \sum_{i\ge k} V_a(H_X,i),$$
$$V_b^{\downarrow}(H_X,k) = \sum_{i\le k} V_b(H_X,i), \quad V_b^{\uparrow}(H_X,k) = \sum_{i\ge k} V_b(H_X,i).$$

It is easy to verify that if $E \in \mathfrak{g}_{H_X}(1,\ge 2)$, then

$$E\left(V^{\uparrow}(H_X,k)\right) \subset V^{\uparrow}(H_X,k+2),$$

and consequently

$$E\left(V_a^{\uparrow}(H_X,k)\right) \subset V_b^{\uparrow}(H_X,k+2), \quad E\left(V_b^{\uparrow}(H_X,k)\right) \subset V_a^{\uparrow}(H_X,k+2).$$

Definition 6.1 By using the notations n_a, n_b, and $X^{(k)}$ introduced in Section 2, we define:

$$r_{k,a}(X) = n_a(X^{(k)}), \quad r_{k,b}(X) = n_b(X^{(k)}),$$

for $k \ge 0$. If k is odd, then one has $r_{k,a}(X) = r_{k,b}(X)$ and we denote this common value simply by $r_k(X)$. We also set, for all k,

$$n_k(X) = r_{k,a}(X) + r_{k,b}(X).$$

Note that, when $k = 0$, we have $r_{0,a}(X) = p$, $r_{0,b}(X) = q$ and $n_0(X) = n$.

The geometric meaning of these numbers was explained in [11, Lemma 11]:

$$r_{k,a}(X) = \dim E^k(V) \cap V_a, \quad r_{k,b}(X) = \dim E^k(V) \cap V_b,$$

where $E \in \mathfrak{g}_{H_X}(1,2)$ is a generic element.

The following formula follows from the $\mathfrak{sl}_2$-theory:

$$(6.1)\qquad r_{k,a}(X) = \dim V_a^{\uparrow}(H_X,k) + \begin{cases} \dim V_a^{\downarrow}(H_X,-k-1) & \text{if } k \text{ is even,} \\ \dim V_b^{\downarrow}(H_X,-k-1) & \text{if } k \text{ is odd.} \end{cases}$$

It can be rewritten as:

$$(6.2)\qquad r_{k,a}(X) = \dim V_a^{\uparrow}(H_X,k+1) + \begin{cases} \dim V_a^{\downarrow}(H_X,-k) & \text{if } k \text{ is even,} \\ \dim V_b^{\downarrow}(H_X,-k) & \text{if } k \text{ is odd.} \end{cases}$$

Similar formulae are valid for $r_{k,b}(X)$.

Lemma 6.2 *Let $X, Y \in \mathcal{D}(p, q)$ with $Y \leq X$ and let $E \in \mathcal{O}_Y^X$. If $r_{k,a}(Y) = r_{k,a}(X)$, then for k even:*

(i) $E^k\big(V_a^{\uparrow}(H_X, -k)\big) = V_a^{\uparrow}(H_X, k)$;
(ii) $E^k\big(V_a^{\uparrow}(H_X, 1-k)\big) = V_a^{\uparrow}(H_X, k+1)$;
(iii) $\ker(E^k) \cap V_a \subset V_a^{\uparrow}(H_X, 1-k)$;

and for k odd:

(iv) $E^k\big(V_b^{\uparrow}(H_X, -k)\big) = V_a^{\uparrow}(H_X, k)$;
(v) $E^k\big(V_b^{\uparrow}(H_X, 1-k)\big) = V_a^{\uparrow}(H_X, k+1)$;
(vi) $\ker(E^k) \cap V_b \subset V_b^{\uparrow}(H_X, 1-k)$.

Similar statements are valid when $r_{k,b}(Y) = r_{k,b}(X)$.

Proof Assume that k is even. Since $E \in \mathcal{O}_Y$, we have $r_{k,a}(Y) = \dim E^k(V_a)$. Assertion (i) follows from

$$V_a = V_a^{\downarrow}(H_X, -k-1) + V_a^{\uparrow}(H_X, -k), \quad E^k\big(V_a^{\uparrow}(H_X, -k)\big) \subset V_a^{\uparrow}(H_X, k),$$

and equality (6.1). Assertion (ii) follows from

$$V_a = V_a^{\downarrow}(H_X, -k) + V_a^{\uparrow}(H_X, 1-k), \quad E^k\big(V_a^{\uparrow}(H_X, 1-k)\big) \subset V_a^{\uparrow}(H_X, k+1),$$

and equality (6.2). As $p = \dim V_a^{\uparrow}(H_X, 1-k) + \dim V_a^{\downarrow}(H_X, -k)$, (6.2) implies that

$$\dim V_a^{\uparrow}(H_X, 1-k) - \dim V_a^{\uparrow}(H_X, k+1) = p - r_{k,a}(X) = \dim\big(V_a \cap \ker(E^k)\big).$$

Now (iii) follows from the last inclusion displayed above.

The proofs for k odd are similar. ■

Let us say that the *length* of the pair (X, Y), with $Y < X$, is the length m of the shortest chain

$$X = X_0 \to X_1 \to \cdots \to X_{m-1} \to X_m = Y \tag{6.3}$$

of elementary moves that joins X to Y.

The following simple observation will be useful. Assume that $X = P+Z$, $Y = Q+Z$ and $Y < X$. Then $Q < P$ and the length of (P, Q) does not exceed that of (X, Y).

Lemma 6.3 *Let (X, Y) be a pure b-pair with X and Y disjoint, and let $2l+3$ or $2l+2$ be the length of the first row of X, where l is a nonnegative integer. Then $r_{2l,a}(Y) = r_{2l,a}(X)$ if l is even and $r_{2l,b}(Y) = r_{2l,b}(X)$ if l is odd.*

Proof Assume that the assertion of the lemma is false and let (X, Y) be a counter-example of minimal length, m. Choose a sequence of elementary moves as in (6.3). It is easy to see that $m > 1$.

We give the detailed proof for l odd.

Since X is a b-vertex and l is odd, if the first row of X has length $2l+3$, it is necessarily of type $(ab)^{l+1}a$. Since m is minimal (i.e., our counter-example is minimal), X_1 has no rows of length $> 2l+1$. Consequently, $X = X^{\wedge} + Z$, where either

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(i) $X^\wedge = (ab)^{l+1}a$ or
(ii) $X^\wedge = (ab)^{l+1}, (ba)^{l+1}$,

and, in both cases, Z has no rows of length $> 2l+1$.

The first elementary step $X \to X_1$ has to move two a-cells from $X^\wedge$ to subsequent rows. Consequently, the first row of X_1 is $(ba)^l b$.

The minimality of m also implies that

$$r_{2l,b}(X) = r_{2l,b}(X_1) = \cdots = r_{2l,b}(X_{m-1}) > r_{2l,b}(Y). \tag{6.4}$$

Denote by s (resp. t) the number of rows of Z (resp. X_1) of length $2l+1$. Such rows are necessarily of type $(ba)^l b$.

In case (i) we must have $s = 0$. Otherwise, the first move $X \to X_1$ must be of type 1 (see Table 1) and X_1 would have rows of length $2l+2$.

In case (ii), we have $t = s+2$ and (6.4) implies that X_{m-1} also has exactly t rows of length $2l+1$. Consequently, Y must have at least s rows of length $2l+1$. As X and Y are disjoint, we infer that $s = 0$.

We conclude that Z has no rows of length $2l+1$.

We claim that, if the first row of Z has even length, say $2d$, then d must be even. Assume that d is odd. As X is a b-vertex, Z has no rows of type $(ba)^{d-1}b$. This implies that X_1 must contain two rows of type $(ab)^d a$, contradicting the fact that X_1 is a b-vertex. Our claim is proved.

In particular, Z has no rows of length $2l$. Hence, the first row of Z has length $\leq 2l-1$.

Assume that the first row of Z is $(ab)^d a$. Then d must be even and $X \to X_1$ implies that Z has at least two rows of length $2d$ if $d > 0$, or else there would not be an elementary move that moves two a-cells from the first row of X.

We deduce that either

$$Z = \left((ab)^d a\right)^s, (ab)^d, (ba)^d, U, \quad d \text{ even}, \quad 0 \leq d \leq l-1, \quad s \geq 0,$$

where U has no rows of length $> 2d$, or

$$Z = (ba)^{d-1}b, U, \quad d \text{ even}, \quad 2 \leq d \leq l-1,$$

where U has no rows of length $> 2d-1$ and we set $s = -1$. Then

$$X_1 = X_1^\wedge, \left((ab)^d a\right)^{s+2}, U \quad \text{and} \quad X_{m-1} = X_1^\wedge + X_{m-1}^\vee,$$

where $X_1^\wedge$ is $(ba)^l b$ in case (i) and $\left((ba)^l b\right)^2$ in case (ii), and $X_{m-1}^\vee$ has no rows of length $> 2d+1$. By comparing X and X_1, we see that

$$r_{2i+1}(X_{m-1}) \leq r_{2i+1}(X_1) = r_{2i+1}(X) - 1, \quad d \leq i \leq l;$$
$$r_{2i,a}(X_{m-1}) \leq r_{2i,a}(X_1) = r_{2i,a}(X) - 2, \quad d+1 \leq i \leq l.$$

The last elementary step, $X_{m-1} \to Y$, will move at least one b-cell from $X_1^{\wedge}$ to $X_{m-1}^{\vee}$. This implies that

$$r_{2i,b}(Y) \leq r_{2i,b}(X) - 1, \quad d+1 \leq i \leq l.$$

Set

$$P = \begin{cases} (ab)^l a, \left((ab)^d a\right)^{s+1}, (ba)^d b, U & \text{in case (i)} \\ (ab)^l a, (ba)^l b, \left((ab)^d a\right)^{s+1}, (ba)^d b, U & \text{in case (ii).} \end{cases}$$

By using the above inequalities, it is easy to check that $Y \leq P < X$. Since P is not a b-vertex, we have a contradiction.

This concludes the proof when l is odd. The case when l is even gives a contradiction by using the same arguments except that the roles of "a" and "b" have to be interchanged while still preserving the hypothesis that (X, Y) is a pure b-pair. ■

The sets introduced in the next definition will play an important role in our proof.

Definition 6.4 For $X, Y \in \mathcal{D}(p,q)$ with $Y \leq X$, we define

$$\mathcal{O}_Y^X = \mathcal{O}_Y \cap \mathfrak{g}_{H_X}(1, \geq 2).$$

Note that these sets are nonempty, the set $\mathcal{O}_X^X$ is the generic orbit of PV (4.4), and that, for $Y < X$, the set $\mathcal{O}_Y^X$ is contained in the singular set $\hat{S}$ of this PV.

Lemma 6.5 *Let $X, Y \in \mathcal{D}(p,q)$, $X \to Y$, and let $E \in \mathcal{O}_Y^X$. Let $\hat{S}_\mu$ be an irreducible component of the singular set $\hat{S}$ of PV (4.4). If $E \in \hat{S}_\mu$, then $\mathcal{O} = K^0 \cdot E$ is the dominating orbit of $\hat{S}_\mu$.*

Proof Let $\mathcal{O}_\mu$ be the dominating K^0-orbit of $\hat{S}_\mu$ and let Z be its ab-diagram. Since E belongs to the closure of $\mathcal{O}_\mu$, we have $Y \leq Z$. As $Z < X$ and $X \to Y$, we infer that $Y = Z$. Hence $\mathcal{O}$ and $\mathcal{O}_\mu$ are connected components of $\mathcal{O}_Y$, and it follows that $\mathcal{O} = \mathcal{O}_\mu$. ■

Lemma 6.6 *Let (X, Y) be a pure b-pair, $Y < Z < X$, and assume that (X, Z) and (Z, Y) split. If $\mathcal{O}_Y^X \subset \overline{\mathcal{O}_Z^{\mathrm{I}}}$, then (X, Y) splits.*

Proof Let $\mathcal{O}$ be the basic component of $\mathcal{O}_X$. By (4.5), we have

$$\begin{aligned} \mathcal{O}_Y \cap \overline{\mathcal{O}} &= \mathcal{O}_Y \cap \left(K^0 \cdot \mathfrak{g}_{H_X}(1, \geq 2)\right) \\ &= K^0 \cdot \mathcal{O}_Y^X \\ &\subset \mathcal{O}_Y \cap \overline{\mathcal{O}_Z^{\mathrm{I}}} = \mathcal{O}_Y^{\mathrm{I}}. \end{aligned}$$

Hence (X, Y) splits. ■

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We shall denote by E_0 the projection $\pi_X(E)$ of $E \in \mathfrak{g}_{H_X}(1, \geq 2)$. If, say $x \in V_a(H_X, j)$ and i is odd, then $E^i(x) \in V_b^{\uparrow}(H_X, j+2i)$ and $E_0^i(x)$ is the component of $E^i(x)$ in $V_b(H_X, j+2i)$. We point out that if X is a b-vertex, then $V(H_X, 4i) = V_a(H_X, 4i)$ and $V(H_X, 4i+2) = V_b(H_X, 4i+2)$. Recall that the basic relative invariants F and/or f (with a variety of subscripts and/or superscripts) of PVs (4.1) and (4.4) have been introduced in the previous section.

Lemma 6.7 *Let (X, Y) be a pure b-pair with $X = P + Z$, $Y = Q + Z$ and P and Q disjoint. Let L be the length of the first row of P and write $L = 2l + 3$ or $L = 2l + 2$, where l is a nonnegative integer. Then $\mathcal{O}_Y^X$ is contained in the (not necessarily irreducible) hypersurface:*

(i) $\hat{F}_{e,2l+1} = 0$ *if $L = 2l + 2$;*
(ii) $\hat{F}_{a,2l+2} = 0$ *if $L = 2l + 3$ and l is odd;*
(iii) $\hat{F}_{b,2l+2} = 0$ *if $L = 2l + 3$ and l is even.*

Proof Let $E \in \mathcal{O}_Y^X$ be arbitrary and denote by E_0 its projection to $\mathfrak{g}_{H_X}(1, 2)$. Clearly, P and Q are not empty and $Q < P$. The first row of Q has length $< L$.

(i) In this case $r_{2l+1}(Y) < r_{2l+1}(X)$ and $n_i(Y) = n_i(X)$ for $i > 2l+1$. We claim that the map $V(H_X, -2l-1) \to V(H_X, 2l+1)$, induced by E_0^{2l+1}, is not an isomorphism.

By (6.1) we have

$$r_{2l+1}(X) = \dim V_b^{\uparrow}(H_X, 2l+1) + \dim V_a^{\downarrow}(H_X, -2l-2).$$

As $n_{2l+2}(Y) = n_{2l+2}(X)$ and $Y < X$, we have $r_{2l+2,a}(Y) = r_{2l+2,a}(X)$ and $r_{2l+2,b}(Y) = r_{2l+2,b}(X)$. Lemma 6.2 shows that $V_a \cap \ker E^{2l+2} \subset V_a^{\uparrow}(H_X, -2l-1)$. Consequently, from the direct decomposition

$$V_a = V_a^{\uparrow}(H_X, -2l-1) \oplus V_a^{\downarrow}(H_X, -2l-2)$$

we obtain the direct decomposition

$$E^{2l+1}(V_a) = E^{2l+1}\left(V_a^{\uparrow}(H_X, -2l-1)\right) \oplus E^{2l+1}\left(V_a^{\downarrow}(H_X, -2l-2)\right).$$

Hence

$$r_{2l+1}(Y) = \dim E^{2l+1}\left(V_a^{\uparrow}(H_X, -2l-1)\right) + \dim V_a^{\downarrow}(H_X, -2l-2).$$

As $r_{2l+1}(Y) < r_{2l+1}(X)$, it follows that $E^{2l+1}\left(V_a^{\uparrow}(H_X, -2l-1)\right)$ is a proper subspace of $V_b^{\uparrow}(H_X, 2l+1)$. On the other hand, as $n_{2l+2}(Y) = n_{2l+2}(X)$, Lemma 6.2 implies that

$$E^{2l+2}\left(V_b^{\uparrow}(H_X, -2l-2)\right) = V_b^{\uparrow}(H_X, 2l+2).$$

As

$$E^{2l+1}\left(V_a^{\uparrow}(H_X, -2l-1)\right) \supset E^{2l+2}\left(V_b^{\uparrow}(H_X, -2l-2)\right),$$

our claim is proved.

It follows that $\hat{F}_{e,2l+1}(E) = F_{e,2l+1}(E_0) = 0$.

(ii) In this case $n_{2l+2}(Y) < n_{2l+2}(X)$ and $n_i(Y) = n_i(X)$ for $i > 2l+2$. We claim that the map $V(H_X, -2l-2) \to V(H_X, 2l+2)$, induced by E_0^{2l+2}, is not an isomorphism.

The first row of P is necessarily of type $(ab)^{l+1}a$. As P and Q are disjoint, we must have $r_{2l+2,a}(Y) < r_{2l+2,a}(X)$. By (6.1) we have

$$r_{2l+2,a}(X) = \dim V_a^{\uparrow}(H_X, 2l+2) + \dim V_a^{\downarrow}(H_X, -2l-3).$$

As $r_{2l+3}(Y) = r_{2l+3}(X)$, Lemma 6.2 shows that $\ker E^{2l+3} \cap V_a \subset V_a^{\uparrow}(H_X, -2l-2)$. Consequently, from the direct decomposition

$$V_a = V_a^{\downarrow}(H_X, -2l-3) \oplus V_a^{\uparrow}(H_X, -2l-2),$$

we obtain the direct decomposition

$$E^{2l+2}(V_a) = E^{2l+2}\big(V_a^{\downarrow}(H_X, -2l-3)\big) \oplus E^{2l+2}\big(V_a^{\uparrow}(H_X, -2l-2)\big).$$

Hence

$$r_{2l+2,a}(Y) = \dim V_a^{\downarrow}(H_X, -2l-3) + \dim E^{2l+2}\big(V_a^{\uparrow}(H_X, -2l-2)\big).$$

As $r_{2l+2,a}(Y) < r_{2l+2,a}(X)$, the image $E^{2l+2}\big(V_a^{\uparrow}(H_X, -2l-2)\big)$ is a proper subspace of $V_a^{\uparrow}(H_X, 2l+2)$. On the other hand, as $r_{2l+3}(Y) = r_{2l+3}(X)$, Lemma 6.2 implies that

$$E^{2l+3}\big(V_b^{\uparrow}(H_X, -2l-3)\big) = V_a^{\uparrow}(H_X, 2l+3).$$

As

$$E^{2l+2}\big(V_a^{\uparrow}(H_X, -2l-2)\big) \supset E^{2l+3}\big(V_b^{\uparrow}(H_X, -2l-3)\big),$$

our claim is proved. It follows that $\hat{F}_{a,2l+2}(E) = F_{a,2l+2}(E_0) = 0$.

(iii) The proof in this case is similar to that of (ii). ■

We can now prove our main result.

Theorem 6.8 *The diagram $\Delta(p,q)$ is the Hasse diagram of the partially ordered set $(\mathcal{N}/K^0, \leq)$.*

Proof By Theorem 3.7, it suffices to prove that every pure pair (X, Y) in $\Gamma(p,q)$ splits. Without any loss of generality, it suffices to do that for pure b-pairs only. We proceed by induction on X by using the partial order "$\leq$" of $\Gamma(p,q)$. Thus our first induction hypothesis is that if $Z < X$ and (Z, U) is a pure b-pair, then (Z, U) splits. For fixed b-vertex X, we shall use downward induction on Y. Thus we assume that if (X, Y) is a pure b-pair and $Y < Z < X$, then (X, Z) splits. Note that if $X \to Y$, then this condition is vacuously satisfied.

In order to show that (X, Y) splits, it suffices to prove that

($*$) there exists an irreducible hypersurface $\hat{S}_\mu$, an irreducible component of the singular set $\hat{S}$ of (4.4), such that $\mathcal{O}_Y^X \subset \hat{S}_\mu$.

Indeed, let $\mathcal{O}$ be the basic component of $\mathcal{O}_X$, and let $\mathcal{O}_\mu$ be the dominating K^0-orbit of $\hat{S}_\mu$ and Z the ab-diagram of $\mathcal{O}_\mu$. Then $Y \leq Z < X$. If $Y < Z$, then our two induction hypotheses imply that (Z, Y) and (X, Z) split. Since $\mathcal{O}_\mu \subset \overline{\mathcal{O}} \cap \mathcal{O}_Z = \mathcal{O}_Z^{\mathrm{I}}$, we have $\mathcal{O}_Y^X \subset \overline{\mathcal{O}_\mu} \subset \overline{\mathcal{O}_Z^{\mathrm{I}}}$. Hence (X, Y) splits by Lemma 6.6. On the other hand, if $Y = Z$ then $\mathcal{O}_Y^X \subset \overline{\mathcal{O}_\mu} \cap \mathcal{O}_Y = \mathcal{O}_\mu$, and consequently (X, Y) splits.

In order to prove the assertion ($*$) we proceed as follows. Let $E \in \mathcal{O}_Y^X$ be arbitrary and denote by E_0 its projection to $\mathfrak{g}_{H_X}(1, 2)$. Let P, Q, L and l be defined as in Lemma 6.7. As in that lemma, we distinguish three cases.

(i) $L = 2l + 2$. Then $r_{2l+1}(Y) < r_{2l+1}(X)$ and $n_i(Y) = n_i(X)$ for $i > 2l + 1$.

By Lemma 6.7, the set $\mathcal{O}_Y^X$ is contained in the hypersurface $\hat{F}_{e,2l+1} = 0$. We are done if the relative invariant $F_{e,2l+1}$ is irreducible, *i.e.*, if $V(H_X, 2l - 1)$ has larger dimension than $V(H_X, 2l + 1)$. From now on we assume that these dimensions are equal.

We assume that l is odd. The l even case can be treated similarly. By Lemma 6.3, $r_{2l,b}(Q) = r_{2l,b}(P)$ and, consequently, $r_{2l,b}(Y) = r_{2l,b}(X)$. By Lemma 6.2, the map $V_b(H_X, 1 - 2l) \to V_b(H_X, 2l + 1)$, induced by E_0^{2l}, is onto. As these two spaces have the same dimension, this map is in fact an isomorphism. By taking the adjoints, we infer that the map $V_b(H_X, -2l - 1) \to V_b(H_X, 2l - 1)$, induced by E_0^{2l}, is also an isomorphism. As $F_{e,2l+1}(E_0) = 0$, it follows that the map $V_b(H_X, 2l-1) \to V_a(H_X, 2l+1)$ induced by E_0 is not an isomorphism, and consequently $\mathcal{O}_Y^X$ is contained in the irreducible hypersurface $\hat{f}^a_{e,2l+1} = 0$.

(ii) $L = 2l + 3$ and l is odd. By Lemma 6.7, $\mathcal{O}_Y^X$ is contained in the hypersurface $\hat{F}_{a,2l+2} = 0$. If $V(H_X, 2l)$ has larger dimension than $V(H_X, 2l + 2)$, then, with one exception which will be dealt with later, the relative invariant $F_{a,2l+2}$ is irreducible, and we are done.

Now assume that the spaces $V(H_X, 2l)$ and $V(H_X, 2l+2)$ have the same dimension. By Lemma 6.3, $r_{2l,b}(Q) = r_{2l,b}(P)$ and, consequently, $r_{2l,b}(Y) = r_{2l,b}(X)$. Then by Lemma 6.2,

$$E^{2l}\big(V_b^{\uparrow}(H_X, -2l)\big) = V_b^{\uparrow}(H_X, 2l)$$

and

$$E^{2l}\big(V_b^{\uparrow}(H_X, -2l + 1)\big) = V_b^{\uparrow}(H_X, 2l + 1).$$

As the spaces $V_b(H_X, -2l)$ and $V_b(H_X, 2l)$ have the same dimension, it follows that the map $V_b(H_X, -2l) \to V_b(H_X, 2l)$, induced by E_0^{2l}, is an isomorphism. Hence $\hat{F}_{a,2l}(E) = F_{a,2l}(E_0) \neq 0$, *i.e.*, $\mathcal{O}_Y^X$ is contained in the irreducible hypersurface $\hat{f}_{a,2l+2} = 0$.

The exceptional case occurs when $X(a)$ consists of exactly two rows: One of type $(ab)^{l+1}a$ and the other of type $(ba)^l b$. Then $F_{a,2l+2} = 2\xi\eta$, where ξ and η are two different irreducible polynomials. Denote by $\hat{S}_\xi$ and $\hat{S}_\eta$ the corresponding hypersurfaces in $\mathfrak{g}_{H_X}(1, \geq 2)$. We know that $\mathcal{O}_Y^X \subset \hat{S}_\xi \cup \hat{S}_\eta$. Let Z be the ab-diagram obtained from X by replacing $X(a)$ with the pair of rows of length $2l + 2$. Obviously, the pair (X, Z) is also a pure b-pair. By using Z instead of Y, we conclude that $\mathcal{O}_Z^X \subset \hat{S}_\xi \cup \hat{S}_\eta$. Since $X \to Z$ and both ${}^{\mathrm{I}}\mathcal{O}_Z^{\mathrm{I}}$ and ${}^{\mathrm{II}}\mathcal{O}_Z^{\mathrm{I}}$ are contained in the closure of $\mathcal{O}_X^{\mathrm{I}}$ (see Theorem 3.3), Lemma 6.5 implies that one of the orbits ${}^{\mathrm{I}}\mathcal{O}_Z^{\mathrm{I}}$ and ${}^{\mathrm{II}}\mathcal{O}_Z^{\mathrm{I}}$ is the dominating orbit of the hypersurface $\hat{S}_\xi$ and the other the dominating orbit of $\hat{S}_\eta$. As $\dim(\mathcal{O}_Z) < \dim(\mathcal{O}_X)$, the induction hypothesis implies that (Z, Y) splits. Consequently, both $\mathcal{O}_Y \cap \hat{S}_\xi$ and $\mathcal{O}_Y \cap \hat{S}_\eta$ are contained in $\mathcal{O}_Y^{\mathrm{I}}$. It follows that $\mathcal{O}_Y^X \subset \mathcal{O}_Y^{\mathrm{I}}$, and so (X, Y) splits.

(iii) $L = 2l + 3$ and l is even. The proof in this case is similar to that of (ii). ■

7 Factorization of Kac's Relative Invariant

This section is supplementary to our main results proved in Section 6, and the reader may wish to proceed directly to Section 8 after reading the definitions of ω and φ_X.

We will need another involution $\omega \in G$. It leaves V_a and V_b invariant and acts on the basis vectors as follows:

$$\omega(a_i) = a_{p+1-i}, \quad 1 \le i \le p; \qquad \omega(b_j) = b_{q+1-j}, \quad 1 \le j \le q.$$

Under the adjoint action, ω leaves invariant $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{h}$ and acts as -1 on $\mathfrak{h}$. In particular, $\omega(H_X) = -H_X$, and consequently $\omega\big(V_a(H_X, i)\big) = V_a(H_X, -i)$ and $\omega\big(V_b(H_X, i)\big) = V_b(H_X, -i)$ for all k's.

By analogy with a construction of Kac [13, Lemma 1.4], for a given $X \in \mathcal{D}(p, q)$ and $x \in \mathfrak{g}_{H_X}(1, 2)$, we define the linear operator $A_x \in \mathrm{End}\big(\mathfrak{g}_{H_X}(1, 2)\big)$ by

$$A_x(y) = \big[x, [x, \omega(y)]\big] \tag{7.1}$$

and set $\varphi_X(x) = \det A_x$.

Lemma 7.1 *The polynomial φ_X is a relative invariant of PV (4.1) with character $\det\big(\mathrm{Ad}(z)|_{\mathfrak{g}_{H_X}(1,2)}\big)^2$, $z \in Z_K(H_X)^0$.*

Proof For $x, y \in \mathfrak{g}_{H_X}(1, 2)$ and $z \in Z_K(H_X)^0$, we have

$$\begin{aligned} A_{z\cdot x}(y) &= \big[z \cdot x, [z \cdot x, \omega(y)]\big] = z \cdot \big[x, [x, z^{-1} \cdot \omega(y)]\big] \\ &= z \cdot A_x\big(\omega\, \mathrm{Ad}(z^{-1})\omega(y)\big)\,. \end{aligned}$$

As

$$\begin{aligned} \det\big(\omega \circ \mathrm{Ad}(z^{-1})|_{\mathfrak{g}_{H_X}(1,-2)} \circ \omega\big) &= \det\big(\mathrm{Ad}(z^{-1})|_{\mathfrak{g}_{H_X}(1,-2)}\big) \\ &= \det\big(\mathrm{Ad}(z)|_{\mathfrak{g}_{H_X}(1,2)}\big)\,, \end{aligned}$$

we have

$$\varphi_X(z \cdot x) = \big(\det \mathrm{Ad}(z)|_{\mathfrak{g}_{H_X}(1,2)}\big)^2 \varphi_X(x).$$ ■

We shall refer to φ_X as *Kac's relative invariant* of PV (4.1). As in [13, Proposition 1.1], one can show that $x \in \mathfrak{g}_{H_X}(1, 2)$ is generic if and only if $\varphi_X(x) \neq 0$. This is also a consequence of the factorization of φ_X which we are going to prove in the next theorem.

If $x \in \mathfrak{g}_{H_X}(1,2)_u$ and $y \in \mathfrak{g}_{H_X}(1,2)_v$ for $u, v \in \{a, e, b\}$ (see (4.10)), it is easy to verify that $[x, \omega(y)] = 0$ if $u \neq v$ and $A_x(y) \in \mathfrak{g}_{H_X}(1,2)_u$ if $u = v$. It follows that Kac's relative invariant has the following factorization:

$$\varphi_X = \varphi_{X(a)}\varphi_{X(e)}\varphi_{X(b)}, \tag{7.2}$$

where $X = X(a) + X(e) + X(b)$ is the canonical decomposition of X (see (5.1)).

We shall now factorize Kac's relative invariant when X is equal to $X(a)$, $X(e)$ or $X(b)$.

Theorem 7.2 *Let $X \in \mathcal{D}(p,q)$.*

(i) *If $X = X(a)$, then*

$$\varphi_X = c_a \Big(\prod_{i\geq 1} F_{a,2i}^{d_{2i-2}-d_{2i+2}}\Big)^2, \tag{7.3}$$

where $d_{2i} = \dim V(H_X, 2i)$.

(ii) *If $X = X(e)$, then*

$$\varphi_X = c_e \Big(\prod_{i\geq 1} F_{e,2i+1}^{d_{2i-1}-d_{2i+3}}\Big)^2, \tag{7.4}$$

where $d_{2i+1} = \dim V_a(H_X, 2i+1) = \dim V_b(H_X, 2i+1)$.

(iii) *If $X = X(b)$, then*

$$\varphi_X = c_b \Big(\prod_{i\geq 1} F_{b,2i}^{d_{2i-2}-d_{2i+2}}\Big)^2, \tag{7.5}$$

where $d_{2i} = \dim V(H_X, 2i)$.

The factors c_a, c_e, c_b denote nonzero constants.

Proof For $z \in Z = Z_K(H_X)^0$, let $z_{a,i}$ (resp. $z_{b,i}$) be the restriction of z to $V_a(H_X, i)$ (resp. $V_b(H_X, i)$), and let $\psi_{a,i}$ (resp. $\psi_{b,i}$) be the character of Z defined by $\psi_{a,i}(z) = \det(z_{a,i})$ (resp. $\psi_{b,i}(z) = \det(z_{b,i})$). Note that $\psi_{a,-i} = -\psi_{a,i}$ and $\psi_{b,-i} = -\psi_{b,i}$. In particular, $\psi_{a,0} = \psi_{b,0} = 0$.

(i) Since a relative invariant is uniquely determined (up to a multiplicative constant) by its character, it suffices to show that the two members of (7.3) have the same character. As $X = X(a)$, we have $V(H_X, i) = 0$ for i odd, $V(H_X, 4i) = V_a(H_X, 4i)$ and $V(H_X, 4i+2) = V_b(H_X, 4i) + 2$. Consequently, $\psi_{a,4i+2} = \psi_{b,4i} = 0$ for all integers i. To simplify the notation, we shall write $\psi_{2i} = \psi_{a,2i}$ if i is even and $\psi_{2i} = \psi_{b,2i}$ if i is odd. It was shown in Section 5 that the character of $F_{a,2i}$ is $2\psi_{2i}$. Hence the character of the second member of (7.3) is

$$2\sum_{i\geq 1}(d_{2i-2} - d_{2i+2})\psi_{2i}.$$

As $X = X(a)$, we have $\mathfrak{g}_{H_X}(1,2) = \mathfrak{g}_{H_X}(1,2)_a$, see (4.11). To simplify the notation, we set $\mathfrak{g}(1,2,2i{+}1) = \mathfrak{g}_{H_X}(1,2,2i{+}1)_a$ for i even and $\mathfrak{g}(1,2,2i{+}1) = \mathfrak{g}_{H_X}(1,2,2i{+}1)_b$ for i odd. We also set $z_{2i} = z_{a,2i}$ if i is even and $z_{2i} = z_{b,2i}$ if i is odd. Each summand $\mathfrak{g}(1,2,2i+1)$ is $\mathrm{Ad}(z)$-invariant for $z \in Z$. Hence

$$\det\big(\mathrm{Ad}(z)|_{\mathfrak{g}_{H_X}(1,2)}\big) = \prod_{i\geq 0} \det\big(\mathrm{Ad}(z)|_{\mathfrak{g}(1,2,2i+1)}\big).$$

For $z \in Z$, $u \in \mathfrak{g}(1,2,2i+1)$, and $y \in V(H_X, 2i)$, we have $\mathrm{Ad}(z)(u)(y) = z_{2i+2}uz_{2i}^{-1}(y)$. Consequently, $\det\big(\mathrm{Ad}(z)|_{\mathfrak{g}(1,2,2i+1)}\big)$, viewed as a character of Z, is equal to $d_{2i}\psi_{2i+2} - d_{2i+2}\psi_{2i}$. The above equality now gives the following equality for characters:

$$\begin{aligned}\det\big(\mathrm{Ad}(z)|_{\mathfrak{g}_{H_X}(1,2)}\big) &= \sum_{i\geq 0}(d_{2i}\psi_{2i+2} - d_{2i+2}\psi_{2i})\\ &= \sum_{i\geq 1}(d_{2i-2} - d_{2i+2})\psi_{2i}.\end{aligned}$$

By Lemma 7.1, we conclude that indeed the two members of (7.3) have the same character.

The proofs of (ii) and (iii) are similar. ∎

We remark that it is easy to obtain the prime factorization of φ_X by using formula (7.2), the above theorem, and the results of Section 5.

The relative invariants $F_{a,2k}, F_{e,2k+1}, F_{b,2k}$ have not been normalized. For instance, $F_{a,2k}(u)$ has been defined in Section 5 as the determinant of a certain linear transformation $v_{2k-1}v_{2k-1}^*$ computed with respect to some unspecified bases of its domain and co-domain spaces. We can now normalize it by defining $F_{a,2k}(u)$ as the determinant of the linear operator $(-1)^k v_{2k-1}v_{2k-1}^*\omega$ on the space $V(H_X, 2k-1)$. The factor $(-1)^k$ is introduced so that the linear transformation $(-1)^k v_{2k-1}v_{2k-1}^*$ coincides with the linear transformation $V(H_X, 1-2k) \to V(H_X, 2k-1)$ induced by u^{2k} (see (4.9)). The same method can be used to normalize the relative invariant $F_{b,2k}$. We normalize the relative invariant $F_{e,2k+1}$ by defining $F_{e,2k+1}(u)$ as the determinant of the linear operator $w_{2k}\omega$, where w_{2k} is the linear transformation defined in Section 5. Note that this time no sign correction is needed.

If we agree to use these normalizations, then the question of determining the constants c_a, c_e, c_b in the above theorem arises.

There is a more general method for construction of relative invariants which generalizes Kac's construction. It is described in the paper [12] of A. Gyoja, who attributes the method to M. Kashiwara. The relative invariants $F_{a,2k}$, $F_{b,2k}$ and $F_{e,2k+1}$, constructed in Section 5, are of that type.

8 Examples and Some Open Questions

Let us consider a few examples in more detail. For the first three examples we take $p = 6$ and $q = 4$. Then $\Gamma(p,q)$ has 25 vertices, enumerated by integers 0 to 24. Their ab-diagrams are listed in Table 2.

No.	X	No.	X
0	a^6, b^4	13	$(aba)^2, bab, a$
1	ab, ba, a^4, b^2	14	$(aba)^3, b$
2	bab, a^5, b^2	15	$abab, baba, a^2$
3	aba, a^4, b^3	16	$ababa, ab, ba, a$
4	$(ab, ba)^2, a^2$	17	$babab, aba, a^2$
5	bab, ab, ba, a^3	18	$ababa, bab, a^2$
6	aba, ab, ba, a^2, b	19	$ababa, aba, a, b$
7	$(bab)^2, a^4$	20	$(ababa)^2$
8	aba, bab, a^3, b	21	$(ba)^3b, a^3$
9	$(aba)^2, a^2, b^2$	22	$(ab)^3a, a^2, b$
10	$(aba)^2, ab, ba$	23	$(ab)^3a, aba$
11	$babab, a^4, b$	24	$(ab)^4a, a$
12	$ababa, a^3, b^2$		

Table 2: The vertices of $\Gamma(6,4)$.

The diagram $\Gamma(6,4)$ is shown in Figure 1. We have written a (resp. b) near a vertex to indicate that it is an a-vertex (resp. b-vertex). There are three a-vertices (10, 14 and 23) and nine b-vertices (4, 5, 7, 15, 16, 18, 20, 21 and 24). All other vertices are stable (there are no ab-vertices). There is only one maximal pure a-pair (14, 10) and four maximal pure b-pairs (7, 4), (20, 7), (21, 16) and (24, 21). On the right hand side of Figure 1 we show the dimension of the orbits $\mathcal{O}_X$ on various levels.

For $X \in \mathcal{D}(p,q)$ and $E \in \mathfrak{g}_{H_X}(1, \geq 2)$, we often have to compute the ab-diagram $Y \in \mathcal{D}(p,q)$ for which $E \in \mathcal{O}_Y$ is true. This diagram is uniquely determined by the dimensions of $V_a \cap \ker(E^k)$ and $V_b \cap \ker(E^k)$ for $k = 0, 1, 2, \ldots$, which can be easily computed.

The involution $\omega \in G = \mathrm{O}_n(\mathbf{C})$ defined in the previous section is given by the matrix

$$\omega = \begin{bmatrix} S_p & 0 \\ 0 & S_q \end{bmatrix},$$

where S_k denotes the k by k matrix having ones on the side diagonal and zeros elsewhere.

The space $\mathfrak{p}$ consists of matrices

$$x = \begin{bmatrix} 0 & \mathbf{x} \\ -S_q{}^t\mathbf{x}S_p & 0 \end{bmatrix}, \tag{8.1}$$

where $\mathbf{x}$ is an arbitrary complex p by q matrix.

Example 8.1 Let $X = (aba)^2, bab, a$ (No. 13), a stable vertex. In this case

$$X(a) = bab, a \quad X(e) = \varnothing, \quad X(b) = (aba)^2.$$

The relative invariant $F_{a,2}$ is reducible (this is the exceptional case), while $F_{b,2}$ is irreducible.

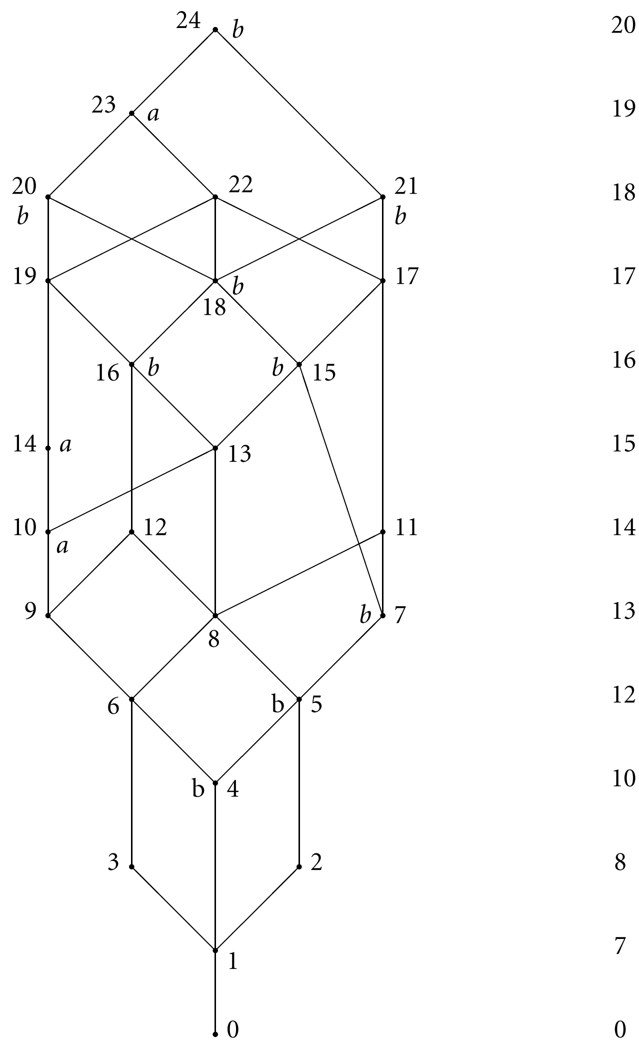

Figure 1: The diagram $\Gamma(6, 4)$.

An arbitrary matrix $x \in \mathfrak{g}_{H_X}(1,2)$ has the form (8.1) where

$$\mathbf{x} = \begin{bmatrix} 0 & x_1 & x_2 & 0 \\ 0 & x_3 & x_4 & 0 \\ 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A computation shows that $F_{a,2}(x) = -2x_5x_6$ and $F_{b,2}(x) = -(x_1x_4 - x_2x_3)^2$. The matrix of the linear operator A_x (see (7.1)) is:

$$\begin{bmatrix} 2\,x_1{}^2 & 0 & 2\,x_1\,x_3 & x_2\,x_3 - x_1\,x_4 & 0 & 0 \\ 0 & 2\,x_2{}^2 & x_1\,x_4 - x_2\,x_3 & 2\,x_2\,x_4 & 0 & 0 \\ 2\,x_1\,x_3 & x_1\,x_4 - x_2\,x_3 & 2\,x_3{}^2 & 0 & 0 & 0 \\ x_2\,x_3 - x_1\,x_4 & 2\,x_2\,x_4 & 0 & 2\,x_4{}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\,x_5{}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,x_6{}^2 \end{bmatrix}$$

and we find that

$$\varphi_X(x) = \det A_x = -12\,x_5{}^2 x_6{}^2 (x_1\,x_4 - x_2\,x_3)^4.$$

Hence, $\varphi_X = -3F_{a,2}^2 F_{b,2}^2$ with $c_a = 1$ and $c_b = -3$.

The dominating nilpotent K^0-orbit for the hypersurface $x_1x_4 = x_2x_3$ in $\mathfrak{g}_{H_X}(1, \geq 2)$ is the orbit $\mathcal{O}_Y$, where $Y = aba, bab, a^3, b$ (No. 8), while those for the hyperplanes $x_5 = 0$ and $x_6 = 0$ are the two connected components ${}^{\mathrm{I}}\mathcal{O}_Z$ and ${}^{\mathrm{II}}\mathcal{O}_Z$ of $\mathcal{O}_Z$, where $Z = (aba)^2, ab, ba$ (No. 10).

In this example, the singular set $\hat{S}$ of PV (4.4) has three irreducible components, and the K^0-orbit $\mathcal{O}_X$ has three children $\mathcal{O}_Y$, ${}^{\mathrm{I}}\mathcal{O}_Z$, ${}^{\mathrm{II}}\mathcal{O}_Z$. Each of the children orbits is the dominating orbit of one of these irreducible components. This is not true in general as shown by the next example.

Example 8.2 Now let $X = (ba)^3 b, a^3$ (No. 21), a proper b-vertex. The orbit $\mathcal{O}_X^{\mathrm{I}}$ has only two children: $\mathcal{O}_Y$ and $\mathcal{O}_Z^{\mathrm{I}}$, where $Y = babab, aba, a, a$ (No. 17) and $Z = ababa, bab, a, a$ (No. 18). On the other hand, PV (4.4) has three basic relative invariants, and consequently its singular set $\hat{S}$ has three irreducible components.

An arbitrary matrix $x \in \mathfrak{g}_{H_X}(1,2)$ has the form (8.1) where

$$\mathbf{x} = \begin{bmatrix} 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & x_4 & 0 \\ 0 & 0 & x_5 & 0 \\ 0 & 0 & 0 & x_6 \end{bmatrix}.$$

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The three irreducible hypersurfaces, the irreducible components of $\hat{S} \subset \mathfrak{g}_{H_X}(1, \geq 2)$, are given by the equations $x_1 = 0$, $x_6 = 0$, and $x_2x_5 + x_3x_4 = 0$. Their respective dominating nilpotent K^0-orbits are $\mathcal{O}_U^{\mathrm{I}}$, $\mathcal{O}_Z^{\mathrm{I}}$, $\mathcal{O}_Y$, respectively, where $U = (bab)^2, a^4$ (No. 7). Two of these dominating orbits are the children of $\mathcal{O}_X^{\mathrm{I}}$, but $\mathcal{O}_U^{\mathrm{I}}$ is not.

The fact that the dominating orbit of the hyperplane $x_1 = 0$ in $\mathfrak{g}_{H_X}(1, \geq 2)$ is $\mathcal{O}_U^{\mathrm{I}}$ is not obvious. A simple computation shows that this dominating orbit must be one of the two connected components of $\mathcal{O}_U$. To prove that this dominating orbit is in fact $\mathcal{O}_U^{\mathrm{I}}$, it suffices to observe that the whole hyperplane $x_1 = 0$ in $\mathfrak{g}_{H_X}(1, \geq 2)$ is also contained in $\mathfrak{g}_{H_U}(1, \geq 2)$.

Example 8.3 Let $X = (ab)^4a, a$ (No. 24), a proper b-vertex. The K^0-orbit $\mathcal{O}_X^{\mathrm{I}}$ has three children: ${}^{\mathrm{I}}\mathcal{O}_Y$, ${}^{\mathrm{II}}\mathcal{O}_Y$ and $\mathcal{O}_Z^{\mathrm{I}}$, where $Y = (ab)^3a, aba$ (No. 23) and $Z = (ba)^3b, a^3$ (No. 21). An arbitrary matrix $x \in \mathfrak{g}_{H_X}(1, 2)$ has the form (8.1) where

$$\mathbf{x} = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The singular set $\hat{S}$ of PV (4.4) is the union of five hyperplanes $x_i = 0$, $1 \leq i \leq 5$, in the space $\mathfrak{g}_{H_X}(1, \geq 2)$. The dominating orbits of these hyperplanes are: $\mathcal{O}_Z^{\mathrm{I}}$ for $x_1 = 0$, $\mathcal{O}_Q^{\mathrm{I}}$ for $x_2 = 0$, $\mathcal{O}_Y^{\mathrm{II}}$ for $x_3 = 0$, $\mathcal{O}_Y^{\mathrm{I}}$ for $x_4 = 0$ and $\mathcal{O}_P^{\mathrm{I}}$ for $x_5 = 0$, where $P = (ababa)^2$ (No. 20) and $Q = ababa, bab, a, a$ (No. 18).

The last four of these hyperplanes are quasi-homogeneous Q_{H_X}-varieties (*i.e.*, they are PVs), but the first one is not. Apart from this exceptional case, and another one that arises from the orbit $\mathcal{O}_X^{\mathrm{II}}$, all irreducible components of the singular sets $\hat{S}$ are quasi-homogeneous varieties in the case when $(p, q) = (6, 4)$.

Example 8.4 The objective of this example is to show that different irreducible components of the singular set $\hat{S}$ may have the same dominating K^0-orbit. We take $p = 7$, $q = 6$, and $X = (ab)^6a$. The K^0-orbit $\mathcal{O}_X^{\mathrm{I}}$ has two children: $\mathcal{O}_Y$ and $\mathcal{O}_Z^{\mathrm{I}}$, where $Y = (ab)^5a, a, b$ and $Z = (ba)^5b, a^2$. An arbitrary matrix $x \in \mathfrak{g}_{H_X}(1, 2)$ has the form (8.1) where

$$\mathbf{x} = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The singular set $\hat{S}$ of PV (4.4) is the union of six hyperplanes $x_i = 0$, $1 \leq i \leq 6$, in the space $\mathfrak{g}_{H_X}(1, \geq 2)$. A calculation shows that the hyperplanes $x_5 = 0$ and $x_6 = 0$ have the same dominating orbit, namely $\mathcal{O}_P$ where $P = (ab)^4a, aba, b$. The second of these two hyperplanes is quasi-homogeneous Q_{H_X}-variety but the first one is not. For the sake of completeness, we mention that $\mathcal{O}_Z^{\mathrm{I}}$ is the dominating orbit for $x_1 = 0$, $\mathcal{O}_Y$ for $x_4 = 0$, while the orbit $\mathcal{O}_Q^{\mathrm{I}}$, with $Q = (ab)^4a, bab, a$, dominates both $x_2 = 0$ and $x_3 = 0$.

In conclusion, we state several open problems.

Problem 1 For $X \in \mathcal{D}(p, q)$, identify the dominating orbits for each irreducible component of the singular set $\hat{S}$ of PV (4.4).

Problem 2 For $X \in \mathcal{D}(p, q)$, determine which irreducible components of the singular set $\hat{S}$ of PV (4.4) fail to be quasi-homogeneous (as Q_{H_X}-varieties).

Problem 3 Determine and/or characterize the maximal pure pairs in $\Gamma(p, q)$.

Problem 4 Determine the coefficients c_a, c_e, c_b in the factorizations of Theorem 7.2.

9 Appendix: Two Special Types of PVs

In this section we construct the basic relative invariants for two important types of PVs and describe their generic points and isotropy subgroups.

9.1 The First Type of PVs

The basic case ($m = 1$) of this type is well known, see [15, §5, Proposition 23]. Many other cases appear in [14, Section 4.1], but we could not find any reference that states the result in the generality and form that we need.

We consider a sequence $V_0, V_2, V_4, \ldots$ of finite-dimensional complex vector spaces such that only finitely many of them are nonzero, and that their dimensions $d_{2i} = \dim(V_{2i})$ satisfy $d_0 \geq d_2 \geq d_4 \geq \cdots$. We also assume that $d_0 \geq 1$ and denote by m the largest integer ≥ 0 such that $d_{2m} \geq 1$. Let $f_0 \colon V_0 \times V_0 \to \mathbf{C}$ be a fixed nondegenerate symmetric bilinear form, and define

$$L = \bigoplus_{i \geq 1} \mathrm{Hom}_{\mathbf{C}}(V_{2i-2}, V_{2i}),$$

$$G = \mathrm{SO}(V_0, f_0) \times \prod_{i \geq 1} \mathrm{GL}(V_{2i}).$$

Introduce the following notation for the components of $u \in L$ and $g \in G$:

$$u = (u_1, u_3, u_5, \ldots), \quad u_{2i-1} \in \mathrm{Hom}_{\mathbf{C}}(V_{2i-2}, V_{2i}),$$

$$g = (g_0, g_2, g_4, \ldots), \quad g_0 \in \mathrm{SO}(V_0);\ g_{2i} \in \mathrm{GL}(V_{2i}),\ i \geq 1.$$

Then L is a G-module via the action:

$$g \cdot u = (g_2 u_1 g_0^{-1}, g_4 u_3 g_2^{-1}, \ldots).$$

We now construct a particular element $e = (e_1, e_3, \dots) \in L$. For that purpose we fix an orthogonal direct decomposition

$$V_0 = \bigoplus_{i \geq 1} U_{2i}$$

such that $\dim(U_{2i}) = d_{2i-2} - d_{2i}$. Furthermore, we set $U_{2i}^{\downarrow} = U_2 + U_4 + \cdots + U_{2i}$ and $U_{2i}^{\uparrow} = U_{2i} + U_{2i+2} + \cdots$. We choose the components $e_{2i-1} \in \operatorname{Hom}_{\mathbf{C}}(V_{2i-2}, V_{2i})$ of e so that

$$U_{2i}^{\downarrow} = \ker(e_{2i-1}e_{2i-3} \cdots e_1), \quad i \geq 1.$$

These conditions imply that each e_{2i-1} is surjective.

Note that each nonzero $\operatorname{Hom}_{\mathbf{C}}(V_{2i-2}, V_{2i})$ is a simple G-module with only one exception: If $d_0 = 2$ and $m \geq 1$ then $\operatorname{Hom}_{\mathbf{C}}(V_0, V_2)$ is the sum of two simple modules. This is due to the fact that, in this case, V_0 itself is the sum of two simple $\mathrm{SO}(V_0)$-modules. Hence, if l is the length of the module L, then $l = m + 1$ in the exceptional case, and $l = m$ otherwise.

The intersection of $\mathrm{SO}(V_0)$ with the direct product of the subgroups $\mathrm{O}(U_{2i})$, $i \geq 1$, will be denoted by $S\bigl(\prod_{i\geq 1} \mathrm{O}(U_{2i})\bigr)$.

Proposition 9.1 *The pair (G, L) is a regular PV, and the element $e \in L$ constructed above is a generic element. The stabilizer G_e consists of all elements $g = (g_0, g_2, g_4, \dots) \in G$, where $g_0 \in \mathrm{SO}(V_0)$ is only subject to the condition that it leaves invariant each of the subspaces U_{2i}, and the other components of g are uniquely determined by the equations*

$$g_{2i}e_{2i-1} = e_{2i-1}g_{2i-2}, \quad i \geq 1. \tag{9.1}$$

Consequently, the projection map

$$\pi\colon G_e \to S\Bigl(\prod_{i\geq 1} \mathrm{O}(U_{2i})\Bigr), \tag{9.2}$$

sending $g = (g_0, g_2, g_4, \dots)$ to g_0, is an isomorphism.

Proof For $g = (g_0, g_2, g_4, \dots) \in G$, we have $g \cdot e = e$ iff equations (9.1) are satisfied.

Assume that $g \in G_e$. If $x \in U_{2i}^{\downarrow}$ then $e_{2i-1}e_{2i-3} \cdots e_1(x) = 0$ and so

$$\begin{aligned} 0 &= g_{2i}e_{2i-1}e_{2i-3} \cdots e_1(x) = e_{2i-1}g_{2i-2}e_{2i-3} \cdots e_1(x) \\ &= \cdots = e_{2i-1}e_{2i-3} \cdots e_1 g_0(x). \end{aligned}$$

This shows that $g_0(U_{2i}^{\downarrow}) = U_{2i}^{\downarrow}$, and consequently g_0 leaves invariant each of the subspaces U_{2i}.

Conversely, assume that $g_0 \in \mathrm{SO}(V_0)$ leaves invariant each of the subspaces U_{2i}. Then the equation $g_2 e_1 = e_1 g_0$ has a unique solution for $g_2 \in \mathrm{GL}(V_2)$. Indeed, since e_1 is surjective, there exists $s_1 \in \operatorname{Hom}_{\mathbf{C}}(V_2, V_0)$ such that $e_1 s_1 = 1$. Therefore

$g_2 = e_1 g_0 s_1$. As $V_0 = \operatorname{im}(s_1) \oplus \ker(e_1)$, and $U_2^{\downarrow} = \ker(e_1)$ is g_0-invariant, we conclude that g_2 is invertible, *i.e.*, $g_2 \in \mathrm{GL}(V_2)$. Similarly, the equation $g_4 e_3 = e_3 g_2$ has a unique solution for $g_4 \in \mathrm{GL}(V_4)$, *etc.* This proves that our description of G_e is correct and that π is an isomorphism.

An easy computation shows that $\dim(L) = \dim(G/G_e)$, and so (G, L) is a PV and e is a generic point. As G_e is reductive, this PV is regular. ∎

Let $\sigma\colon V_0^* \to V_0$ be the isomorphism induced by f_0. Define the polynomial functions $F_{2i}\colon L \to \mathbf{C}$, $i \geq 1$, by:

$$F_{2i}(u) = \det(u_{2i-1} u_{2i-3} \cdots u_1 \sigma\, {}^t u_1 \cdots {}^t u_{2i-3}\, {}^t u_{2i-1}),$$

where ${}^t u_{2j-1}\colon V_{2j}^* \to V_{2j-2}^*$ is the transpose of $u_{2j-1}\colon V_{2j-2} \to V_{2j}$, and the determinant is computed with respect to some fixed bases of V_{2i}^* and V_{2i}. It is easy to check that F_{2i} is a relative invariant of (G, L) with character

$$\chi_{2i}(g) = \det(g_{2i})^2.$$

Proposition 9.2 *Assume that if $d_0 = 2$ then also $d_m = 2$. Then the basic relative invariants of (G, L) are:*

(i) *F_{2i} for i's such that $d_{2i-2} > d_{2i}$ and $1 \leq i \leq m$.*
(ii) *$\det(u_{2i-1})$ for i's such that $d_{2i-2} = d_{2i}$ and $1 \leq i \leq m$.*

Hence, the number of basic relative invariants is equal to m.

Proof Let X be the character group of G. Define $\varphi_{2i} \in X$, $i \geq 0$, by $\varphi_{2i}(g) = \det(g_{2i})$. Note that φ_{2i} is the trivial character if $i = 0$ or $i > m$, and that X is a free Abelian group of rank m with free generators φ_{2i}, $1 \leq i \leq m$. Let Λ be the set of integers $i \geq 1$ for which $d_{2i-2} > d_{2i}$. Denote by X_L the character group of $G/G'G_e$, viewed as a subgroup of X. Finally, let Y be the subgroup of X_L generated by the characters:

(i)′ $\chi_{2i} = 2\varphi_{2i}$ for $i \in \Lambda$;
(ii)′ $\varphi_{2i} - \varphi_{2i-2}$ for $i \notin \Lambda$.

In order to prove the proposition, we have to show that $Y = X_L$.

Note that the index $[X : Y] = 2^{r-1}$, where $r = |\Lambda|$. We claim that $G'G_e/G'$ is an elementary Abelian group of order 2^{r-1}. This can be proved by using Proposition 9.1 and the fact that $G'G_e/G' \cong G_e/(G_e \cap G')$. Indeed, let $g = (g_0, g_2, g_4, \dots) \in G_e$ and denote by $g_{0,2i}$ the restriction of g_0 to U_{2i}. We have $g \in G'$ iff $\det(g_{2i}) = 1$ for all $i \geq 1$. On the other hand, $\det(g_{2i})$ is equal to the determinant of the restriction of g_0 to $U_{2i+2}^{\uparrow}$. It follows that $g \in G'$ iff for each i, $g_{0,2i} \in \mathrm{SO}(U_{2i})$. Now our claim follows.

By applying the character group functor to the short exact sequence

$$1 \to G'G_e/G' \to G/G' \to G/G'G_e \to 1,$$

we conclude that X/X_L is also an elementary Abelian group of order 2^{r-1}. Hence $[X : X_L] = 2^{r-1} = [X : Y]$, which implies that $X_L = Y$. ∎

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Remark 9.3 We conclude with the remark concerning the exceptional case: $d_0 = 2$ and $d_{2m} = 1$. Then there is a unique k, $1 \le k \le m$, such that $d_{2k-2} = 2$ and $d_{2k} = 1$. The only new feature of this case is that the relative invariant F_{2k} is not irreducible: It has a factorization $F_{2k} = 2\xi\eta$ where ξ and η are two different irreducible polynomials. Let $V_0 = V_0' \oplus V_0''$, where V_0' and V_0'' are the two 1-dimensional isotropic subspaces of V_0. Then, say ξ is the determinant of the linear map $u_{2k-1}u_{2k-3}\cdots u_1 u'$, where $u'\colon V_0' \to V_0$ is the inclusion map. To get η, we just use V_0'' instead of V_0'. Hence, to summarize, the basic relative invariants in the exceptional case are:

$$\det(u_1), \dots, \det(u_{2k-3}), \xi, \eta, \det(u_{2k+1}), \dots, \det(u_{2m}).$$

Note that, in this case, the number of basic relative invariants is $m + 1$.

9.2 The Second Type of PVs

We consider a doubly infinite sequence

$$\dots, V_{-5}, V_{-3}, V_{-1}, V_1, V_3, V_5, \dots$$

of finite-dimensional complex vector spaces. We assume that only finitely many of them are nonzero, and that their dimensions $d_{2i-1} = \dim(V_{2i-1})$ satisfy $d_{-2i+1} = d_{2i-1}$ and $d_1 \ge d_3 \ge d_5 \ge \cdots$. We also assume that $d_1 \ge 1$, and denote by m the largest integer ≥ 1 such that $d_{2m-1} \ge 1$. We set

$$L = \bigoplus_{i \in \mathbf{Z}} \operatorname{Hom}_{\mathbf{C}}(V_{2i-1}, V_{2i+1}),$$
$$G = \prod_{i \in \mathbf{Z}} \operatorname{GL}(V_{2i+1}),$$

and introduce the following notation for the components of $u \in L$ and $g \in G$:

$$u = (\dots, u_{-4}, u_{-2}, u_0, u_2, u_4, \dots), \quad u_{2i} \in \operatorname{Hom}_{\mathbf{C}}(V_{2i-1}, V_{2i+1}),$$
$$g = (\dots, g_{-3}, g_{-1}, g_1, g_3, \dots), \quad g_{2i-1} \in \operatorname{GL}(V_{2i-1}).$$

Then L is a G-module via the action:

$$g \cdot u = (\dots, g_{-1}u_{-2}g_{-3}^{-1}, g_1 u_0 g_{-1}^{-1}, g_3 u_2 g_1^{-1}, \dots).$$

We now construct a particular element $e = (\dots, e_{-2}, e_0, e_2, \dots) \in L$. For that purpose we fix a direct decomposition

$$V_1 = \bigoplus_{i \ge 1} U_{2i} \tag{9.3}$$

with $\dim(U_{2i}) = d_{2i-1} - d_{2i+1}$. Furthermore, we set $U_{2i}^{\downarrow} = U_2 + U_4 + \cdots + U_{2i}$ and $U_{2i}^{\uparrow} = U_{2i} + U_{2i+2} + \cdots$.

We now choose the components $e_{2i} \in \mathrm{Hom}_{\mathbf{C}}(V_{2i-1}, V_{2i+1})$ of e so that

$$U_{2i}^{\downarrow} = \ker(e_{2i}e_{2i-2}\cdots e_2), \tag{9.4}$$

$$U_{2i+2}^{\uparrow} = \mathrm{im}(e_0 e_{-2}\cdots e_{-2i}), \tag{9.5}$$

for all i's. In particular, e_0 is an isomorphism, and the e_{2i}'s are surjective (resp. injective) if i is positive (resp. negative). Note also that each nonzero space $\mathrm{Hom}_{\mathbf{C}}(V_{2i-1}, V_{2i+1})$ is a simple G-module.

Proposition 9.4 *The pair (G, L) is a regular PV and the element $e \in L$ constructed above is a generic element. The stabilizer G_e consists of all elements $g = (\ldots, g_{-3}, g_{-1}, g_1, g_3, \ldots) \in G$, where $g_1 \in \mathrm{GL}(V_1)$ is only subject to the condition that it leaves invariant each of the subspaces U_{2i}, and the other components of g are uniquely determined by the equations*

$$g_{2i+1}e_{2i} = e_{2i}g_{2i-1}, \quad i \in \mathbf{Z}. \tag{9.6}$$

Consequently, the projection map

$$\pi\colon G_e \to \prod_{i\geq 1} \mathrm{GL}(U_{2i}), \tag{9.7}$$

sending $g = (\ldots, g_{-3}, g_{-1}, g_1, g_3, \ldots)$ to g_1, is an isomorphism.

Proof For $g = (\ldots, g_{-3}, g_{-1}, g_1, g_3, \ldots) \in G$, we have $g \cdot e = e$ iff equations (9.6) are satisfied.

Assume that $g \in G_e$. If $x \in U_{2i}^{\downarrow}$ then, by (9.4), $e_{2i}e_{2i-2}\cdots e_2(x) = 0$ and so

$$\begin{aligned} 0 &= g_{2i+1}e_{2i}e_{2i-2}\cdots e_2(x) = e_{2i}g_{2i-1}e_{2i-2}\cdots e_2(x) \\ &= \cdots = e_{2i}e_{2i-2}\cdots e_2 g_1(x). \end{aligned}$$

This shows that g_1 leaves invariant $U_{2i}^{\downarrow}$. Similarly, by using (9.5), one can show that g_1 leaves invariant $U_{2i}^{\uparrow}$. It follows that g_1 leaves invariant each U_{2i}.

Conversely, if $g \in G$ leaves invariant each U_{2i}, then the equation $g_3 e_2 = e_2 g_1$ has a unique solution for $g_3 \in \mathrm{GL}(V_3)$ (see the proof of Proposition 9.1). Next, the equation $g_5 e_4 = e_4 g_3$ has a unique solution for $g_5 \in \mathrm{GL}(V_5)$, *etc.* As e_0 is an isomorphism, the equation $g_1 e_0 = e_0 g_{-1}$ gives $g_{-1} = e_0^{-1} g_1 e_0$. By using (9.5), one can show that the equation $g_{-3}e_{-2} = e_{-2}g_{-1}$ has a unique solution for $g_{-3} \in \mathrm{GL}(V_{-3})$, *etc.* This proves that our description of G_e is correct and that π is an isomorphism.

An easy computation shows that $\dim(L) = \dim(G/G_e)$, and so (G, L) is a PV and e is a generic point. As G_e is reductive, this PV is regular. ■

Define the polynomial functions $F_{2i+1}\colon L \to \mathbf{C}$, $0 \leq i \leq m-1$, by:

$$F_{2i+1}(u) = \det(u_{2i}u_{2i-2}\cdots u_0 \cdots u_{-2i+2}u_{-2i}),$$

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where the determinant is computed with respect to some fixed bases. It is easy to check that F_{2i+1} is a relative invariant of (G, L) with character

$$\chi_{2i+1}(g) = \det(g_{2i+1}) \det(g_{-2i-1})^{-1}.$$

Proposition 9.5 *The basic relative invariants of (G, L) are:*

(i) *F_{2i+1} if $d_{2i-1} > d_{2i+1}$ or $i = 0$;*
(ii) *$\det(u_{2i})$ and $\det(u_{-2i})$ if $d_{2i-1} = d_{2i+1}$ and $i > 0$.*

The number of basic relative invariants is $2m - d$, where d is the number of integers $i \geq 1$ for which $d_{2i-1} > d_{2i+1}$.

Proof Let X be the character group of G. It is a free Abelian group of rank $2m$. The characters φ_{2i+1}, $-m \leq i \leq m-1$, of G defined by $\varphi_{2i+1}(g) = \det(g_{2i+1})$ form a basis of X. Let e be the generic element of L constructed in the previous proposition. Then we know that G_e is isomorphic to the direct product of the groups $\mathrm{GL}(U_{2i})$. Denote by Λ the set of integers $i \geq 1$ for which $d_{2i-1} > d_{2i+1}$. Then $|\Lambda| = d$ and $U_{2i} \neq 0$ iff $i \in \Lambda$.

Denote by X_L the character group of $G/G'G_e$, viewed as a subgroup of X. In order to prove the proposition, we have to show that the characters

(i)′ $\chi_{2i+1} = \varphi_{2i+1} - \varphi_{-2i-1}$ for $i \in \Lambda$;
(ii)′ $\chi'_{2i+1} = \varphi_{2i+1} - \varphi_{2i-1}$ and $\chi''_{2i+1} = \varphi_{-2i+1} - \varphi_{-2i-1}$ for $i \notin \Lambda$;

generate X_L.

Let Y be the subgroup of X_L generated by these characters. Observe that every coset of Y in X has the unique representative of the form

$$\chi = \sum_{i \in \Lambda} k_{2i+1} \varphi_{2i+1}.$$

Assume now that this representative χ belongs to X_L. Let $r \geq 1$ be the least integer such that $r \in \Lambda$. Then $U_{2i} = 0$ for $i < r$ while $U_{2r} \neq 0$. Consider the 1-dimensional torus in G_e which acts as an arbitrary scalar on U_{2r} and acts trivially on all other spaces U_{2i}. If $i \in \Lambda$ and $i > r$, then the character φ_{2i+1} is trivial on T_1. Since χ is also trivial on T_1 but φ_{2r+1} is not, we infer that the coefficient k_{2r+1} is zero. Similarly, one can show that all the coefficients k_{2i+1} are zero, *i.e.*, $\chi \in Y$. Hence $Y = X_L$ and the proof is completed. ■

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