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THE MAXIMAL EXTENSION OF A ZERO-DIMENSIONAL PRODUCT SPACE

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ABSTRACT. It is known that if a topological property $\mathcal{P}$ of Tychonoff spaces is closed-hereditary, productive and possessed by all compact Hausdorff spaces, then each (0-dimensional) Tychonoff space X is a dense subspace of a (0-dimensional) Tychonoff space $\mathcal{P}X(\mathcal{P}_0X)$ with $\mathcal{P}$ such that each continuous map from X to a (0-dimensional) Tychonoff space with $\mathcal{P}$ admits a continuous extension over $\mathcal{P}X(\mathcal{P}_0X)$. In response to Broverman's question [Canad. Math. Bull. **19** (1), (1976), 13–19], we prove that if for every two 0-dimensional Tychonoff spaces X and Y, $\mathcal{P}(X\times Y)=\mathcal{P}X\times\mathcal{P}Y$ if and only if $\mathcal{P}_0(X\times Y)=\mathcal{P}_0X\times\mathcal{P}_0Y$, then $\mathcal{P}$ is contained in countable compactness.

1. **Introduction.** All spaces considered are assumed to be Tychonoff spaces. Following [16], we call a topological property $\mathcal{P}$ of spaces an *extension property* if it is closed hereditary, productive and each $\mathcal{P}$-regular space has a $\mathcal{P}$-regular compactification, where a *$\mathcal{P}$-regular space* is a subspace of a space having $\mathcal{P}$. It is known ([8], [16]) that if $\mathcal{P}$ is an extension property, then each $\mathcal{P}$-regular space R is densely embedded in a space $\mathcal{P}R$ with $\mathcal{P}$ such that each continuous map from R to a space with $\mathcal{P}$ admits a continuous extension over $\mathcal{P}R$. The space $\mathcal{P}R$ is called the *maximal $\mathcal{P}$-extension* of R. For example, if $\mathcal{P}$ is compactness, then $\mathcal{P}R$ is the Stone–Čech compactification βR of R. For details of extension properties, the reader is referred to [16]. Throughout this paper, X and Y denote 0-dimensional spaces (i.e., spaces having a base consisting of clopen sets), and $\mathcal{P}$ denotes an extension property such that the $\mathcal{P}$-regular spaces are just Tychonoff spaces. Every compact space has $\mathcal{P}$. Since 0-dimensionality $\mathcal{Z}$ is known to be an extension property, the property $\mathcal{P}_0$ defined by "R has $\mathcal{P}_0$ if and only if R has both $\mathcal{P}$ and $\mathcal{Z}$" is an extension property such that the $\mathcal{P}_0$-regularity is $\mathcal{Z}$. In [3, p. 19], Broverman proved that if $\mathcal{P}$ is contained in pseudocompactness, then $\mathcal{P}(X\times Y)=\mathcal{P}X\times\mathcal{P}Y$ if and only if $\mathcal{P}_0(X\times Y)=\mathcal{P}_0X\times\mathcal{P}_0Y$, and he asked whether his result holds for any extension property $\mathcal{P}$. The main purpose of this paper is to answer the question

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in the negative by proving that if $\mathscr{P}_0(X\times Y)=\mathscr{P}_0X\times\mathscr{P}_0Y$ implies $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$, then $\mathscr{P}$ is contained in countable compactness. Combining with his result, we have the following theorem.

THEOREM 1. *The following conditions on $\mathscr{P}$ are equivalent:*

(a) *Either $\mathscr{P}$ is contained in countable compactness or $\mathscr{P}_0=\mathscr{Z}$.*

(b) *Either $\mathscr{P}$ is contained in pseudocompactness or $\mathscr{P}_0=\mathscr{Z}$.*

(c) *For two spaces X and Y, $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$ if and only if $\mathscr{P}_0(X\times Y)=\mathscr{P}_0X\times\mathscr{P}_0Y$.*

(d) *If for two spaces X and Y, $\mathscr{P}_0(X\times Y)=\mathscr{P}_0X\times\mathscr{P}_0Y$, then $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$.*

A space R is called *ultrarealcompact* (resp. *$P_z(\aleph_1)$-compact*) if for each $p\in\beta R-R$ there is a countable, disjoint cover $\mathfrak{U}$ of R by open sets (resp. zero-sets) in R such that $p\notin cl_{\beta R}U$ for each $U\in\mathfrak{U}$ (cf. [13], [15]). Every ultrarealcompact space is $P_z(\aleph_1)$-compact, and every $P_z(\aleph_1)$-compact space is realcompact. All of these properties are extension properties which do not satisfy (a). Recall from [7] that a space R is N-compact if R is homeomorphic to a closed subspace of $N^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$, where N is the space of natural numbers. It is known ([13]) that R is ultrarealcompact if and only if R is homeomorphic to a closed subspace of the product of a compact space with an N-compact space, and that if $\mathscr{P}$ is ultrarealcompactness, then $\mathscr{P}_0$ is N-compactness. It follows from these facts and [16, Theorem 2.9] that (a) is equivalent to the following condition (a′).

(a′) *Either there exists an ultrarealcompact space which does not have $\mathscr{P}$ or $\mathscr{P}_0=\mathscr{Z}$.*

It is natural to ask whether (d) is equivalent to the following condition (e): *If for two spaces X and Y, $\mathscr{P}(X\times Y)=\mathscr{P}X\times\mathscr{P}Y$, then $\mathscr{P}_0(X\times Y)=\mathscr{P}_0X\times\mathscr{P}_0Y$.* The answer is negative. In fact, we prove in Section 4 that ultrarealcompactness satisfies (e); however, somewhat weaker condition than (a′) follows from (e):

THEOREM 2. *If $\mathscr{P}$ satisfies* (e), *then either there exists a $P_z(\aleph_1)$-compact space which does not have $\mathscr{P}$ or $\mathscr{P}_0=\mathscr{Z}$.*

These theorems will be proved in Section 3. Terminology and notation will be used as in [6].

2. **Preliminaries.** We denote the closed unit interval by I, and $D=\{0,1\}\subset I$. Let R and S be spaces. If $M\subset R$ and each continuous map $f:M\to I$ ($f:M\to D$) admits a continuous extension over R, then M is said to be *C^*-embedded* (*D-embedded*) in R. For every X, there is a unique 0-dimensional compactification β_0X of X in which X is dense and D-embedded

(cf. [2]). It is known ([8], [16]) that $\mathcal{P}M = \bigcap \{S \mid S$ has $\mathcal{P}$ and $M \subset S \subset \beta M\}$ and $\mathcal{P}_0 X = \bigcap \{S \mid S$ has $\mathcal{P}$ and $X \subset S \subset \beta_0 X\}$. The first lemma follows from these facts.

LEMMA 1. *Let R be a* (0-*dimensional*) *space with $\mathcal{P}$ and let M be a dense C^*-* (D-) *embedded subspace of R such that there is no space S with $\mathcal{P}$ for which $M \subset S \subsetneqq R$. Then $R = \mathcal{P}M$ ($R = \mathcal{P}_0 M$).*

For $r \in R$, $\chi(r, R)$ denotes the minimal cardinality of a neighborhood base of r in R, and $\mathfrak{m}$, $\mathfrak{n}$ denote infinite cardinals. Two subsets A and B of R are said to be *completely separated* (D-*separated*) in R if there is a continuous map $f: R \to I$ ($f: R \to D$) such that $f(A) = \{0\}$ and $f(B) = \{1\}$. A family $\{F_\alpha\}_{\alpha \in A}$ of subsets of R is called $D(\mathfrak{m})$- ($D_0(\mathfrak{m})$-) *expandable* if there is a locally finite family $\{G_\alpha\}_{\alpha \in A}$ of open sets in R with $F_\alpha \subset G_\alpha$ for each $\alpha \in A$ and each F_α is a union of at most $\mathfrak{m}$ subsets each of which is completely separated (D-separated) from $R - G_\alpha$ (cf. [12]). Recall from [9] that R is *pseudo*-$\mathfrak{m}$-*compact* if each locally finite family of nonempty open sets in R has cardinality less than $\mathfrak{m}$. Pseudocompact spaces are known to be precisely pseudo-$\aleph_0$-compact spaces. The following lemmas will be proved quite similarly to [12, Theorem 1.2].

LEMMA 2. *If $R \times S$ is C^*-embedded in $R \times \mathcal{P}S$ and there exists a $D(\mathfrak{m})$-expandable family $\mathfrak{F}$ in S with $|\mathfrak{F}| = \mathfrak{n}$ such that $\bigcap_{F \in \mathfrak{F}} cl_{\mathcal{P}S} F \neq \varnothing$, then each $r \in R$ with $\chi(r, R) \leq \mathfrak{n}$ has a pseudo-$\mathfrak{m}$-compact neighborhood.*

LEMMA 3. *If $X \times Y$ is D-embedded in $X \times \mathcal{P}_0 Y$ and there exists a $D_0(\mathfrak{m})$-expandable family $\mathfrak{F}$ in Y with $|\mathfrak{F}| = \mathfrak{n}$ such that $\bigcap_{F \in \mathfrak{F}} cl_{\mathcal{P}_0 Y} F \neq \varnothing$, then each $x \in X$ with $\chi(x, X) \leq \mathfrak{n}$ has a pseudo-$\mathfrak{m}$-compact neighborhood.*

3. **Proofs of theorems.** Let W^* denote the linearly ordered space of ordinals less than or equal to the first uncountable ordinal ω_1, and let $W = W^* - \{\omega_1\}$. Then $\beta W = W^*$.

Proof of Theorem 1. The implication (b) $\to$ (c) is the result of Broverman quoted in the introduction, and (a) $\to$ (b) and (c) $\to$ (d) are obvious. It remains to prove (d) $\to$ (a). Suppose that $\mathcal{P}$ is not contained in countable compactness and $\mathcal{P}_0 \neq \mathcal{Z}$ Then by [16, Theorem 2.9] every N-compact space has $\mathcal{P}$. Let S be a 0-dimensional space which does not have $\mathcal{P}$. Since S is homeomorphic to the diagonal of $\prod \{\beta_0 S - \{s\} \mid s \in \beta_0 S - S\}$, we can find a point $s^* \in \beta_0 S - S$ such that $\beta_0 S - \{s^*\}$ does not have $\mathcal{P}$.

CLAIM 1. *$\beta_0 S - \{s^*\}$ is pseudocompact.*

Proof. Suppose not. Then there is a discrete family $\{O_n\}_{n \in N}$ of nonempty clopen sets in $\beta_0 S - \{s^*\}$. Define a map $f: S \to D$ by $f(s) = 0$ if $s \in \bigcup_{n \in N} (O_{2n} \cap S)$ and $f(s) = 1$ otherwise; then f is continuous. But f cannot be extended continuously to s^* since each neighborhood of s^* meets all but finitely many O_n's, which is a contradiction. $\square$

We utilize the 0-dimensional space M constructed by Dowker in [5]. Let Q be the set of rational numbers in I, and choose a disjoint family $\{Q_\alpha\}_{\alpha<\omega_1}$ of countable dense subsets in $I-Q$. Consider the product space $W^*\times I$ and its subspace

$$M^*=\left\{(\alpha, r)\,\middle|\,(\alpha, r)\in W^*\times I, r\notin \bigcup_{\beta\geq\alpha} Q_\beta\right\}.$$

The Dowker's space M is a dense subspace of M^* defined by $M=M^*\cap(W\times I)$. He proved that M is 0-dimensional but $\dim M>0$. Let $P^*=W^*\times\beta_0S$. Let R^* be the quotient space obtained from the disjoint sum $P^*\oplus M^*$ by identifying $(\alpha, s^*)\in P^*$ and $(\alpha, 0)\in M^*$ for each $\alpha\leq\omega_1$, and let $\pi: P^*\oplus M^*\to R^*$ be the quotient map. Let us set $R=R^*-\pi(\{(\omega_1, s^*)\}\oplus(M^*-M))$.

Claim 2. *R is C^*-embedded in R^*.*

Proof. Let $f: R\to I$ be a continuous map. It is known that M is C^*-embedded in M^* (cf. [6, 6.2.20]). On the other hand, since $W\times\beta_0S$ is pseudocompact, it follows from the Glicksberg's theorem ([6, 3.12.20]) that $\beta(W\times\beta_0S)=\beta W\times\beta_0S=P^*$. Thus $f\circ(\pi\mid\pi^{-1}(R))$ extends to a continuous map $h: P^*\oplus M^*\to I$. Define a map $f_1: R^*\to I$ by $f_1(p)=h(\pi^{-1}(p))$ for $p\in R^*$. Then f_1 is well-defined since $h((\alpha, s^*))=h((\alpha, 0))$ for each $\alpha\leq\omega_1$. It follows from [6, 2.4.2] that f_1 is a continuous extension of f over R^*. $\square$

We denote a neighborhood system of s^* in β_0S by $\mathfrak{U}$. For each $\alpha<\omega_1$ and each $U\in\mathfrak{U}$, set

$$G(\alpha, U)=R\cap\pi(\{(\beta, t)\mid\alpha<\beta\leq\omega_1, t\in U\oplus I\}),$$

and set

$$E_i'=\{\pi((\beta, 1/i))\mid(\beta, 1/i)\in M^*,\ \beta<\omega_1\},\qquad i\in N,$$
$$F=\{\pi((\beta, r))\mid(\beta, r)\in M^*, \beta<\omega_1, 0<r\leq 1\}.$$

Let Y^* be the quotient space obtained from $R^*\times N$ by collapsing the set $\{\pi((\omega_1, s))\}\times N$ to a point $y(s)\in Y^*$ for each $s\in\beta_0S$, and let $\phi: R^*\times N\to Y^*$ be the quotient map. Let us set

$$Y=Y^*-\phi((R^*-R)\times N),$$

$y^*=y(s^*)$ and $T=\{y(s)\mid s\in\beta_0S-\{s^*\}\}$. Then Y is 0-dimensional and T is homeomorphic to $\beta_0S-\{s^*\}$. Next, let $Y_0=\{y_0\}\cup Y$ be the quotient space obtained from Y^* by collapsing the set Y^*-Y to a point y_0. If we set

$$H(\alpha, U)=\{y_0\}\cup\phi(G(\alpha, U)\times N),$$

then $\{H(\alpha, U)\mid \alpha<\omega_1, U\in\mathfrak{U}\}$ is a neighborhood base of y_0 in Y_0.

CLAIM 3. *$\mathcal{P}_0 Y=Y_0$ and $y^*\in\mathcal{P}Y$.*

Proof. It is easily checked that Y_0 is 0-dimensional. To show that $Y_0\subset\beta_0 Y$ and $Y^*\subset\beta Y$, we prove that Y is C^*- (D-) embedded in Y^* (Y_0). Let $f: Y\to I$ ($g: Y\to D$) be a continuous map. By Claim 2, $f\circ(\phi\mid\phi^{-1}(Y))$ extends to a continuous map h on $R^*\times N$. Since h is constant on $\phi^{-1}(y(s))$ for each $s\in\beta_0 S$, it follows from [6, 2.4.2] that a map $f_1: Y^*\to I$ defined by $f_1(p)=h(\phi^{-1}(p))$ for $p\in Y^*$ is a continuous extension of f over Y^*. Similarly, g extends to a continuous map $g_1: Y^*\to D$ on Y^*. Since Y^*-Y is connected, g_1 must be constant on Y^*-Y. Again using [6, 2.4.2], we can define a continuous extension of g over Y_0. Thus $Y_0\subset\beta_0 Y$ and $Y^*\subset\beta Y$. Since T does not have $\mathcal{P}$, T is not closed in $\mathcal{P}_0 Y$ and in $\mathcal{P}Y$, so $Y_0\subset\mathcal{P}_0 Y$ and $y^*\in\mathcal{P}Y$. Since Y_0 is a 0-dimensional Lindelöf space, it is N-compact by [11, 2.1], and hence Y_0 has $\mathcal{P}$. It follows from Lemma 1 that $\mathcal{P}_0 Y=Y_0$. $\square$

Let us set $X=Q$. Since Q is N-compact, it has $\mathcal{P}$, and hence $X=\mathcal{P}X=\mathcal{P}_0 X$. The following Claims 4 and 5 complete the proof.

CLAIM 4. *$\mathcal{P}_0(X\times Y)=\mathcal{P}_0 X\times\mathcal{P}_0 Y(=X\times\mathcal{P}_0 Y)$.*

Proof. We show that $X\times\mathcal{P}_0 Y\subset\beta_0(X\times Y)$ by proving that $X\times Y$ is D-embedded in $X\times\mathcal{P}_0 Y$. Let $f: X\times Y\to D$ be a continuous map. For each $x\in X$, if we define a map $f_x: Y\to D$ by $f_x(y)=f((x, y))$, then f_x admits a continuous extension g_x over $\mathcal{P}_0 Y$. Define a map $g: X\times\mathcal{P}_0 Y\to D$ by $g((x, y))=g_x(y)$; then $g\mid(X\times Y)=f$. To prove that g is continuous, let $(x_0, y_0)\in X\times\mathcal{P}_0 Y$. We may assume that $g((x_0, y_0))=0$. For each $x\in X$, set $A(x)=g_x^{-1}(0)$ and $B(x)=g_x^{-1}(1)$. Then $y_0\in A(x_0)$. Let $\{V_n\}_{n\in N}$ be a countable neighborhood base of x_0 in X such that $V_n\supset V_{n+1}$. Assume that for each $n\in N$ there is $x_n\in V_n$ with $y_0\in B(x_n)$. Then, since each nonempty G_δ-set in Y_0 meets Y, there is a point $y\in Y\cap A(x_0)\cap(\bigcap_{n\in N} B(x_n))$. Since

$$0=f((x_0, y))=f\left(\lim_n (x_n, y)\right)\neq\lim_n f((x_n, y))=1,$$

this contradicts continuity of f. Thus there exists $k\in N$ such that $y_0\notin B(x)$ for each $x\in V_k$. For each $j\geq k$, set

$$C_j=\bigcup\{A(x_0)\cap B(x)\mid x\in V_j\}.$$

Then $\{C_j\}_{j\geq k}$ is a decreasing sequence of open sets in Y such that $\bigcap_{j\geq k} cl_Y C_j=\varnothing$. In fact, for each $y\in Y\cap A(x_0)$ there exists $l\geq k$ and a neighborhood V of y in Y such that $V_l\times V\subset f^{-1}(0)$, so $V\cap C_l=\varnothing$, and hence $y\notin cl_Y C_l$. Since T

is pseudocompact by Claim 1, $T\cap C_m=\varnothing$ for some $m\geq k$. Thus $(T\cup\{y_0\})\cap C_m=\varnothing$. For each $x\in V_m$, since $A(x_0)\cap B(x)$ is closed in Y_0 and $T\cup\{y_0\}$ is compact, we can find $\alpha_x<\omega_1$ such that

$$H(\alpha_x, \beta_0 S)\cap(A(x_0)\cap B(x))=\varnothing.$$

Let $\alpha=\sup\{\alpha_x \mid x\in V_m\}$; then $\alpha<\omega_1$ because V_m is countable. Set $H=H(\alpha, \beta_0 S)\cap A(x_0)$. Then H is a neighborhood of y_0 in Y_0 such that $V_m\times H\subset g^{-1}(0)$, since $H\cap B(x)=\varnothing$ for each $x\in V_m$. So g is continuous, hence $X\times\mathcal{P}_0Y\subset\beta_0(X\times Y)$. If $X\times Y\subset Z\subset X\times\mathcal{P}_0Y$ and Z has $\mathcal{P}$, then $X\times\mathcal{P}_0Y\subset Z$. For, if there is $x'\in X$ such that $(x', y_0)\in(X\times\mathcal{P}_0Y)-Z$, then $(\{x'\}\times\mathcal{P}_0Y)\cap Z$ has $\mathcal{P}$ and is properly contained in $\{x'\}\times\mathcal{P}_0Y$, which is impossible. Hence it follows from Lemma 1 that $\mathcal{P}_0(X\times Y)=X\times\mathcal{P}_0Y$. □

CLAIM 5. $\mathcal{P}(X\times Y)\neq\mathcal{P}X\times\mathcal{P}Y(=X\times\mathcal{P}Y)$.

Proof. For each $n\in N$, set

$$E_n=\bigcup_{i\in N}\phi(E_i'\times\{n\}) \quad\text{and}\quad F_n=\phi(F\times\{n\}).$$

Then $\{F_n\}_{n\in N}$ is a locally finite family of open sets in Y with $E_n\subset F_n$. Since each $\phi(E_i'\times\{n\})$ is completely separated from $Y-F_n$, $\{E_n\}_{n\in N}$ is a $D(\aleph_0)$-expandable family in Y. Since $y^*\in\bigcap_{n\in N} cl_{\mathcal{P}Y}E_n$ and each point of X has no pseudocompact neighborhood, it follows from Lemma 2 that $\mathcal{P}(X\times Y)\neq X\times\mathcal{P}Y$. Hence the proof of Theorem 1 is complete. □

REMARK 1. Let $\mathcal{Q}$ be an extension property satisfying the following conditions (1)–(3):

(1) $\mathcal{Q}$ is hereditary.
(2) The two-point discrete space has $\mathcal{Q}$.
(3) There is a space which does not have $\mathcal{Q}$.

Then a property $\mathcal{P}_{\mathcal{Q}}$ defined by "R has $\mathcal{P}_{\mathcal{Q}}$ if and only if R has both $\mathcal{P}$ and $\mathcal{Q}$" is an extension property such that the $\mathcal{P}_{\mathcal{Q}}$-regularity is $\mathcal{Q}$. By [16, Proposition 1.4] every 0-dimensional space has $\mathcal{Q}$, and every space R with $\mathcal{Q}$ contains no copy of I. For, if the space R contains I, then every space S can be embedded in $R^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$, so S has $\mathcal{Q}$ by (1), which contradicts (3). Therefore if Y is the space Y constructed above, then $\mathcal{P}_0Y=\mathcal{P}_{\mathcal{Q}}Y$. Hence Theorem 1 remains true, with essentially the same proof, if $\mathcal{P}_0$ is replaced by $\mathcal{P}_{\mathcal{Q}}$. Let E be an hereditarily indecomposable continuum (cf. [10]). Then E-complete regularity in the sense of Engelking and Mrówka [7] is an example of such a property $\mathcal{Q}$ which is not 0-dimensionality.

Before proving Theorem 2, we prove the following.

THEOREM 2′. *If $\mathcal{P}$ satisfies condition* (e) *and $\mathcal{P}_0\neq\mathcal{X}$, then every connected, separable, metric space with $\mathcal{P}$ is compact.*

Proof. Let T be a connected, separable, metric space with $\mathcal{P}$. As noted in the introduction, either $\mathcal{P}$ is contained in countable compactness or $\mathcal{P}$ contains ultrarealcompactness. In the former case, T is compact, since each countably compact metric space is compact, so it remains to settle the later case. Suppose that T is not compact. By [14, Ch. 5, 4.4], there is a family $\{P_\alpha\}_{\alpha<\omega_1}$ of dense 0-dimensional subspaces of T such that $P_\alpha \subset P_\beta$ if $\alpha < \beta$ and $T = \bigcup_{\alpha<\omega_1} P_\alpha$. Consider the product space $W^* \times T$ and its subspace

$$L^* = \left\{(\alpha, t) \,\middle|\, (\alpha, t) \in W^* \times T, t \notin \bigcup_{\beta>\alpha} P_\beta\right\}.$$

It was proved in [14] that $L^* \cap (W \times T)$ is 0-dimensional. Since $\mathcal{P}_0 \neq \mathcal{Z}$, there is a 0-dimensional space S which does not have $\mathcal{P}$. Similarly to the proof of Theorem 1, we can find $s^* \in \beta_0 S - S$ such that $\beta_0 S - \{s^*\}$ does not have $\mathcal{P}$. Let $P^* = W^* \times \beta_0 S$, and pick $t^* \in P_0$. Let Y^* be the quotient space obtained from the disjoint sum $P^* \oplus L^*$ by identifying $(\alpha, s^*) \in P^*$ and $(\alpha, t^*) \in L^*$ for each $\alpha \leq \omega_1$, and let $\psi : P^* \oplus L^* \to Y^*$ be the quotient map. Let us set

$$Y = Y^* - \psi(\{(\omega_1, s^*)\} \oplus (\{\omega_1\} \times T)).$$

Then Y is 0-dimensional. Next, let $Y_0 = \{y_0\} \cup Y$ be the quotient space obtained from Y^* by collapsing the set $Y^* - Y$ to a point y_0.

CLAIM 1. *$\mathcal{P}_0 Y = Y_0$ and $\mathcal{P} Y \subset Y^*$.*

Proof. It can be proved quite similarly to the proof of Theorem 1 that $\mathcal{P}_0 Y = Y_0$ and Y is C^*-embedded in Y^*. Thus $Y^* \subset \beta Y$. By Lemma 1, it suffices to show that Y^* has $\mathcal{P}$. Suppose not; then there is a point $p \in \mathcal{P} Y^* - Y^*$. If we set $R = Y^* - Y$, then R has $\mathcal{P}$ and is a retract of Y^*, which shows that $R = \mathcal{P} R = cl_* R$, where $cl_* R$ denotes the closure of R in $\mathcal{P} Y^*$. Thus there exist open neighborhoods U and V of p in $\mathcal{P} Y^*$ such that $cl_* U \cap R = \emptyset$ and $cl_* V \subset U$. Set $J = cl_* U \cap Y$. As is easily seen, J is a 0-dimensional Lindelöf space, so we can find a clopen set H in J with $cl_* V \cap Y \subset H \subset U \cap Y$ and a countable disjoint open cover $\{G_n\}_{n \in N}$ of J such that $p \notin cl_* G_n$ for each $n \in N$. Note that H is clopen in Y^*. Set

$$Z = cl_{\beta Y^*}(Y^* - H) \cup \left(\bigcup_{n \in N} cl_{\beta Y^*}(H \cap G_n)\right).$$

Then $Y^* \subset Z \subset \beta Y^*$ and $p \notin Z$. Since Z is a countable union of disjoint compact open subspaces, it is ultrarealcompact, so Z has $\mathcal{P}$ by our assumption, and hence $\mathcal{P} Y^* \subset Z$ by Lemma 1. This contradicts the fact that $p \in \mathcal{P} Y^*$. Consequently, Y^* has $\mathcal{P}$. $\square$

Let X be the countable product of discrete spaces of cardinality $\aleph_1$. Since X is N-compact, it is ultrarealcompact, so $X = \mathcal{P} X = \mathcal{P}_0 X$. Recall from [1] that a space is of *pointwise countable type* if each point is contained in a compact subset of countable character. It is known ([1]) that spaces of pointwise countable type are k-spaces and are preserved by countable products.

CLAIM 2. $\mathcal{P}(X\times Y)=\mathcal{P}X\times\mathcal{P}Y(=X\times\mathcal{P}Y)$.

Proof. Observe that Y^* is a Lindelöf space. Since $Y^*\subset\beta Y$ and since each nonempty G_δ-set in Y^* meets Y, Y^* is the Hewitt realcompactification υY of Y by [6, 3.11.10]. It is easily checked that Y^* is of pointwise countable type, and hence so is $X\times Y^*$. It follows from [4, Theorem 2.7] that $X\times Y$ is C^*-embedded in $X\times\upsilon Y(=X\times Y^*)$, so $X\times Y^*\subset\beta(X\times Y)$. Since $\mathcal{P}Y\subset Y^*$ and there is no space Z' having $\mathcal{P}$ for which $X\times Y\subset Z'\subsetneqq X\times\mathcal{P}Y$, $\mathcal{P}(X\times Y)=X\times\mathcal{P}Y$ by Lemma 1. □

CLAIM 3. $\mathcal{P}_0(X\times Y)\neq\mathcal{P}_0X\times\mathcal{P}_0Y(=X\times\mathcal{P}_0Y)$.

Proof. Since T is not compact, there is a countable locally finite family $\{E_n\}_{n\in N}$ of open sets in T such that $t^*\notin\bigcup_{n\in N}E_n$. For each $n\in N$, let $F_n=\psi((W\times E_n)\cap L^*)$. Then $\{F_n\}_{n\in N}$ is a locally finite family of open sets in Y with $y_0\in\bigcap_{n\in N}cl_{\mathcal{P}_0Y}F_n$. Since each F_n is a union of $\aleph_1$ many clopen sets in Y, $\{F_n\}_{n\in N}$ is a $D_0(\aleph_1)$-expandable family in Y. Since each point of X has no pseudo-$\aleph_1$-compact neighborhood, it follows from Lemma 3 that $\mathcal{P}_0(X\times Y)\neq X\times\mathcal{P}_0Y$. Hence the proof of Theorem 2′ is complete. □

Proof of Theorem 2. By Theorem 2′, it suffices to show that there exists a $P_z(\aleph_1)$-compact, connected, separable, metric space T which is not compact. Consider the subspace $T=\bigcup_{n\in N}(I_n\cup J_n)$ of the Euclidean plane, where

$$I_n=\{(x,y)\mid x=1/n,\ 0\leq y\leq 1\},$$
$$J_n=\{(x,y)\mid x^2+y^2=1/n^2,\text{ and } x\leq 0 \text{ or } y\leq 0\}.$$

Then T is connected, separable, non-compact, and $P_z(\aleph_1)$-compact, since it is a countable union of disjoint compact zero-sets. Hence the proof is complete. □

REMARK 2. If $\mathcal{P}$ contains $P_z(\aleph_1)$-compactness and $\mathcal{P}_0\neq\mathcal{Z}$, then it follows from Theorems 1 and 2 that $\mathcal{P}$ satisfies neither (d) nor (e). Examples of such extension properties are $P_z(\aleph_1)$-compactness, realcompactness, almost realcompactness and Dieudonné completeness.

4. **Extension properties satisfying (e) but not (d).** The embedding i_X of X in $\mathcal{P}_0X$ extends to a unique continuous map $\mathcal{P}i_X:\mathcal{P}X\to\mathcal{P}_0X$. The following lemma was essentially proved by Broverman in the proof of [3, Theorem 2.2].

LEMMA 4. *If $\mathcal{P}(X\times Y)=\mathcal{P}X\times\mathcal{P}Y$ and if $\mathcal{P}i_X\times\mathcal{P}i_Y$ is a quotient map from $\mathcal{P}X\times\mathcal{P}Y$ onto $\mathcal{P}_0X\times\mathcal{P}_0Y$, then $\mathcal{P}_0(X\times Y)=\mathcal{P}_0X\times\mathcal{P}_0Y$.*

Let us say that a space R is *ultracomplete* if for each $p \in \beta R - R$ there is a disjoint open cover $\mathfrak{U}$ of R such that $p \notin cl_{\beta R} U$ for each $U \in \mathfrak{U}$. As noted in [13], ultracompleteness is an extension property, and every ultracomplete space is Dieudonné complete. By Theorem 1, neither ultrarealcompactness nor ultracompleteness satisfies (d), while we have the following theorem.

THEOREM 3. *Both ultrarealcompactness and ultracompleteness satisfy* (e).

Proof. Let $\mathcal{P}$ be ultracompleteness (ultrarealcompactness). By Lemma 4 and [6, 3.7.7], it suffices to show that for each $X, \mathcal{P}i_X : \mathcal{P}X \to \mathcal{P}_0 X$ is perfect onto. The embedding i_X extends to a continuous map $\beta i_X : \beta X \to \beta_0 X$. Note that $\mathcal{P}i_X = (\beta i_X) \mid \mathcal{P}X$. If $\mathcal{P}i_X$ is not perfect onto, then there is a point $p \in \beta X - \mathcal{P}X$ with $(\beta i_X)(p) \in \mathcal{P}_0 X$. Since $\mathcal{P}X$ has $\mathcal{P}$, there is a (countable) disjoint open cover $\{U_\alpha\}_{\alpha \in A}$ of $\mathcal{P}X$ such that $p \notin cl_{\beta X} U_\alpha$ for each $\alpha \in A$. Set $V_\alpha = cl_{\beta_0 X} i_X(U_\alpha \cap X)$ and $q = (\beta i_X)(p)$. Then V_α is clopen in $\beta_0 X$. If $q \in V_\alpha$, then $W = (\beta i_X)^{-1}(V_\alpha) - cl_{\beta X} U_\alpha$ is an open neighborhood of p in βX with $W \cap X = \varnothing$, which is impossible. Thus $q \notin V_\alpha$ for each $\alpha \in A$. Let us set $Z = \bigcup_{\alpha \in A} V_\alpha$. Then $q \notin Z$, and Z has $\mathcal{P}$ since it is a (countable) union of disjoint compact open subspaces, so $\mathcal{P}_0 X \subset Z$ by Lemma 1, and hence $q \notin \mathcal{P}_0 X$. This contradiction completes the proof. $\square$

The following problem is still open.

PROBLEM. *Characterize an extension property $\mathcal{P}$ satisfying* (e).

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